Due Tuesday February 26, in class

1. A coin is flipped four times. Let X denote the number of heads obtained. Plot the probability mass function of the random variable $X - 2$.

   <u>Solution:</u> Let Y=X-2.

$$
\begin{aligned}
P(X=0) &= P(Y=-2) = \binom{4}{0}(\tfrac{1}{2})^4 = \frac{1}{16} \\
P(X=1) &= P(Y=-1) = \binom{4}{1}(\tfrac{1}{2})^4 = \frac{1}{4} \\
P(X=2) &= P(Y=0) = \binom{4}{2}(\tfrac{1}{2})^4 = \frac{3}{8} \\
P(X=3) &= P(Y=1) = \binom{4}{3}(\tfrac{1}{2})^4 = \frac{1}{4} \\
P(X=4) &= P(Y=2) = \binom{4}{4}(\tfrac{1}{2})^4 = \frac{1}{16}
\end{aligned}
$$

2. If X has distribution function F, what is the distribution function of e^X? (Distribution function is a synonymn for c.d.f.)

   <u>Solution:</u> Let $Y = e^X$. Since we know that X has a distribution function F,

$$
\begin{aligned}
F_Y(y) &= P(Y \le y) = P(e^X \le y) \\
&= P(X \le ln(y)) \\
&= F(ln(y))
\end{aligned}
$$

3. Suppose that two teams play a series of games that ends when one of them has won i games. Suppose that each game played is, independently, won by team A with probability p. Find the expected number of games that are played when (a) $i = 2$, and (b) $i = 3$. Also show that, in both cases, this number is maximized when $p = 1/2$.

   <u>Solution:</u> Let X denote the number of games played.

   (a) When $i = 2$,

$$
\begin{aligned}
E(X) &= 2[p^2 + (1-p)^2] + 3[2p^2(1-p) + 2p(1-p)^2] \\
&= -2p^2 + 2p + 2
\end{aligned}
$$

   By letting

$$
\frac{d}{dp}EX = 0,
$$

   we can get $p = \frac{1}{2}$ at which EX is maximized. ($\frac{d^2}{dp^2}EX|_{p=\frac{1}{2}} < 0$)

(b) When $i = 3$,

$$
\begin{aligned}
E(X) &= 3[p^3 + (1-p)^3] + 4[3p^3(1-p) + 3p(1-p)^3] + 5[6p^3(1-p)^2 + 6p^2(1-p)^3] \\
&= 6p^4 - 12p^3 + 3p^2 + 3p + 3
\end{aligned}
$$

By letting
$$
\frac{d}{dp}EX = 0,
$$

we can get $p = \frac{1}{2}$ at which EX is maximized. ($\frac{d^2}{dp^2}EX|_{p=\frac{1}{2}} < 0$)

4. You have \$1000 and a certain commodity presently sells for \$2 per ounce. Suppose that after one week the commodity will sell for either \$1 or \$4 an ounce, with both these possibilities being equally likely. A trading strategy is a plan specifying amounts of the commodity to buy or sell at the beginning and end of the week.

(a) If your objective is to maximize the expected amount of money that you posess at the end of the week, what strategy should you employ?

(b) If your objective is to maximize the expected amount of the commodity that you possess at the end of the week, what strategy should you employ?

Solution: Suppose that you buy x ounces during the first week, and then you will have $1000 - 2x$ dollars until next week.

(a) Let M denote the amount of money that you possess at the end of the week.

$$
\begin{aligned}
EM &= (1000 - 2x + x)(\frac{1}{2}) + (1000 - 2x + 4x)(\frac{1}{2}) \\
&= 1000 + \frac{1}{2}x
\end{aligned}
$$

The expected amount of money would be larger if you bought more during the first week. Therefore, the best strategy would be to use your all money to get 500 ounces of the commodity and sell them after one week.

(b) Let N denote the amount of the commodity.

$$
\begin{aligned}
EN &= (x + 1000 - 2x)\frac{1}{2} + (x + \frac{1000 - 2x}{4})\frac{1}{2} \\
&= -\frac{1}{4}x + 625
\end{aligned}
$$

The expected amount of the commodity would be smaller if you bought more during the first week. Therefore, the best strategy would be not to buy anything during the first week, and spend all the money after one week.

5. Each night different meterologists give us the probability that it will rain the next day. To judge how well these people predict, we will score each of them as follows: if a meteorologist says that it will rain with probability p, then he or so will receive a score of

$$
\begin{array}{ll}
1 - (1-p)^2 & \text{if it does rain} \\
1 - p^2 & \text{if it does not rain}
\end{array}
$$

We will then keep track of scores over a certain time span and conclude that the meteorologist with the highest average score is the best predictor of the weather. Suppose now that a given meterologist is aware of this and so wants to maximize his or her expected score. If this person truly believes that it will rain tomorrow with probability p^*, what value of p should he or she assert so as to maximize the expected score?

Solution:

$$E[score] = p^*[1 - (1-p)^2] + (1-p^*)[1-p^2]$$

$$\frac{dE}{dp} = 2(1-p)p^* - 2p(1-p^*) = 0$$

$$p = p^*$$

6. If $\mathbb{E}(X) = 1$ and $\text{Var}(X) = 5$, find (a) $\mathbb{E}[(2+X)^2]$, and (b) $\text{Var}(4+3X)$.

Solution:

(a)

$$E[(2+X)^2] = Var(2+X) + (E[2+X])^2 = Var(X) + 9 = 14$$

(b)

$$Var(4+3X) = 9Var(X) = 45$$