Statistics 100A
Homework 8 Solutions
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Part 1: Chapter 7

1. *A player throws a fair die and simultaneously flips a fair coin. If the coin lands heads, then she wins twice, and if tails, the one-half of the value that appears on the die. Explain her expected winnings.*

   First, this problem is poorly worded. If the coin lands heads, then she wins twice the value on the die, otherwise she wins $\frac{1}{2}$ the value that appears on the die.

   Let X represent the number of pips that show on the die. Let Y be the result of the toss of the fair coin, such that $Y = 1$ if the coin lands heads, and $Y = 0$ otherwise.

   Then,

   $$E(XY) = \begin{cases} 2X, & Y = 1 \\ \frac{1}{2}, & Y = 0 \end{cases}$$

   Each value of Y occurs with probability $\frac{1}{2}$. So,

   $$
   \begin{aligned}
   E(XY) &= \frac{1}{2}(2X) + \frac{1}{2}\left(\frac{1}{2}X\right) \\
   &= \frac{1}{2}\left(2\left[\frac{1}{6} + 2 \times \frac{1}{6} + \ldots + 6 \times \frac{6}{6}\right]\right) + \frac{1}{2}\left(\frac{1}{2}\left[\frac{1}{6} + 2 \times \frac{1}{6} + \ldots + 6 \times \frac{6}{6}\right]\right) \\
   &= \boxed{4.375}
   \end{aligned}
   $$

5. *The county hospital is located at the center of a square whose sides are 3 miles wide. If an accident occurs within this square, then the hospital sends out an ambulance. The road network is rectangular so the travel distance from the hospital, whose coordinates are (0,0), to the point (x, y) is $|x| + |y|$. If an accident occurs at a point that is uniformly distributed in the square, find the expected travel distance of the ambulance.*

   First, let $Z = |X| + |Y|$ for simplicity. We want to find $E(Z)$. Note that $-\frac{3}{2} < X < \frac{3}{2}$ and $-\frac{3}{2} < Y < \frac{3}{2}$ since the sides of the square have length 3. Thus, $|X| < \frac{3}{2}$ and $|Y| < \frac{3}{2}$.

   $$
   \begin{aligned}
   E(Z) &= E\left(|X| + |Y|\right) \\
   &= E|X| + E|Y| \\
   &= \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{x}{\frac{3}{2} - \left(-\frac{3}{2}\right)} dx + \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{y}{\frac{3}{2} - \left(-\frac{3}{2}\right)} dy \\
   &= 2\int_{0}^{\frac{3}{2}} \frac{x}{3} dx + 2\int_{0}^{\frac{3}{2}} \frac{y}{3} dy \\
   &= \boxed{\frac{3}{2}}
   \end{aligned}
   $$

6. *A fair die is rolled 10 times. Calculate the expected sum of the 10 rolls.*

This is a sum X of 10 independent random variables X_i.

$$
\begin{aligned}
E(X) &= E\left(\sum_{i=1}^{10} X_i\right) \\
&= \sum_{i=1}^{10} E(X_i) \\
&= \sum_{i=1}^{10} \frac{1}{6} \times 1 + \frac{1}{6} \times 2 + \ldots + \frac{1}{6} \times 6 \\
&= \sum_{i=1}^{10} \frac{21}{6} \\
&= 10\left(\frac{21}{6}\right) \\
&= \boxed{35}
\end{aligned}
$$

7. *Suppose that A and B each randomly, and independently, choose 3 of 10 objects. Find the expected number of objects*

(a) *chosen by both A and B.*

Let X be the number of objects that are selected by *both* A and B. To further simplify the problem we use indicator variables X_i. Let $X_i = 1$ if object i is selected by both A and B, and $X_i = 0$ otherwise, where $1 \leq i \leq 10$. Then,

$$
E(X) = E\left(\sum_{i=1}^{10} X_i\right) = \sum_{i=1}^{10} E(X_i)
$$

Now we must find $E(X_i)$. We know that X_i only takes on one of two values, $X_i = 1$ or $X_i = 0$. So, for the case of a sum of independent random indicator variables, $E(X_i) = P(X_i = 1)$.

Each person can choose 3 of the 10 items. There are 3 ways to choose the item of interest, since a person can draw 3 objects. Since person A and B draw independently,

$$
P(X_i = 1) = \left(\frac{3}{10}\right)^2
$$

Then,

$$
E(X) = \sum_{i=1}^{10} E(X_i) = \sum_{i=1}^{10} \left(\frac{3}{10}\right)^2 = 10\left(\frac{3}{10}\right)^2 = \boxed{0.9}
$$

(b) *not chosen by either A or B.*

The principle is similar to part (a). Let $X_i = 1$ if object i is not chosen by A *and* is not chosen by B. $P(X_i = 1) = \left(\frac{7}{10}\right)^2$, because the probability that an arbitrary person does not choose object i is $\frac{7}{10}$ and person A and person B draw independently. Then,

$$E(X) = \sum_{i=1}^{10} E(X_i) = \sum_{i=1}^{10} \left(\frac{7}{10}\right)^2 = 10 \left(\frac{7}{10}\right)^2 = \boxed{4.9}$$

(c) *Chosen by exactly one of A or B*

In this case, either person A draws object i and person B does not, *or* person B draws object i and person A does not. Again, let $X_i = 1$ if exactly one of A or B draws object i, $X_i = 0$ otherwise. The person that eventually draws object i had probability $\frac{3}{10}$ of drawing the object and the person that does not draw object i had probability $\frac{7}{10}$ of not drawing object i. But there are two ways to arrange this situation. A can draw the object, and B does not, or B draws the object and A does not. Thus,

$$E(X_i) = P(X_i = 1) = 2 \times \frac{3}{10} \times \frac{7}{10}$$

and

$$E(X) = 10 \left(2 \times \frac{3}{10} \times \frac{7}{10}\right) = \boxed{4.2}$$

9. *A total of n balls, numbered 1 through n, are put into n urns, also numbered 1 through n in such a way that ball i is equally likely to go into any of the urns $1, 2, \ldots, i$. Find*

(a) *the expected number of urns that are empty.*

Let X be the number of empty urns. Define an indicator variable $X_i = 1$ if urn i is empty, $X_i = 0$ otherwise. We must find $E(X_i) = P(X_i = 1)$.

The ith urn remains empty as the first $i - 1$ balls are deposited into the other urns. On the ith drop, urn i must remain empty. Since a ball can land in any of the i urns with equal probability, the probability that the ith ball will not land in urn i is $1 - \frac{1}{i}$. To remain empty, the urn must not receive a ball on the $i+1$st drop etc. so the probability that the $i + 1$st ball will not land in urn i, is $1 - \frac{1}{i+1}$. So,

$$\begin{aligned}
E(X_i) &= \left(1 - \frac{1}{i}\right)\left(1 - \frac{1}{i+1}\right)\left(1 - \frac{1}{i+2}\right)\cdots\left(1 - \frac{1}{n}\right) \\
&= \left(\frac{i-1}{i}\right)\left(\frac{i}{i+1}\right)\cdots\left(\frac{n-1}{n}\right) \\
&= \frac{i-1}{n}
\end{aligned}$$

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So,

$$
\begin{aligned}
E(X) &= \sum_{i=1}^{n} E(X_i) \\
&= \sum_{i=1}^{n} \frac{i-1}{n} \\
&= \frac{1}{n} \sum_{i=1}^{n} i - 1 \\
&= \frac{1}{n} \sum_{j=0}^{n-1} j \\
&= \frac{1}{n} \cdot \frac{n(n-1)}{2} \\
&= \boxed{\frac{n-1}{2}}
\end{aligned}
$$

(b) *the probability that none of the urns is empty.*

For all of the urns to have at least one ball in them, the nth ball must be dropped into the nth urn, which has probability $\frac{1}{n}$. The $n-1$st ball must be placed in the $n-1$st urn which has probability $\frac{1}{n-1}$ and so on. So, the probability that none of the urns will be empty is

$$
\frac{1}{n} \times \frac{1}{n-1} \times \frac{1}{n-2} \times \ldots \times \frac{1}{2} \times \frac{1}{1} = \boxed{\frac{1}{n!}}
$$

17. *A deck of n cards, numbered 1 through n, is thoroughly shuffled so that all possible $n!$ orderings can be assumed to be equally likely. Suppose you are to make n guesses sequentially, where the ith one is a guess of the card in position i. Let N denote the number of correct guesses.*

   (a) *If you are not given any information about your earlier guesses show that, for any strategy, $E(N) = 1$.*

   Proof. Let N_i be an indicator variable such that $N_i = 1$ if the ith guess is correct. The probability that the guess is correct is simply $\frac{1}{n}$. Then,

$$
E(N) = \sum_{i=1}^{n} E(N_i) = \sum_{i=1}^{n} P(N_i = 1) = \sum_{i=1}^{n} \frac{1}{n} = n \times \frac{1}{n} = 1
$$

$\square$

   (b) *Suppose that after each guess you are shown the card that was in the position in question. What do you think is the best strategy? Show that under this strategy*

$$
E(N) = \frac{1}{n} + \frac{1}{n-1} + \ldots + 1 \approx \int_{1}^{n} \frac{1}{x} dx = \log n
$$

   Once the card is shown, and assuming the person is not a complete idiot, the player has $n-1$ cards remaining to guess from. Therefore, the best strategy (or is it just common sense?) is to not guess any cards that have already been shown.

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Proof. After a card is shown, there is one less card to guess from given our elegant strategy, so the denominator in the probability dereases by one at each guess. So,

$$E(N) = \sum_{i=1}^{n} E(N_i) = \frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \ldots + \frac{1}{1}$$

which is a discrete sum. For large n, the sum can be approximated as an integral.

$$\int_{1}^{n} \frac{1}{x} dx = \log n$$

$\square$

(c) *Suppose that you are told after each guess whether you are right or wrong. In this case it can be shown that the strategy that maximizes $E(N)$ is one which keeps on guessing the same card until you are told you are correct and then changes to a new card. For this strategy show that*

$$E(N) = 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} \approx e - 1$$

Proof. Let N_i be an indicator variable such that $N_i = 1$ if the guess about card i is eventually correct. This occurs only if of cards 1 through i, the first card is card 1, the second card is card 2 and so on, and the last card is card i. There is only one possible way this arrangement can occur, and the total number of arrangements of cards is $n!$. Therefore,

$$E(N) = \sum_{i=1}^{n} E(N_i) = \sum_{j=1}^{n} \frac{1}{j!}$$

Note that this is the power series representation of $e - 1$.

$\square$

21. *For a group of 100 people, compute*

(a) *the expected number of days of the year that are birthdays of exactly 3 people.*

Let X be the number of days of the year that are birthdays for exactly 3 people. Let $X_i = 1$ if day i is a birthday for 3 people. This is a **binomial** problem with $n = 100$ and $p = \left(\frac{1}{365}\right)^3$. Then,

$$E(X_i) = P(X_i = 1) = \binom{100}{3} \left(\frac{1}{365}\right)^3 \left(\frac{364}{365}\right)^{97}$$

Recall that i represents a day of the year, so in the following calculation, we sum $i = 1$ to $i = 365$.

$$E(X) = \sum_{i=1}^{365} E(X_i) = \boxed{365 \times \binom{100}{3} \left(\frac{1}{365}\right)^3 \left(\frac{364}{365}\right)^{97}}$$

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(b) *the expected number of distinct birthdays.*

You may be tempted to think of this problem as "the number of people that have different birthdays," but that is incorrect. Let X be the number of distinct birthdays; that is, the number of days of the year that are "occupied" by a birthday. Let $X_i = 1$ if someone has a birthday on day i. We use the complement. We find the probability that all 100 people have different birthdays and subtract from 1.

$$E(X_i) = 1 - \left(\frac{364}{365}\right)^{100}$$

Then,

$$E(X) = \sum_{i=1}^{365} E(X_i) = \boxed{365 \times \left[1 - \left(\frac{364}{365}\right)^{100}\right]}$$

38. *The random variables X and Y have a joint density function given by*

$$f(x, y) = \begin{cases} \frac{2e^{-2x}}{x} & 0 \leq x < \infty, 0 \leq y < x \\ 0 & \text{otherwise} \end{cases}$$

Compute $Cov(X, Y)$.

By definition, $\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$.

$$\begin{aligned} f_X(x) &= \int_0^x \frac{2e^{2x}}{x} dy = 2e^{-2x} \\ \\ E(X) &= \int_0^x 2xe^{-2x} dx \\ &\quad \text{let } u = x, dv = e^{-2x} \text{ so } du = dx, v = -\tfrac{1}{2}e^{-2x} \\ &= \frac{1}{2} \\ \\ E(Y) &= \int_0^\infty \int_y^\infty 2y\frac{e^{-2x}}{x} dx dy \\ &= \int_0^\infty \int_0^x \frac{2ye^{-2x}}{x} dy dx \\ &= \int_0^\infty \frac{y^2 e^{-2x}}{x}\Big|_0^x dx \\ &= \int_0^\infty xe^{-2x} dx \\ &\quad \text{take } t = 2x \text{ then } dt = 2dx. \\ &= \frac{1}{2}\int_0^\infty \frac{t}{2}e^{-t} dt \\ &= \frac{1}{4}\Gamma(2) = \frac{1}{4} \end{aligned}$$

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Finally,

$$
\begin{aligned}
E(XY) &= \int_0^\infty \int_0^x xy \frac{2e^{-2x}}{x} \, dy \, dx \\
&= \int_0^\infty \int_0^x 2ye^{-2x} \, dy \, dx \\
&= \int_0^\infty y^2 \, e^{-2x} \big|_0^x \, dx \\
&= \int_0^\infty x^2 e^{-2x} \, dx \\
&\quad \text{let } t = 2x \text{ then } dt = 2dx. \\
&= \frac{1}{2} \cdot \left(\frac{1}{2} \right)^2 \int_0^\infty t^2 e^{-t} \, dt \\
&= \frac{1}{8} \Gamma(3) \\
&= \frac{1}{4}
\end{aligned}
$$

So at last,

$$
\text{Cov}\,(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{4} = \boxed{\frac{1}{8}}
$$

40. *The joint density function of X and Y is given by*

$$
f(x,y) = \frac{1}{y} e^{\left(y + \frac{x}{y} \right)}
$$

Find $E(X)$, $E(Y)$ and show that $Cov\,(X,Y) = 1$.

Proof. By definition,

$$
\text{Cov}\,(X,Y) = E(XY) - E(X)E(Y)
$$

This problem involves a trick, and we must solve $E(Y)$ first.

$$
\begin{aligned}
f_Y(y) &= \int_0^\infty \frac{1}{y} e^{\left(y + \frac{x}{y} \right)} \, dx \\
&= \frac{e^{-y}}{y} \int_0^\infty e^{-\frac{x}{y}} \, dx \\
&= -e^{-y} \left[e^{-\frac{x}{y}} \right]_0^\infty \\
&= e^{-y}
\end{aligned}
$$

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Then,

$$
\begin{aligned}
E(Y) &= \int_0^\infty y e^{-y} dy \\
&= \boxed{1}
\end{aligned}
$$

Using $E(Y)$, we can find $E(X)$. Using properties of conditional expectation,

$$E(X) = E(E(X|Y))$$

We can find the distribution of $X|Y$ by taking the joint density function and treating y as a constant. To make it clearer, let $y = c$, some constant. Then,

$$f_{X|Y}(x|y) = \frac{1}{c} e^{-c-\frac{x}{c}} = \frac{1}{c} e^{-c} e^{-\frac{1}{c}x}$$

which "looks" exponential with parameter $\lambda = \frac{1}{c}$. The expected value of an exponential random variable is $\frac{1}{\lambda}$. But, $\lambda = \frac{1}{c} = \frac{1}{Y}$, so

$$E(X) = E(E(X|Y)) = E(Y) = \boxed{1}$$

Now we use another property of conditional expectation to find $E(XY)$.

$$E(XY) = E(E(XY|Y)) = E(YE(X|Y))$$

but $E(X|Y) = Y$, so $E(XY) = E(YE(X|Y)) = E\left(Y^2\right)$.

We know that the moment-generating function of Y is

$$M_Y(t) = \frac{\lambda}{\lambda - t}$$

To find the second moment, take the second derivative and set $t = 0$.

$$M_Y''(t) = \frac{2\lambda}{(\lambda - t)^2}$$

$$E(Y^2) = M_Y''(0) = \frac{2}{\lambda} = 2$$

because $E(Y) = \frac{1}{\lambda} = 1$ so $\lambda = 1$.

So, $E(XY) = 2$. Then,

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 2 - 1 \times 1 = 1$$

$\square$

45. *If X_1, X_2, X_3, X_4 are (pairwise) uncorrelated random variables each having mean 0 and variance 1, compute the correlations of*

Recall that *pairwise uncorrelated* means that any pair of these variables are uncorrelated.

(a) $X_1 + X_2$ *and* $X_2 + X_3$.

By definition,

$$
\begin{aligned}
\mathrm{Corr}\,(X_1 + X_2, X_2 + X_3) &= \frac{\mathrm{Cov}\,(X_1 + X_2, X_2 + X_3)}{\sigma_{X_1 + X_2}\sigma_{X_2 + X_3}} \\
&= \frac{\mathrm{Cov}\,(X_1, X_2) + \mathrm{Cov}\,(X_1, X_3) + \mathrm{Cov}\,(X_2, X_2) + \mathrm{Cov}\,(X_2, X_3)}{\sqrt{\mathrm{Var}\,(X_1) + \mathrm{Var}\,(X_2) - 2\mathrm{Cov}\,(X_1, X_2)}\sqrt{\mathrm{Var}\,(X_2) + \mathrm{Var}\,(X_3) - 2\mathrm{Cov}\,(X_2, X_3)}} \\
&\quad \mathrm{Cov}\,(X_i, X_j) = 0 \text{ for } i \neq j \text{ due to pairwise uncorrelatedness.} \\
&= \frac{\mathrm{Cov}\,(X_2, X_2)}{\sqrt{\mathrm{Var}\,(X_1) + \mathrm{Var}\,(X_2)}\sqrt{\mathrm{Var}\,(X_2) + \mathrm{Var}\,(X_3)}} \\
&\quad \mathrm{Cov}\,(X_i, X_i) = \mathrm{Var}\,(X_i) \\
&= \frac{\mathrm{Var}\,(X_2)}{\sqrt{\mathrm{Var}\,(X_1) + \mathrm{Var}\,(X_2)}\sqrt{\mathrm{Var}\,(X_2) + \mathrm{Var}\,(X_3)}} \\
&= \frac{1}{\sqrt{2}\sqrt{2}} \\
&= \boxed{\frac{1}{2}}
\end{aligned}
$$

(b) $X_1 + X_2$ *and* $X_3 + X_4$.

$$
\begin{aligned}
\mathrm{Corr}\,(X_1 + X_2, X_3 + X_4) &= \frac{\mathrm{Cov}\,(X_1 + X_2, X_3 + X_4)}{\sigma_{X_1 + X_2}\sigma_{X_3 + X_4}} \\
&= \frac{\mathrm{Cov}\,(X_1, X_3) + \mathrm{Cov}\,(X_1, X_4) + \mathrm{Cov}\,(X_2, X_3) + \mathrm{Cov}\,(X_2, X_4)}{\sqrt{\mathrm{Var}\,(X_1) + \mathrm{Var}\,(X_2) - 2\mathrm{Cov}\,(X_1, X_2)}\sqrt{\mathrm{Var}\,(X_3) + \mathrm{Var}\,(X_4) - 2\mathrm{Cov}\,(X_3, X_4)}} \\
&\quad \mathrm{Cov}\,(X_i, X_j) = 0 \text{ for } i \neq j \text{ due to pairwise uncorrelatedness.} \\
&= \boxed{0}
\end{aligned}
$$

Part 2

1. *Suppose that the random variable X is described by the pdf $f_X(x) = 5x^{-6}, x > 1$. What is the highest moment of X that exists?*

   First, there was a bad typo in this problem that may have confused students. There should not be a y in the problem. Everything is in terms of x. Second, the key to solving this problem is to not use the moment-generating function because we end up with an integral that cannot easily be solved. Instead, we compute the moments using $E(X^r)$. We integrate over $x > 1$.

$$\begin{aligned} E(X^r) &= \int_1^\infty x^r 5x^{-6} dx \\ &= 5 \int_1^\infty x^{r-6} dx \\ &= 5 \left[\frac{x^{r-5}}{r-5} \right]_1^\infty \end{aligned}$$

   We want to find the largest value of r such that $5 \left[\frac{x^{r-5}}{r-5} \right]_1^\infty$ exists. First, $r \neq 5$ because then the quantity is undetermined. Now consider the numerator. If $r > 5$, then we have divergence. If $r < 5$, we get a negative exponent for x and it may as well be in the denominator, which will make the whole quantity go to 0 as $x \to \infty$. Thus, $r < 5$, so the boxed fourth moment, $E(X^4)$ is the largest moment that exists.

2. *Two chips are drawn at random without replacement from a box that contains five chips numbered 1 through 5. If the sum of chips drawn is even, the random variable X equals 5; if the sum of chips drawn is odd, $X = -3$.*

   First, let's define Y, which is the sum of the two chips X_1 and X_2. $Y = X_1 + X_2$. Then,

$$X = \begin{cases} 5, & Y\,\text{even} \\ -3, & Y\,\text{odd} \end{cases}$$

   (a) *Find the moment-generating function (mgf) for X.*

   Since $E(X) = \sum_{i=1}^n x_i P(X_i = x_i)$, we need to find the probabilities $P(X = 5)$ and $P(X = -3)$. The easiest way to find these is to just enumerate all of the possible drawings.

$$P(X = -3) = \frac{12}{20} = \frac{3}{5} \qquad\qquad P(X = 5) = \frac{8}{20} = \frac{2}{5}$$

   Then,

$$\boxed{M_X(t) = E\left(e^{tx}\right) = e^{-3t}\left(\tfrac{3}{5}\right) + e^{5t}\left(\tfrac{2}{5}\right)}$$

(b) *Use the mgf to find the first and second non-central moments.*

To find the first moment, $E(X)$, take the derivative of the moment-generating function and evaluate at $t = 0$.

$$M_X'(t) = -\frac{9}{5}e^{-3t} + 2e^{5t}$$

$$M_X'(0) = -\frac{9}{5} + 2 = \frac{1}{5}$$

Thus, $\boxed{E(X) = \frac{1}{5}}$.

$$M_X''(t) = \frac{27}{5}e^{-3t} + 10e^{5t}$$

$$M_X''(0) = \frac{27}{5} + 10 = 15.4$$

So, $\boxed{E\left(X^2\right) = 15.4}$.

(c) *Find the expected value and variance of X.*

The expected value is just the first moment from part (a). $\boxed{E(X) = \frac{1}{5}}$.

$$\text{Var}\,(X) = E\left(X^2\right) - E(X)^2$$

where $E\left(X^2\right)$ is the second moment from part (a).

$$\text{Var}\,(X) = 15.4 - \left(\frac{1}{5}\right)^2 = \boxed{15.36}$$

3. *Let X have pdf*

$$f_X(x) = \begin{cases} x & 0 \le x \le 1 \\ 2 - x & 1 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

(a) *Find the moment-generating function (mgf) $M_X(t)$ for X.*

$$\begin{aligned} E\left(e^{tx}\right) &= \int_0^1 xe^{tx}\,dx + \int_1^2 (2 - x)e^{tx}\,dx \\ &= \int_0^1 xe^{tx}\,dx + 2\int_1^2 e^{tx}\,dx - \int_1^2 xe^{tx}\,dx \\ &\quad \text{take } u = x, dv = e^{tx} \text{ so } du = dx, v = \frac{1}{t}e^{tx}. \\ &= \ldots \\ &= \boxed{\frac{1}{t^2}\left(e^t - 1\right)^2} \end{aligned}$$

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(b) *Use mgf to find the first and second non-central moments.*

This part is anticlimactic. We note that the $M_X(t)$ has a singularity at $t = 0$. If we take the first and second derivatives, we are still going to have that obnoxious singularity. The first and second moments do not exist.

(c) *Find the expected value and variance of X.*

Since the first and second moments of X do not exist, we have to compute $E(X)$ and $\mathrm{Var}\,(X)$ using the definition.

$$\mathrm{Var}\,(X) = E(X^2) - E(X)^2$$

$$
\begin{aligned}
E(X) &= \int_0^1 x^2 dx + \int_1^2 x(2-x)dx \\
&= \left.\frac{x^3}{3}\right|_0^1 + \left(x^2 - \frac{x^3}{3}\right)\Big|_1^2 \\
&= \boxed{1}
\end{aligned}
$$

$$
\begin{aligned}
E(X^2) &= \int_0^1 x^3 dx + \int_1^2 x^2(2-x)dx \\
&= \left.\frac{x^4}{4}\right|_0^1 + \left[\frac{2}{3}x^3 - \frac{x^4}{4}\right]_1^2 \\
&= \frac{4}{3}
\end{aligned}
$$

The variance is then

$$\mathrm{Var}\,(X) = E(X^2) - E(X)^2 = \frac{4}{3} - 1^2 = \boxed{\tfrac{1}{3}}$$

4. *Find the moment generating function (mgf) for X, if $X \sim Uniform(a,b)$.*

Since $X \sim U(a,b)$, we know that $f_X(x) = \frac{1}{b-a}$, $a \le x \le b$.

$$
\begin{aligned}
E\left(e^{tx}\right) &= \int_a^b \frac{e^{tx}}{b-a} dx \\
&= \frac{1}{b-a} \int_a^b e^{tx} dx \\
&= \frac{1}{t(b-a)} \left[e^{tx}\right]_a^b \\
&= \boxed{\tfrac{e^{tb}-e^{ta}}{t(b-a)}}
\end{aligned}
$$

5. *Find the moment-generating function for X, if X is a Bernoulli random variable with the pdf*

$$f_X(x) = \begin{cases} p^x(1-p)^{1-x} & x = 0, x = 1 \\ 0 & \text{otherwise} \end{cases}$$

Use the mgf to find the expected value and variance of X.

Note that X can only take on one of two possible values.

$$\begin{aligned} E\left(e^{tx}\right) &= e^{0t}p^0(1-p)^{1-0} + e^{1t}p(1-p)^{1-1} \\ &= 1 - p + pe^t \\ M_X(t) &= 1 + p\left(e^t - 1\right) \end{aligned}$$

So,

$$E(X) = M_X'(0) = pe^0 = \boxed{p}$$

$$\text{Var}(X) = E(X^2) - E(X)^2 = M_X''(0) - p^2 = p - p^2 = \boxed{p(1-p)}$$

6. *Find the moment-generating function for X, if X is a Poisson random variable with the pdf*
$f_X(x) = \frac{\lambda^x}{x!}e^{-\lambda}$.

$$\begin{aligned} M_X(t) &= E\left(e^{tx}\right) \\ &= \sum_{x=0}^{\infty} e^{tx}e^{-\lambda}\frac{\lambda^x}{x!} \\ &= \sum_{x=0}^{\infty} e^{tx-\lambda}\frac{\lambda^x}{x!} \\ &= e^{-\lambda}\lambda + e^{t-\lambda}\lambda + e^{2t-\lambda}\frac{\lambda^2}{2!} + \dots \\ &= e^{-\lambda}\sum_{x=0}^{\infty} \frac{\left(\lambda e^t\right)^x}{x!} \\ &\quad \text{which resembles the power series for } e^y \text{ where } y = \lambda e^t. \\ &\rightarrow e^{-\lambda}e^{\lambda e^t} \\ &= \boxed{e^{\lambda(e^t-1)}} \end{aligned}$$

7. *Find the moment-generating function for Z, if Z is a standard normal random variable with the pdf*

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

Use your result to find the moment-generating function for a normally distributed random variable $X \sim N(\mu, \sigma)$.

This problem is very tricky, and it is a classic example of how to work with the normal distribution in advanced statistics. If you continue in Statistics, or ever take Bayesian Statistics, you will encounter working with the normal distribution in this way. **The goal is to make the computation of $M_Z(t)$ look like the original normal distribution.**

By definition,

$$M_Z(t) = E\left(e^{tz}\right) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{tz - \frac{z^2}{2}} dz$$

which is simple enough so far. In the next step, I factor out the negative sign in the exponential because the normal distribution has that pesky negative sign in front of the quadratic.

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{z^2}{2} - tz\right)} dz$$

This integral would be a nightmare to integrate because of the quadratic in the exponent, and it does not have the form of the normal distribution. To remedy this, we **complete the square**.

$$\frac{z^2}{2} - tz = \frac{(z-t)^2}{2} - \frac{t^2}{2}$$

I can now break up that exponential and factor part of it out of the integral.

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{(z-t)^2}{2} - \frac{t^2}{2}\right)} dz = \frac{e^{\frac{t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z-t)^2}{2}} dz$$

Believe it or not, we can integrate this! Recall that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} = 1 \qquad \Rightarrow \qquad \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du = \sqrt{2\pi}$$

So take $u = z - t$, then $du = dz$ and we have

$$\frac{e^{\frac{t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du = \frac{e^{\frac{t^2}{2}}}{\sqrt{2\pi}} \cdot \sqrt{2\pi} = e^{\frac{t^2}{2}}$$

Thus, $\boxed{M_Z(t) = e^{\frac{t^2}{2}}}$.

Recall that $Z = \frac{X-\mu}{\sigma}$ so, $X = Z\sigma + \mu$. To find the moment-generating function of X, we use the hint.

$$
\begin{aligned}
M_X(t) &= M_{Z\sigma+\mu}(t) \\
&= e^{\mu t} M_Z(\sigma t) \\
&= e^{\mu t} e^{\frac{(\sigma t)^2}{2}} \\
&= \boxed{e^{\mu t + \frac{(\sigma t)^2}{2}}}
\end{aligned}
$$

If you continue with Statistics, you will see that derivations involving the normal distribution always follow a similar recipe: 1) factor out a negative sign, 2) complete the square in the exponent, 3) compare to the standard normal distribution to get to the result.

8. *Let X and Y be independent random variables, and let α and β be scalars. Find an expression for the mgf of $Z = \alpha X + \beta Y$ in terms of mgfs of X and Y.*

We just use the definition of moment generating function and use independence.

$$
\begin{aligned}
M_Z(t) &= E\left(e^{tz}\right) \\
&= E\left(e^{t(\alpha x + \beta y)}\right) \\
&= E\left(e^{t\alpha x} e^{t\beta y}\right) \\
&= E\left(e^{t\alpha x}\right) E\left(e^{t\beta y}\right) \\
&\quad \text{by independence.} \\
&= \boxed{M_X(\alpha t) M_Y(\beta t)}
\end{aligned}
$$

9. *Use the mgf to show that if X follows the exponential distribution, cX $(c > 0)$ does also.*

To show that two random variables follow the same distribution, show that the moment-generating functions are equal.

Proof. From a previous problem, we know that $M_X(t) = \frac{\lambda}{\lambda - t}$.

$$
\begin{aligned}
M_{cX}(t) &= E\left(e^{ctx}\right) = \int_0^\infty e^{ctx} \lambda e^{-c\lambda x} dx \\
&= \lambda \int_0^\infty e^{ctx - c\lambda x} dx \\
&= \lambda \int_0^\infty e^{xc(t-\lambda)} d(cx) \\
&= \frac{c\lambda}{c(t-\lambda)} \left[e^{cx(t-\lambda)}\right]_0^\infty = \frac{\lambda}{t-\lambda}\left[e^{-cx(\lambda-t)}\right]_0^\infty \\
&= \frac{\lambda}{t-\lambda}(-1) = \frac{\lambda}{\lambda - t}
\end{aligned}
$$

Therefore, since $M_X(t) = M_{cX}(t)$, $cX \sim \text{Exp}(\lambda)$. $\qquad\square$

10. *Show that*

  (a) *If X_i follows a binomial distribution with n_i trials and probability of success $p_i = p$ where $i = 1, 2, \ldots, n$ and the X_i are independent then $\sum_{i=1}^{n} X_i$ follows a binomial distribution.*

  The probability of success p is the same for all i.

  Proof. We know that the moment generating function for a Bernoulli random variable Y_i is $M_{Y_i}(t) = 1 - p + pe^t$. We also know that a binomial random variable is just a sum of Bernoulli random variables, $Y = Y_1 + \ldots + Y_{n_i}$. Then by the properties of moment-generating functions,

  $$M_Y(t) = M_{Y_1 + \ldots + Y_{n_i}}(t) = M_{Y_1}(t) \times \ldots \times M_{Y_{n_i}}(t)$$

  So analogously,

  $$M_{X_i}(t) = E\left(e^{tx_i}\right) = (1 - p + pe^t)^{n_i}$$

  All of the x_is are independent, then similarly,

  $$
  \begin{aligned}
  M_{\sum_{i=1}^{n} X_i} &= \prod_{i=1}^{n} M_{X_i}(t) \\
  &= (1 - p + pe^t)^{n_1} \ldots (1 - p + pe^t)^{n_n} \\
  &= (1 - p + pe^t)^{n_1 + \ldots + n_n} \\
  &= (1 - p + pe^t)^{\sum_{i=1}^{n} n_i}
  \end{aligned}
  $$

  Therefore, $\sum_{i=1}^{n} X_i$ is binomial with parameters $N = \sum_{i=1}^{n} n_i$ and p. $\qquad\square$

  (b) *Referring to part a, show that if p_i are unequal, the sum $\sum_{i=1}^{n} X_i$ does not follow a binomial distribution.*

  Now the probability of success is not the same for each i. Replace p in the previous problem with p_i.

  Proof.

  $$
  \begin{aligned}
  M_{\sum_{i=1}^{n} X_i} &= \prod_{i=1}^{n} M_{X_i}(t) \\
  &= (1 - p_1 + p_1 e^t)^{n_1} \ldots (1 - p_n + p_n e^t)^{n_n}
  \end{aligned}
  $$

  which is the best we can do, and does not resemble the binomial. Since we do not have a constant probability of success p for all i, the sum is not a binomial random variable.

  $\qquad\square$