ISyE 6761 — Fall 2012

Homework #1 Solutions (revised 10/6/12)

1. The probability of winning on a single toss of the dice is p. Player A starts, and if he fails, he passes the dice to B, who then attempts to win on her toss. They continue tossing back and forth until one of them wins. What are their probabilities of winning?

   Solution: Let S and F denote "success" and "failure", respectively.

$$\begin{aligned} P(\text{A wins}) &= P(S) + P(FFS) + P(FFFFS) + \cdots \\ &= p + (1-p)(1-p)p + (1-p)^4 p \\ &= p\sum_{i=0}^{\infty}(1-p)^{2i} = \frac{p}{1-(1-p)^2} = \frac{1}{2-p}. \quad \diamondsuit \end{aligned}$$

2. Suppose that all n men at a party throw their hats in the center of the room. Each man then randomly selects a hat. What's the probability that at least one of the men gets his own hat? What happens as $n \to \infty$?

   Solution: Let A_i be the event that man i gets his own hat, for $i = 1, 2, \ldots, n$. Then

$$\begin{aligned} &P(\text{At least one of the men gets his own hat}) \\ &= P(A_1 \cup A_2 \cup \cdots \cup A_n) \\ &= \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_iA_j) + \sum_{i<j<k} P(A_iA_jA_k) + \cdots + (-1)^{n+1}P(A_1A_2\cdots A_n) \\ &= n\cdot\frac{1}{n} - \binom{n}{2}\frac{1}{n(n-1)} + \binom{n}{3}\frac{1}{n(n-1)(n-2)} + \cdots + (-1)^{n+1}\frac{1}{n!} \\ &= 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n+1}\frac{1}{n!}. \end{aligned}$$

   Note that this quantity goes to $1 - \frac{1}{e}$ as $n \to \infty$. $\diamondsuit$

3. A fair coin is continually tossed. What's the probability that the pattern $THHH$ occurs before the pattern $HHHH$?

   Solution: When I first did this problem, I did it a very general, beautiful way that ended up taking me 30 minutes. Then I saw the trivial answer! Namely, the *only* way for $HHHH$ to occur first is if you get that papttern on your first four flips; otherwise, you are guaranteed that $THHH$ will occur first. Therefore, the answer is 15/16. $\diamondsuit$

4. A gambler has in his pocket a fair coin and a two-headed coin.

   (a) He selects one of the coins at random, and when he flips it, it comes up heads. What's the probability that it's the fair coin?

   Solution: Let F, U denote fair and unfair, respectively. We use Bayes' Rule to find

$$\begin{aligned}\mathsf{Pr}(F|H) &= \frac{\mathsf{Pr}(H|F)\mathsf{Pr}(F)}{\mathsf{Pr}(H|F)\mathsf{Pr}(F)+\mathsf{Pr}(H|U)\mathsf{Pr}(U)} \\ &= \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2}+1\cdot\frac{1}{2}} = 1/3. \quad \diamondsuit\end{aligned}$$

   (b) Suppose that he flips the coin n times, and it comes up heads each time. What's the probability that it's fair?

   Solution: As above,

$$\begin{aligned}\mathsf{Pr}(F|HH\cdots H) &= \frac{\mathsf{Pr}(HH\cdots H|F)\mathsf{Pr}(F)}{\mathsf{Pr}(HH\cdots H|F)\mathsf{Pr}(F)+\mathsf{Pr}(HH\cdots H|U)\mathsf{Pr}(U)} \\ &= \frac{\frac{1}{2^n}\cdot\frac{1}{2}}{\frac{1}{2^n}\cdot\frac{1}{2}+1\cdot\frac{1}{2}} = 1/(2^n+1). \quad \diamondsuit\end{aligned}$$

5. A die is thrown 7 times. Find

   (a) Pr('6' comes up at least once).

   Solution: $1 - \mathsf{Pr}(\text{no 6's appear}) = 1-(5/6)^7 \quad \diamondsuit$.

   (b) Pr(each face appears at least once).

   Solution: Denote the six faces by A,B,C,D,E,F. Thus, we need to find the number of tosses of the form A,A,B,C,D,E,F. We then see that

   i. The # ways to choose A is 6.

ii. The # ways to place the two A's is $\binom{7}{2}$.
iii. The # ways to permute B,C,D,E,F is 5!.
iv. The # ways to toss the die 7 times is 6^7.

Thus,

$$\mathsf{Pr}(\text{each face appears at least once}) = 6 \cdot \binom{7}{2} \cdot 5!/6^7. \quad \diamondsuit$$

6. If X is a nonnegative continuous random variable, and g is a differentiable function with $g(0) = 0$, prove that $\mathsf{E}[g(X)] = \int_0^\infty g'(t)\mathsf{Pr}(X > t)\,dt$. [We'll also assume that $\mathsf{E}[g(X)]$ is finite.]

Solution: By the Law of the Unconscious Statistician,

$$\begin{aligned}
\mathsf{E}[g(X)] &= \int_0^\infty g(x)f(x)\,dx \\
&= \int_0^\infty f(x)[g(x) - g(0)]\,dx \\
&= \int_0^\infty f(x)\int_0^x g'(t)\,dt\,dx \\
&= \int_0^\infty \int_0^x g'(t)f(x)\,dt\,dx \\
&= \int_0^\infty g'(t)\int_t^\infty f(x)\,dx\,dt \quad \text{(by Fubini)} \\
&= \int_0^\infty g'(t)\,\Pr(X > t)\,dt. \quad \diamondsuit
\end{aligned}$$

Alternatively, you can use integration by parts to obtain

$$\begin{aligned}
\int_0^\infty g'(t)\,\Pr(X > t)\,dt &= g(t)\,\Pr(X > t)\,|_0^\infty - \int_0^\infty g(t)[-f(t)]\,dt \\
&= \int_0^\infty g(t)f(t)\,dt = \mathsf{E}[g(X)],
\end{aligned}$$

where we have assumed that $\lim_{t\to\infty} g(t)\,\Pr(X > t) = 0$. $\diamondsuit$

7. Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. Exp(λ). What is the p.d.f. of $\min_i X_i$? $\max_i X_i$?

Solution: Let $Y \equiv \min_i X_i$. Then

$$P(Y > y) = P(\min(X_1, X_2, \ldots, X_n) > y)$$

$$
\begin{aligned}
&= P(X_1 > y, X_2 > y, \ldots, X_n > y) \\
&= \prod_{i=1}^{n} P(X_i > y) \\
&= e^{-n\lambda y}.
\end{aligned}
$$

This implies that the p.d.f. of Y is $g(y) = n\lambda e^{-n\lambda y}$ for $y > 0$; and so $Y \sim \text{Exp}(n\lambda)$. $\diamondsuit$

Now let $Z \equiv \max_i X_i$. Then

$$
\begin{aligned}
P(Z < z) &= P(\max(X_1, X_2, \ldots, X_n) < z) \\
&= P(X_1 < z, X_2 < z, \ldots, X_n < z) \\
&= \prod_{i=1}^{n} P(X_i < z) \\
&= [1 - e^{-\lambda z}]^n.
\end{aligned}
$$

This implies that the p.d.f. of Z is $h(z) = n\lambda[1 - e^{-\lambda z}]^{n-1}$ for $y > 0$. $\diamondsuit$

8. Calculate the m.g.f. of the Unif(a, b) distribution and use it to calculate the mean and variance.

Solution: If $X \sim \text{Unif}(a, b)$, then the m.g.f. is

$$
M_X(t) = \mathsf{E}[e^{tX}] = \int_a^b \frac{e^{tx}}{b-a}\,dx = \frac{e^{bt} - e^{at}}{t(b-a)}. \quad \diamondsuit
$$

Now use L'Hôpital's Rule to obtain

$$
\begin{aligned}
\mathsf{E}[X] &= \frac{d}{dt} M_X(t)\Big|_{t=0} \\
&= \frac{d}{dt}\frac{e^{bt} - e^{at}}{t(b-a)}\Big|_{t=0} \\
&= \frac{t(b-a)(be^{bt} - ae^{at}) - (e^{bt} - e^{at})(b-a)}{t^2(b-a)^2}\Big|_{t=0} \\
&= \frac{t(be^{bt} - ae^{at}) - (e^{bt} - e^{at})}{t^2(b-a)}\Big|_{t=0} \\
&= \frac{(be^{bt} - ae^{at}) + t(b^2e^{bt} - a^2e^{at}) - (be^{bt} - ae^{at})}{2t(b-a)}\Big|_{t=0} \\
&= \frac{b^2e^{bt} - a^2e^{at}}{2(b-a)}\Big|_{t=0} = \frac{a+b}{2}. \quad \diamondsuit
\end{aligned}
$$

Similarly (but more tediously), you can calculate $\mathsf{E}[X^2]$ and then $\mathsf{Var}(X) = (b-a)^2/12$. $\diamondsuit$

9. Show that the sum of i.i.d. exponential random variables is a gamma random variable.

Solution: Let $X_1, X_2, \ldots, X_n$ be i.i.d. Exp(λ). Then from class, we know that the m.g.f. of X_i is $M_{X_i}(t) = \lambda/(\lambda - t)$, $i = 1, 2, \ldots, n$; and so the m.g.f. of $Y \equiv \sum_{i=1}^n X_i$ is

$$M_Y(t) \;=\; [M_{X_i}(t)]^n \;=\; \left(\frac{\lambda}{\lambda - t}\right)^n, \quad \text{for } t < \lambda.$$

Meanwhile, the p.d.f. of a Gamma(n, λ) (or Erlang) random variable Z is given by

$$g(z) \;=\; \frac{\lambda^n z^{n-1} e^{-\lambda z}}{\Gamma(n)}, \quad \text{for } z > 0.$$

Thus, the m.g.f. is

$$\begin{aligned}
M_Z(t) &= \int_0^\infty \frac{e^{tz} \lambda^n z^{n-1} e^{-\lambda z}}{\Gamma(n)}\, dz \\
&= \frac{\lambda^n}{\Gamma(n)} \int_0^\infty z^{n-1} e^{-(\lambda - t)z}\, dz \\
&= \frac{\lambda^n}{(\lambda - t)^n \Gamma(n)} \int_0^\infty u^{n-1} e^{-u}\, du \quad \text{(where } u = (\lambda - t)z, \text{ with } t < \lambda) \\
&= \left(\frac{\lambda}{\lambda - t}\right)^n \quad \text{(by definition of } \Gamma(n)).
\end{aligned}$$

Since $M_Y(t) = M_Z(t)$, the uniqueness of m.g.f.'s gives our result. $\diamondsuit$.

10. Suppose that $U \sim \text{Unif}(0, 1)$. Find the p.d.f. of $\frac{-1}{\lambda}\ell\text{n}(U)$.

Solution: (This problem was actually done in class.) The c.d.f. of $Y = \frac{-1}{\lambda}\ell\text{n}(U)$ is

$$\begin{aligned}
G(y) &= \Pr(Y \le y) \\
&= \Pr\left(\frac{-1}{\lambda}\ell\text{n}(U) \le y\right) \\
&= \Pr\left(\ell\text{n}(U) \ge -\lambda y\right) \\
&= \Pr\left(U \ge e^{-\lambda y}\right) \\
&= 1 - e^{-\lambda y}.
\end{aligned}$$

This immediately implies that $Y \sim \text{Exp}(\lambda)$; and so the p.d.f. is $g(y) = \lambda e^{-\lambda y}$, for $y > 0$. $\diamondsuit$

11. Suppose X, Y have joint p.d.f. $f(x,y) = cxy$ for $0 < x < y < 1$ for some c. Find $\mathsf{Corr}(X,Y)$.

Solution: First of all, note that

$$1 \;=\; \int\int_{\Re^2} f(x,y)\,dx\,dy \;=\; \int_0^1 \int_0^y cxy\,dx\,dy \;=\; \frac{c}{8}.$$

Thus, $c = 8$, and we can really get going. In particular,

$$f_X(x) \;=\; \int_{\Re} f(x,y)\,dy \;=\; \int_x^1 8xy\,dy \;=\; 4(x - x^3), \quad 0 < x < 1.$$

$$\mathsf{E}[X] \;=\; \int_{\Re} x f_X(x)\,dx \;=\; \int_0^1 4(x^2 - x^4)\,dx \;=\; \frac{8}{15}.$$

$$\mathsf{E}[X^2] \;=\; \int_{\Re} x^2 f_X(x)\,dx \;=\; \int_0^1 4(x^3 - x^5)\,dx \;=\; \frac{1}{3}.$$

$$\mathsf{Var}(X) \;=\; \mathsf{E}[X^2] - (\mathsf{E}[X])^2 \;=\; \frac{11}{225}.$$

$$f_Y(y) \;=\; \int_{\Re} f(x,y)\,dx \;=\; \int_0^y 8xy\,dy \;=\; 4y^3, \quad 0 < y < 1.$$

$$\mathsf{E}[Y] \;=\; \int_{\Re} y f_Y(y)\,dy \;=\; \int_0^1 4y^4\,dy \;=\; \frac{4}{5}.$$

$$\mathsf{E}[Y^2] \;=\; \int_{\Re} y^2 f_Y(y)\,dy \;=\; \int_0^1 4y^5)\,dy \;=\; \frac{2}{3}.$$

$$\mathsf{Var}(Y) \;=\; \mathsf{E}[Y^2] - (\mathsf{E}[Y])^2 \;=\; \frac{2}{75}.$$

$$\mathsf{E}[XY] \;=\; \int\int_{\Re^2} xyf(x,y)\,dx\,dy \;=\; \int_0^1\int_0^y 8x^2y^2\,dx\,dy \;=\; \frac{4}{9}.$$

All of this stuff implies that

$$\mathsf{Corr}(X,Y) \;=\; \frac{\mathsf{Cov}(X,Y)}{\sqrt{\mathsf{Var}(X)\mathsf{Var}(Y)}} \;=\; \frac{\mathsf{E}[XY]-\mathsf{E}[X]\mathsf{E}[Y]}{\sqrt{\mathsf{Var}(X)\mathsf{Var}(Y)}} \;=\; 0.4924. \quad \diamondsuit$$

12. Use Chebychev's inequality to prove the WLLN, i.e., if $X_1, X_2, \ldots, X_n$ are i.i.d. with mean μ and finite variance, then for any $\epsilon > 0$, we have

$$\mathsf{Pr}\left(|\bar{X}-\mu| > \epsilon\right) \to 0 \quad \text{as } n \to \infty.$$

Solution: (This was done in class.) By Chebychev, we have

$$\mathsf{Pr}\left(|\bar{X}-\mu| > \epsilon\right) \;\le\; \frac{\mathsf{Var}(\bar{X})}{\epsilon^2} \;=\; \frac{\mathsf{Var}(X_i)}{n\epsilon^2} \to 0 \quad \text{as } n \to \infty. \quad \diamondsuit$$

13. Suppose that $X_1, X_2, \ldots, X_{10}$ are i.i.d. Pois(1).

(a) Use the Markov inequality to bound $\mathsf{Pr}(X_1 + \cdots + X_{10} \ge 15)$.

Solution: Let $Y = \sum_{i=1}^{10} X_i$. Markov states that $\Pr(Y \ge \epsilon) \le \mathsf{E}[Y]/\epsilon$. Then since $\mathsf{E}[X_i] = 1$, we have

$$\Pr(Y \ge 15) \;\le\; \frac{10\mathsf{E}[X_i]}{15} \;=\; \frac{2}{3}. \quad \diamondsuit$$

(b) Use the CLT to approximate $\mathsf{Pr}(X_1 + \cdots + X_{10} \ge 15)$.

Solution: By the CLT, we have

$$\begin{aligned}\Pr(Y \ge 15) \;&=\; \Pr\left(\frac{Y-\mathsf{E}[Y]}{\sqrt{\mathsf{Var}(Y)}} \ge \frac{15-\mathsf{E}[Y]}{\sqrt{\mathsf{Var}(Y)}}\right) \\ &\approx\; \Pr\left(\mathsf{Nor}(0,1) \ge \frac{15-10}{\sqrt{10}}\right) \\ &=\; 1-\Phi(1.581) \;=\; 0.0569. \quad \diamondsuit\end{aligned}$$

Note that you might want to improve upon the above solution by employing a *continuity correction* to take into account the fact that the Poisson is a discrete distribution. This would result in the slightly different approximation

$$\begin{aligned}\Pr(Y \geq 15) &= \Pr(Y \geq 14.5) \\ &\approx \Pr\Big(\mathsf{Nor}(0,1) \geq \frac{14.5-10}{\sqrt{10}}\Big) \\ &= 1-\Phi(1.423) \;=\; 0.0774. \quad \diamondsuit\end{aligned}$$

Of course, if you *really* want to check your answer, you can do so exactly, by noting that $Y \sim \text{Pois}(10)$. Then

$$\Pr(Y \geq 15) = 1 - \sum_{y=0}^{14} \frac{e^{-10}(10)^y}{y!} \;=\; 0.08346. \quad \diamondsuit$$

14. Show that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} \;=\; \frac{1}{2}.$$

Solution: Suppose that $Y \sim \text{Pois}(n)$. As implied by the previous problem, you can write $Y = \sum_{i=0}^{n} X_i$ where the X_i's are i.i.d. Pois(1). Thus, by the CLT, Y becomes approximately normal as n becomes large. Now let's use this fact...

$$\begin{aligned}\sum_{k=0}^{n} \frac{e^{-n}n^k}{k!} &= \Pr(Y \leq n) \\ &\approx \Pr\left(\mathsf{Nor}(0,1) \leq \frac{n-\mathsf{E}[Y]}{\sqrt{\mathsf{Var}(Y)}}\right) \\ &= \Pr\left(\mathsf{Nor}(0,1) \leq 0\right) \;=\; 0.5. \quad \diamondsuit\end{aligned}$$

15. Two vendors offer functionally identical products with mean lifetime 10 months. The distribution of the lifetime of the product from the first vendor is $\text{Exp}(\lambda)$, while the distribution of the lifetime of the product from the second vendor is $\text{Erlang}_2(\mu)$. If the objective is to maximize the probability that the lifetime of a product is greater than 8 months, which of the two vendors should be chosen?

Solution: Let X_1 and X_2 be lifetimes of products from the first and second vendors, respectively. Since $X_1 \sim \text{Exp}(\lambda)$ and $X_2 \sim \text{Erlang}_2(\mu)$, we have $\mathsf{E}[X_1] = 1/\lambda = 10$ and $\mathsf{E}[X_2] = 2/\mu = 10$. Thus, $\lambda = 0.1$ and $\mu = 0.2$. This immediately implies that

$$\begin{aligned} P(X_1 \geq 8) &= e^{-0.1\times 8} = 0.449 \\ P(X_2 \geq 8) &= \sum_{i=0}^{1} e^{-0.2\times 8}\frac{(0.2\times 8)^i}{i!} = 2.6e^{-1.6} = 0.525. \end{aligned}$$

Hence, the second vendor should be chosen. $\diamondsuit$

16. A system consists of n components. Each component functions with probability p independently of the others. The system as a whole functions if at least k components function. (This is a k-out-of-n system.)

    (a) What is the probability that the system functions?

    (b) Suppose that the lifetime of each component is exponential with mean one week. Compute the probability that the system functions after t weeks when $n = 3$ and $k = 2$.

    (c) Suppose we visit the system in item (b) at the end of 3 weeks and replace all failed components at a cost of \$75 each. If the system has already failed, it costs us an additional \$1,000. Compute the mean total cost incurred at the end of 3 weeks.

Solution:

(a) $\sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}$. $\diamondsuit$

(b) Set $p = e^{-t}$. Then $\sum_{i=2}^{3} \binom{n}{i} p^i (1-p)^{n-i} = 3e^{-2t} - 2e^{-3t}$. $\diamondsuit$

(c) Set $p = e^{-3}$. Then the total cost is

$$0p^3 + 75\binom{3}{1}p^2(1-p) + (75\times 2 + 1000)\binom{3}{2}p^1(1-p)^2 + (75\times 3 + 1000)(1-p)^3 = 1206.6. \quad \diamondsuit$$

17. Suppose a machine has 3 components with i.i.d. Exp(0.1) lifetimes. Compute the mean lifetime of the machine if it needs all 3 components to function.

Solution: Let X be a lifetime of the machine, and X_i be that of the ith component, where $X_i \sim \text{Exp}(0.1)$, $i = 1, 2, 3$. Then

$$\begin{aligned} P(X > x) &= P(\min(X_1, X_2, X_3) > x) \\ &= P(X_1 > x, X_2 > x, X_3 > x) \\ &= \prod_{i=1}^{3} P(X_i > x) \\ &= e^{-0.3x}. \end{aligned}$$

This implies that $X \sim \text{Exp}(0.3)$, and so the mean lifetime of the machine is $1/0.3$. ♢

18. A statistical experiment consists of starting 10 machines at time 0, and recording the number of operating machines after 10 hours. If the lifetimes (in hours) of these machines are i.i.d. Exp(0.125), compute the mean and variance of the number of operating machines after 10 hours.

    Solution: Let X be the number of operating machines after 10 hours. The probability that a typical machine is alive after 10 hours is $e^{-10\times 0.125} = 0.2865$. Then clearly, $X \sim \text{Bin}(10, 0.2865)$. Hence, $\mathsf{E}[X] = np = 2.865$, and $\mathsf{Var}(X) = npq = 2.044$. ♢

19. Let $X_1, X_2, \ldots$ be i.i.d. $\text{Exp}(\lambda)$ random variables, and let N be a $\text{Geom}(p)$ random variable that is independent of the X_i's. Find the distribution of the random sum $Z = X_1 + \cdots + X_N$.

    Solution: Given that $N = k$, we know that $Z \sim \text{Erlang}_k(\lambda)$. Thus,

$$\begin{aligned} P(Z > x) &= \sum_{k=1}^{\infty} P(Z > x | N = k) P(N = k) = \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} e^{-\lambda x} \frac{(\lambda x)^i}{i!} (1-p)^{k-1} p \\ &= \sum_{i=0}^{\infty} \sum_{k=i+1}^{\infty} e^{-\lambda x} \frac{(\lambda x)^i}{i!} (1-p)^{k-1} p = \sum_{i=0}^{\infty} e^{-\lambda x} \frac{(\lambda x)^i}{i!} p \sum_{k=i+1}^{\infty} (1-p)^{k-1} \\ &= \sum_{i=0}^{\infty} e^{-\lambda x} \frac{(\lambda x)^i}{i!} (1-p)^i = e^{-\lambda x} e^{(1-p)\lambda x} \sum_{i=0}^{\infty} e^{-(1-p)\lambda x} \frac{((1-p)\lambda x)^i}{i!} \\ &= e^{-p\lambda x}. \end{aligned}$$

    Hence, Z has an exponential distribution with parameter $p\lambda$. ♢

20. The lifetimes of two car batteries (brands A and B) are independent exponential random variables with means 12 hours and 10 hours, respectively. What is the probability that a brand B battery lasts longer than a brand A one?

Solution: Let A (resp., B) be lifetime of battery A (resp., B). Then

$$\begin{aligned} P(B > A) &= \int_0^\infty P(B > A | A = x) f_A(x)\, dx \\ &= \int_0^\infty e^{-\lambda_2 x} \lambda_1 e^{-\lambda_1 x}\, dx \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2} = 0.455. \quad \diamondsuit \end{aligned}$$

21. (Bonus Question) A couple has two kids and at least one is a boy born on a Tuesday. What's the probability that *both* are boys?

Solution: Let the events B_x $[G_x]$ = 'Boy [Girl] born on day x,' $x = 1, 2, \ldots, 7$ ($x = 3$ is Tuesday). The sample space for this experiment is

$$S = \{(G_x, G_y), (G_x, B_y), (B_x, G_y), (B_x, B_y),\ x, y = 1, 2, \ldots, 7\}$$

(so $|S| = 4 \times 49 = 196$).

Let C be the event that both are boys (with at least one born on a Tuesday) $= \{(B_x, B_3),\ x = 1, 2, \ldots, 7\} \cup \{(B_3, B_y),\ y = 1, 2, \ldots, 7\}$. Note that $|C| = 13$ (to avoid double counting (B_3, B_3)).

Let D be the event that there is at least one boy born on a Tuesday $= \{(G_x, B_3), (B_3, G_y),\ x, y = 1, 2, \ldots, 7\} \cup C$. So $|D| = 27$ (list them out if you don't believe me). Then

$$\Pr(C|D) = \frac{\Pr(C \cap D)}{\Pr(D)} = \frac{\Pr(C)}{\Pr(D)} = \frac{13/196}{27/196} = 13/27. \quad \diamondsuit$$