CH.9 Tests of Hypotheses for a Single Sample

- Hypotheses testing
- Tests on the mean of a normal distribution-*variance known*
- Tests on the mean of a normal distribution-*variance unknown*
- Tests on the variance and standard deviation of a normal distribution
- Tests on a population proportion
- Testing for goodness of fit

9-1 Hypothesis Testing

9-1.1 Statistical Hypotheses

Statistical inference may be divided into two major areas:

- Parameter estimation
- Hypothesis testing

Statistical hypothesis testing and confidence interval estimation of parameters are the fundamental methods used at the data analysis stage of a **comparative experiment,** in which the engineer is interested, for example, in comparing the mean of a population to a specified value.

A **statistical hypothesis** is a statement about the parameters of one or more populations.

9-1 Hypothesis Testing

9-1.1 Statistical Hypotheses

For example, suppose that we are interested in the burning rate of a solid propellant used to power aircrew escape systems.

- Now burning rate is a random variable that can be described by a probability distribution.
- Suppose that our interest focuses on the **mean** burning rate (a parameter of this distribution).
- Specifically, we are interested in deciding **whether or not the mean burning rate is 50 centimeters per second**.

9-1 Hypothesis Testing

9-1.1 Statistical Hypotheses

Two-sided Alternative Hypothesis

H_0: $\mu = 50$ centimeters per second   null hypothesis

H_1: $\mu \neq 50$ centimeters per second   alternative hypothesis

One-sided Alternative Hypotheses

H_0: $\mu = 50$ centimeters per second

H_1: $\mu < 50$ centimeters per second

or

H_0: $\mu = 50$ centimeters per second

H_1: $\mu > 50$ centimeters per second

9-1 Hypothesis Testing

9-1.1 Statistical Hypotheses

Test of a Hypothesis

•Hypotheses are statements about population or distribution under study.

•A procedure leading to a decision about a particular hypothesis

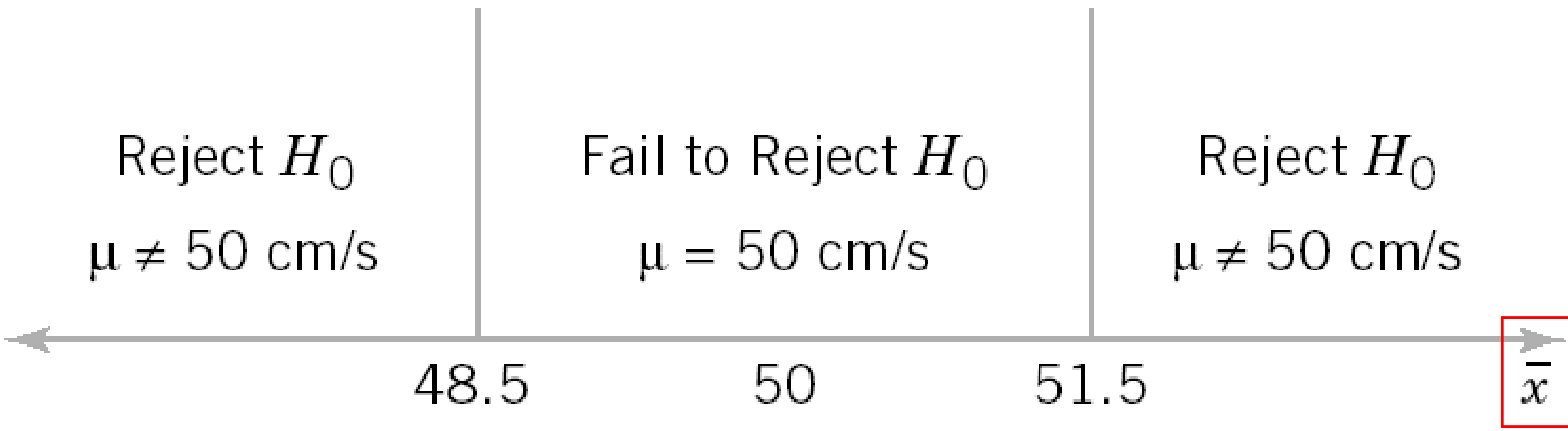
• Hypothesis-testing procedures rely on using the information in a **random sample from the population of interest**.

• If this information is *consistent* with the hypothesis, then we will conclude that the hypothesis is **true (fail to reject hypothesis)**; if this information is *inconsistent* with the hypothesis, we will conclude that the hypothesis is **false (reject hypothesis)**.

9-1 Hypothesis Testing

9-1.2 Tests of Statistical Hypotheses

$$H_0: \mu = 50 \text{ centimeters per second}$$
$$H_1: \mu \neq 50 \text{ centimeters per second}$$



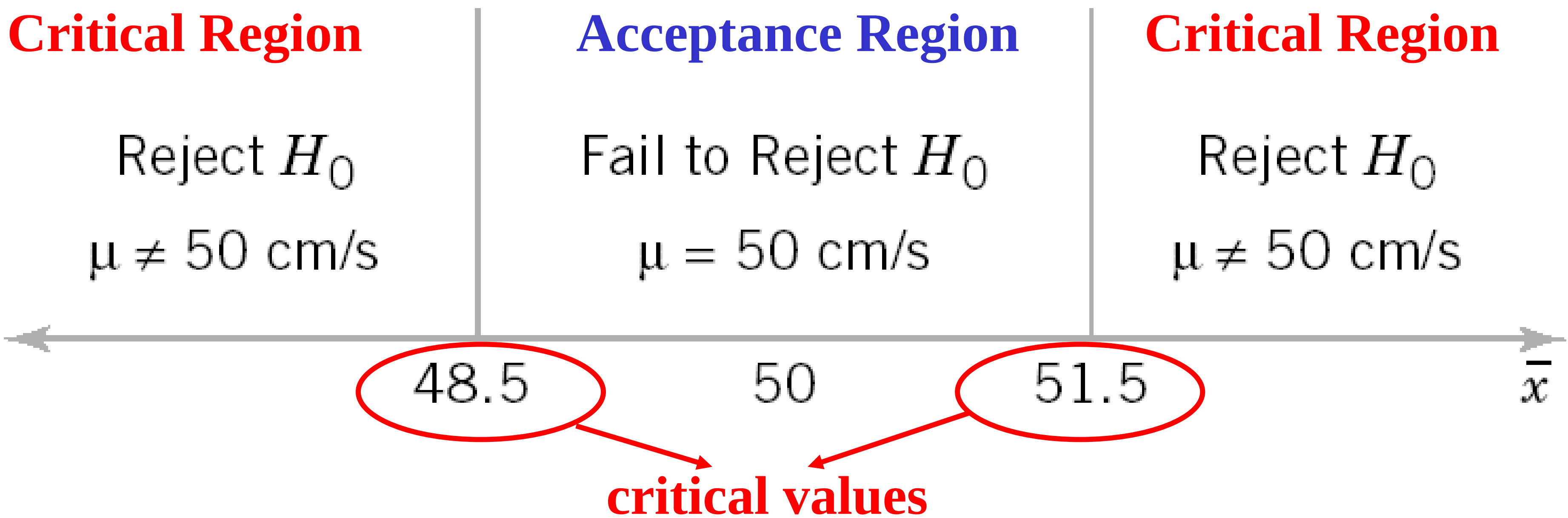
Decision criteria for testing $H_0: \mu = 50$ cm/s versus $H_1: \mu \neq 50$ cm/s.

9-1 Hypothesis Testing

9-1.2 Tests of Statistical Hypotheses

$$H_0\text{: } \mu = 50 \text{ centimeters per second}$$

$$H_1\text{: } \mu \neq 50 \text{ centimeters per second}$$



Decision criteria for testing $H_0{:}\mu = 50$ cm/s versus $H_1{:}\mu \neq 50$ cm/s.

9-1 Hypothesis Testing

9-1.2 Tests of Statistical Hypotheses

Two wrong conclusions are possible:

Type I Error

Rejecting the null hypothesis H_0 when it is true is defined as a **type I error.**

Type II Error

Failing to reject the null hypothesis when it is false is defined as a **type II error.**

9-1 Hypothesis Testing

9-1.2 Tests of Statistical Hypotheses

Decision	H_0 Is True	H_0 Is False
Fail to reject H_0	no error	type II error
Reject H_0	type I error	no error

$$\alpha = P(\text{type I error}) = P(\text{reject } H_0 \text{ when } H_0 \text{ is true})$$

Sometimes the type I error probability is called the **significance level**, or the **α-error**, or the **size** of the test.

9-1 Hypothesis Testing

9-1.2 Tests of Statistical Hypotheses

Ex: σ =2.5   n=10

Standard deviation of the sample mean:

$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{2.5}{\sqrt{10}} = 0.79$$

$$\alpha = P(\overline{X} < 48.5 \text{ when } \mu = 50) + P(\overline{X} > 51.5 \text{ when } \mu = 50)$$

The z-values that correspond to the critical values 48.5 and 51.5 are

$$z_1 = \frac{48.5 - 50}{0.79} = -1.90 \quad \text{and} \quad z_2 = \frac{51.5 - 50}{0.79} = 1.90$$

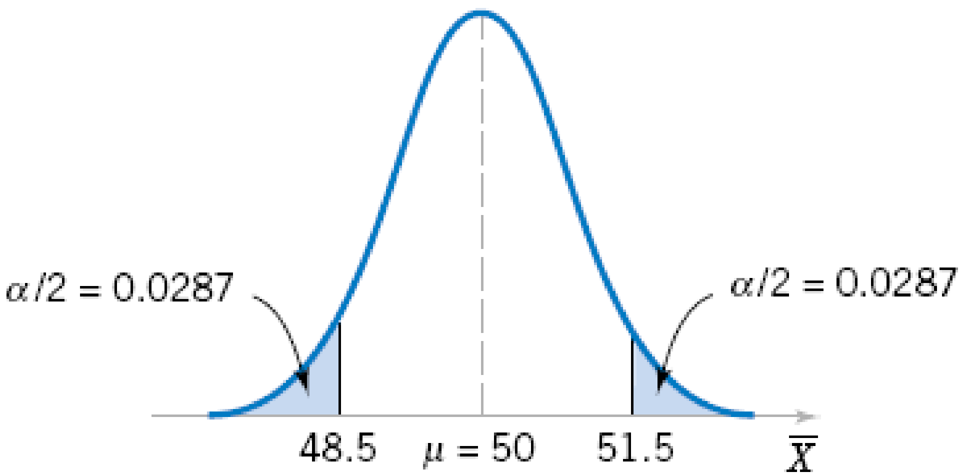
(50 → $\mu_{\overline{X}}$; 0.79 → $\sigma_{\overline{X}}$)

Therefore

$$\alpha = P(Z < -1.90) + P(Z > 1.90) = 0.028717 + 0.028717 = 0.057434$$

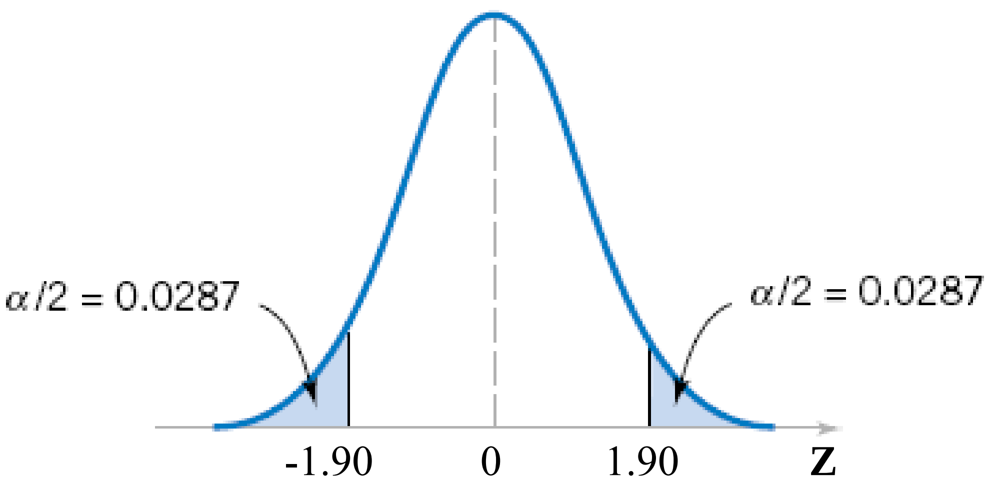
9-1 Hypothesis Testing

Critical Region for $H_0 : \mu = 50 \qquad H_1 : \mu \neq 50$



Probability distribution of $\overline{X}$

Corresponding probability distribution of Z



9-1 Hypothesis Testing

$$\alpha = P(\text{type I error}) = P(\text{reject } H_0 \text{ when } H_0 \text{ is true}) \quad (9\text{-}3)$$

How can we reduce α ?

- by widening acceptance region

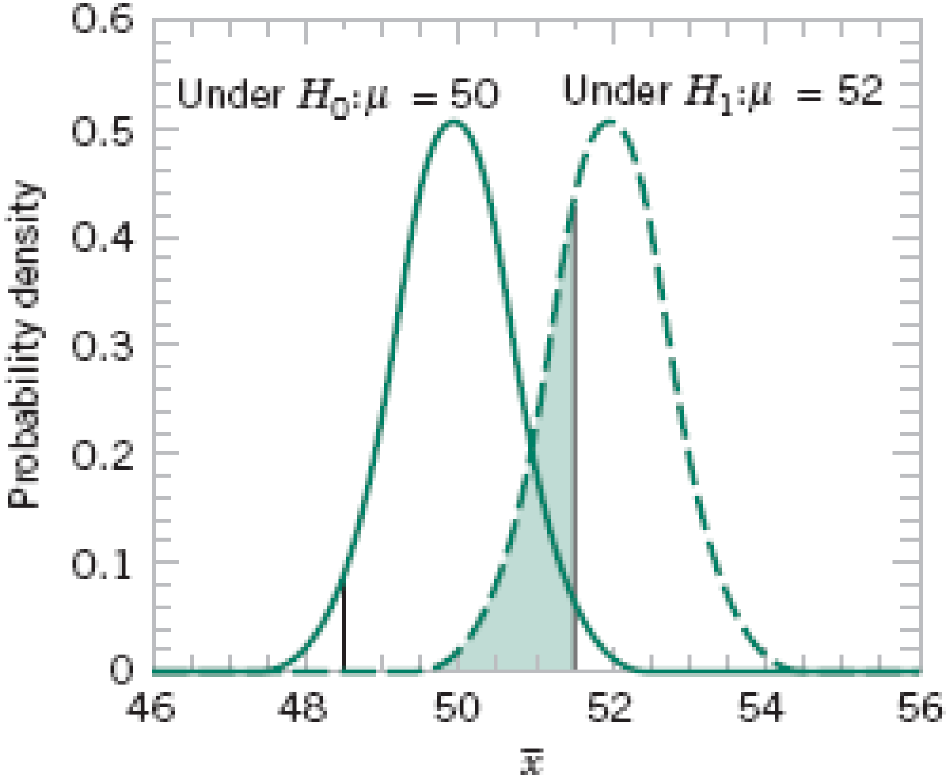
(if take critical values 48 and 52, α =0.0114, Verify !)

- by increasing sample size

(if take n=16, α =0.0164 Verify !)

9-1 Hypothesis Testing

$$\beta = P(\text{type II error}) = P(\text{fail to reject } H_0 \text{ when } H_0 \text{ is false}) \qquad (9\text{-}4)$$



The probability of type II error when $\mu = 52$ and $n = 10$.

9-1 Hypothesis Testing

$$\beta = P(48.5 \leq \overline{X} \leq 51.5 \text{ when } \mu = 52)$$

The z-values corresponding to 48.5 and 51.5 when $\mu = 52$ are

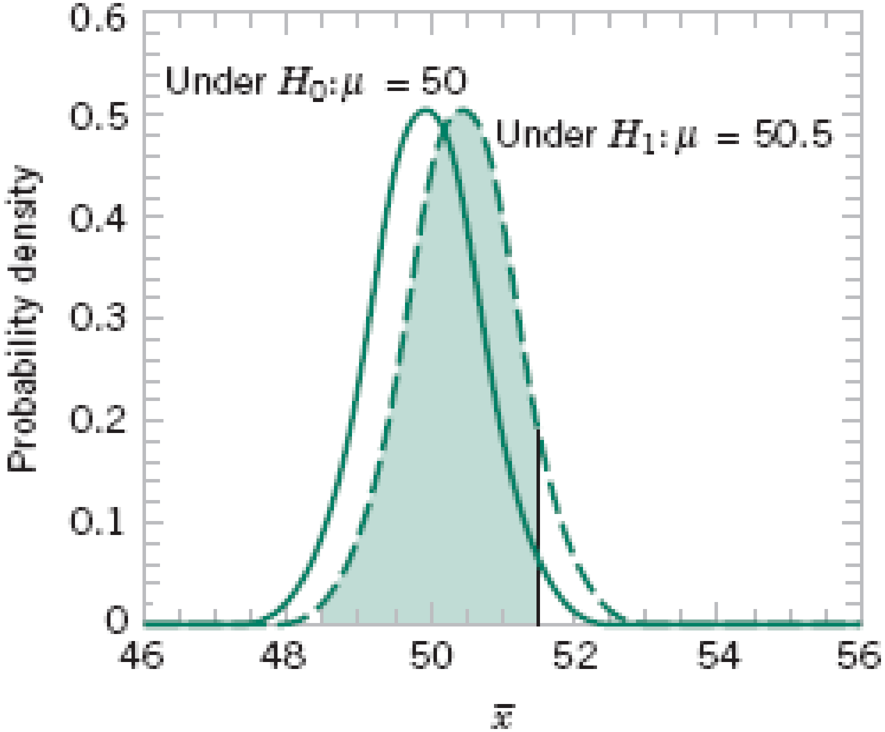
$$z_1 = \frac{48.5 - 52}{0.79} = -4.43 \quad \text{and} \quad z_2 = \frac{51.5 - 52}{0.79} = -0.63$$

Therefore

$$\beta = P(-4.43 \leq Z \leq -0.63) = P(Z \leq -0.63) - P(Z \leq -4.43)$$
$$= 0.2643 - 0.0000 = 0.2643$$

9-1 Hypothesis Testing

$$\beta = P(48.5 \leq \overline{X} \leq 51.5 \text{ when } \mu = 50.5)$$



The probability of type II error when $\mu = 50.5$ and *n* = 10.

9-1 Hypothesis Testing

$$\beta = P(48.5 \le \bar{X} \le 51.5 \text{ when } \mu = 50.5)$$

As shown in Fig. 9-4, the z-values corresponding to 48.5 and 51.5 when $\mu = 50.5$ are

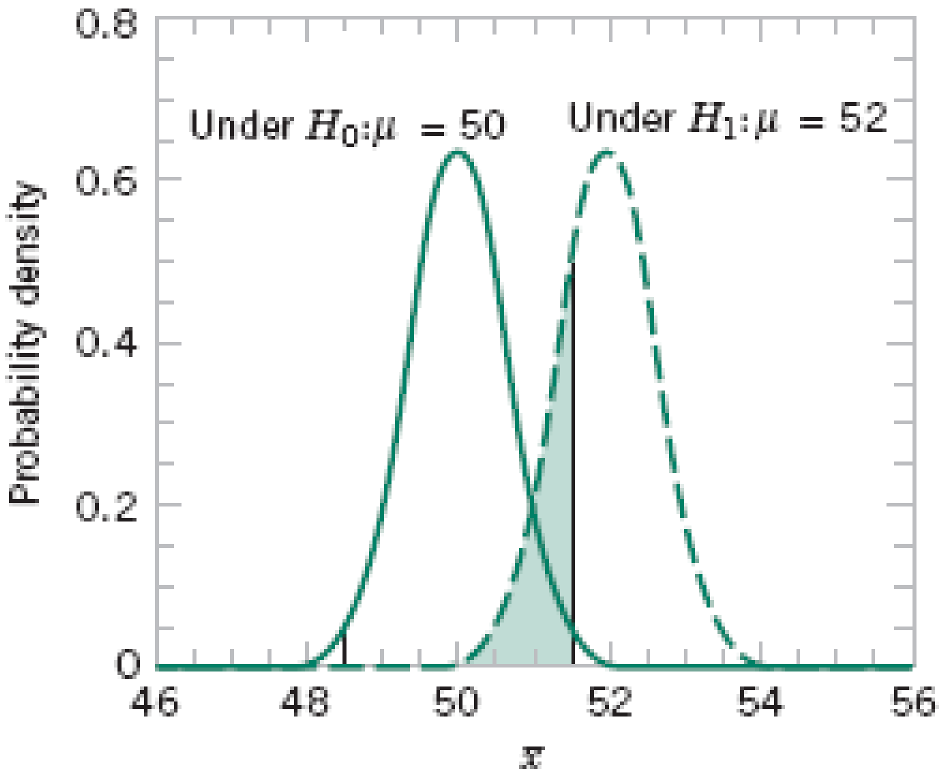
$$z_1 = \frac{48.5 - 50.5}{0.79} = -2.53 \qquad \text{and} \qquad z_2 = \frac{51.5 - 50.5}{0.79} = 1.27$$

Therefore

$$\begin{aligned}\beta &= P(-2.53 \le Z \le 1.27) = P(Z \le 1.27) - P(Z \le -2.53) \\ &= 0.8980 - 0.0057 = 0.8923\end{aligned}$$

9-1 Hypothesis Testing

$$\beta = P(48.5 \leq \overline{X} \leq 51.5 \text{ when } \mu = 52)$$



The probability of type II error when μ = 52 and n = 16.

9-1 Hypothesis Testing

$$\beta = P(48.5 \leq \overline{X} \leq 51.5 \text{ when } \mu = 52)$$

When $n = 16$, the standard deviation of $\overline{X}$ is $\sigma/\sqrt{n} = 2.5/\sqrt{16} = 0.625$, and the z-values corresponding to 48.5 and 51.5 when $\mu = 52$ are

$$z_1 = \frac{48.5 - 52}{0.625} = -5.60 \quad \text{and} \quad z_2 = \frac{51.5 - 52}{0.625} = -0.80$$

Therefore

$$\beta = P(-5.60 \leq Z \leq -0.80) = P(Z \leq -0.80) - P(Z \leq -5.60)$$
$$= 0.2119 - 0.0000 = 0.2119$$

9-1 Hypothesis Testing

- Generally α is controlable when critical values are selected.
- Thus, rejection of null hypothesis H_0 is a strong conclusion.
- β is not constant but depends on the true value of the parameter and sample size.
- Accepting H_0 is a weak conclusion unless β is acceptably small.
- Prefer the terminology "**fail to reject H_0**" rather than "accept H_0"
- Fail to reject H_0
  - implies we have not found sufficient evidence to reject H_0.
  - does not necessarily mean there is a high probability that H_0 is true.
  - means more data are required to reach a strong conclusion.

9-1 Hypothesis Testing

Power

The **power** of a statistical test is the probability of rejecting the null hypothesis H_0 when the alternative hypothesis is true.

- The power is computed as $1 - \beta$, and power can be interpreted as *the probability of correctly rejecting a false null hypothesis.* We often compare statistical tests by comparing their power properties.

- For example, consider the propellant burning rate problem when we are testing $H_0: \mu = 50$ cm/s against $H_1: \mu$ not equal 50 cm/s . Suppose that the true value of the mean is $\mu = 52$. When $n = 10$, we found that $\beta = 0.2643$, so the power of this test is $1 - \beta = 1 - 0.2643 = 0.7357$ when $\mu = 52$.

9-1 Hypothesis Testing

9-1.3 One-Sided and Two-Sided Hypotheses

Two-Sided Test:

$$H_0\colon \mu = \mu_0$$

$$H_1\colon \mu \neq \mu_0$$

One-Sided Tests:

$$H_0\colon \mu = \mu_0 \qquad\qquad H_0\colon \mu = \mu_0$$

or

$$H_1\colon \mu > \mu_0 \qquad\qquad H_1\colon \mu < \mu_0$$

9-1 Hypothesis Testing

Example 9-1

- Suppose if the propellant burning rate is less than 50 cm/s
- Want to show this with a strong conclusion ➔ **CLAIM**
- Hypotheses should be stated as
  H_0: $\mu = 50$ cm/s
  H_1: $\mu < 50$ cm/s
- Since the rejection of H_0 is always a strong conclusion, this statement of the hypotheses will produce the desired outcome if H_0 is rejected.
- Although H_0 is stated with an equal sign, it is understood to include any value of μ not specified by H_1.
- Failing to reject H_0 does not mean $\mu = 50$ cm/s exactly.
- Failing to reject H_0 means we do not have strong evidence in support of H_1.

9-1 Hypothesis Testing

The bottler wants to be sure that the bottles meet the specification on mean internal pressure or bursting strength, which for 10-ounce bottles is a minimum strength of 200 psi.

The bottler has decided to formulate the decision procedure for a specific lot of bottles as a hypothesis testing problem.

There are two possible formulations for this problem: either

$$H_0: \mu = 200 \text{ psi} \qquad \text{or} \qquad H_0: \mu = 200 \text{ psi}$$
$$H_1: \mu > 200 \text{ psi} \qquad \qquad H_1: \mu < 200 \text{ psi}$$

Which is correct? Depends on the objective of the analysis.

9-1 Hypothesis Testing

9-1.4 P-Values in Hypothesis Tests

•When H_0 is rejected at a specified α level, this gives no idea about whether the computed value of the test statistic

- is just barely in the rejection region
- or it is very far into this region.

•Thus, P-value has been adopted widely in practice

> The ***P*-value** is the smallest level of significance that would lead to rejection of the null hypothesis H_0 with the given data.

9-1 Hypothesis Testing

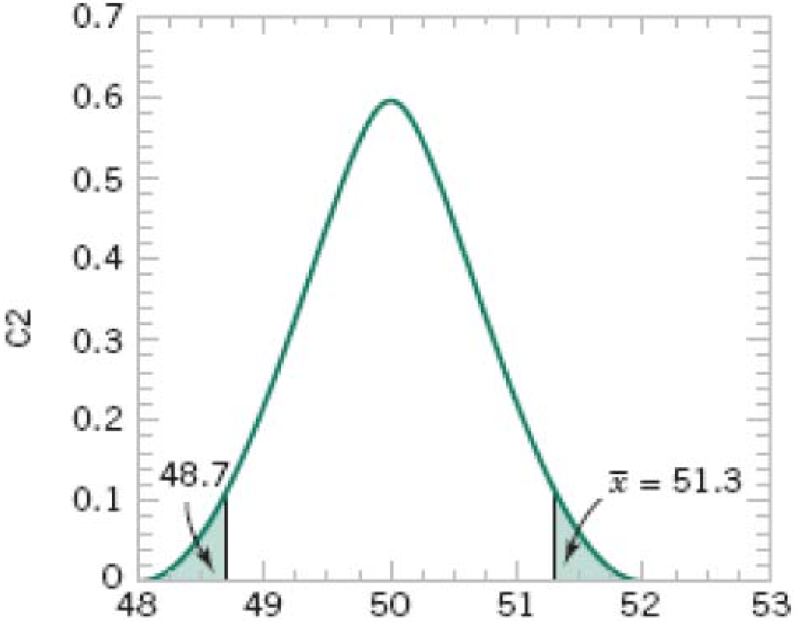
9-1.4 P-Values in Hypothesis Tests

Consider the two-sided hypothesis test for burning rate:

$H_0 : \mu = 50$ cm/s
$H_1 : \mu \neq 50$ cm/s

n=16, σ=2.5, $\bar{x} = 51.3$

P-value?



$$
\begin{aligned}
P\text{-value} &= 1 - P(48.7 < \bar{X} < 51.3) \\
&= 1 - P\left(\frac{48.7 - 50}{2.5/\sqrt{16}} < Z < \frac{51.3 - 50}{2.5/\sqrt{16}}\right) \\
&= 1 - P(-2.08 < Z < 2.08) \\
&= 1 - 0.962 = 0.038
\end{aligned}
$$

9-1 Hypothesis Testing

9-1.5 Connection between Hypothesis Tests and Confidence Intervals

Close relation between **hypothesis tests** and **confidence intervals**

$H_0 : \mu = 50$ cm/s n=16, σ=2.5, α=0.05, $\bar{x} = 51.3$
$H_1 : \mu \neq 50$ cm/s

Critical z values are $z_{\alpha/2}=z_{0.025}=1.96$ and $-z_{0.025}=-1.96$ which corresponds to

Critical values $50 \pm 1.96 \dfrac{2.5}{\sqrt{16}}$ $= [48.775 ; 51.225]$

$\bar{x} = 51.3$ is not in the acceptance region [48.775 ; 51.225]. So reject null hypothesis.

Confidence interval for μ at α=0.05 is $51.3 \pm 1.96 \dfrac{2.5}{\sqrt{16}}$

same conclusion !

That is $50.075 \leq \mu \leq 52.525$

μ=50 is not in the confidence interval [50.075 ; 52.525]. So reject null hypothesis.

9-1 Hypothesis Testing

9-1.5 Connection between Hypothesis Tests and Confidence Intervals

There is a close relationship between the test of a hypothesis about any parameter, say θ, and the confidence interval for θ. If $[l, u]$ is a $100(1-\alpha)\%$ confidence interval for the parameter θ, the test of size α of the hypothesis

$$H_0: \theta = \theta_0$$
$$H_1: \theta \neq \theta_0$$

will lead to rejection of H_0 if and only if θ_0 is **not** in the $100(1-\alpha)\%$ CI $[l, u]$. As an illustration, consider the escape system propellant problem with $\bar{x} = 51.3$, $\sigma = 2.5$, and $n = 16$. The null hypothesis H_0: $\mu = 50$ was rejected, using $\alpha = 0.05$. The 95% two-sided CI on μ can be calculated using Equation 8-7. This CI is $51.3 \pm 1.96(2.5/\sqrt{16})$ and this is $50.075 \leq \mu \leq 52.525$. Because the value $\mu_0 = 50$ is not included in this interval, the null hypothesis H_0: $\mu = 50$ is rejected.

9-1 Hypothesis Testing

9-1.6 General Procedure for Hypothesis Tests

1. From the problem context, identify the parameter of interest.
2. State the null hypothesis, H_0 .
3. Specify an appropriate alternative hypothesis, H_1 .
4. Choose a significance level, α.
5. Determine an appropriate test statistic.
6. State the rejection region for the statistic.
7. Compute any necessary sample quantities, substitute these into the equation for the test statistic, and compute that value.
8. Decide whether or not H_0 should be rejected and report that in the problem context.

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.1 Hypothesis Tests on the Mean

We wish to test:

$$H_0: \mu = \mu_0$$
$$H_1: \mu \neq \mu_0$$

The **test statistic** is:

$$Z_0 = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \qquad (9\text{-}8)$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.1 Hypothesis Tests on the Mean

Reject H_0 if the observed value of the test statistic z_0 is either:

$$z_0 > z_{\alpha/2} \text{ or } z_0 < -z_{\alpha/2}$$

Fail to reject H_0 if

$$-z_{\alpha/2} < z_0 < z_{\alpha/2}$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

Example 9-2

Aircrew escape systems are powered by a solid propellant. The burning rate of this propellant is an important product characteristic. Specifications require that the mean burning rate must be 50 centimeters per second. We know that the standard deviation of burning rate is $\sigma = 2$ centimeters per second. The experimenter decides to specify a type I error probability or significance level of $\alpha = 0.05$ and selects a random sample of $n = 25$ and obtains a sample average burning rate of $\bar{x} = 51.3$ centimeters per second. What conclusions should be drawn?

9-2 Tests on the Mean of a Normal Distribution, Variance Known

Example 9-2

We may solve this problem by following the eight-step procedure outlined in Section 9-1.4. This results in

1. The parameter of interest is μ, the mean burning rate.

2. H_0: $\mu = 50$ centimeters per second

3. H_1: $\mu \neq 50$ centimeters per second

4. $\alpha = 0.05$

5. The test statistic is

$$z_0 = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

Example 9-2

6. Reject H_0 if $z_0 > 1.96$ or if $z_0 < -1.96$. Note that this results from step 4, where we specified $\alpha = 0.05$, and so the boundaries of the critical region are at $z_{0.025} = 1.96$ and $-z_{0.025} = -1.96$.

7. Computations: Since $\bar{x} = 51.3$ and $\sigma = 2$,

$$z_0 = \frac{51.3 - 50}{2/\sqrt{25}} = 3.25$$

8. Conclusion: Since $z_0 = 3.25 > 1.96$, we reject $H_0\colon \mu = 50$ at the 0.05 level of significance. Stated more completely, we conclude that the mean burning rate differs from 50 centimeters per second, based on a sample of 25 measurements. In fact, there is strong evidence that the mean burning rate exceeds 50 centimeters per second.

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.1 Hypothesis Tests on the Mean

We may also develop procedures for testing hypotheses on the mean μ where the alternative hypothesis is one-sided. Suppose that we specify the hypotheses as

$$H_0: \mu = \mu_0$$
$$H_1: \mu > \mu_0 \tag{9-11}$$

In defining the critical region for this test, we observe that a negative value of the test statistic Z_0 would never lead us to conclude that H_0: $\mu = \mu_0$ is false. Therefore, we would place the critical region in the **upper tail** of the standard normal distribution and reject H_0 if the computed value of z_0 is too large. That is, we would reject H_0 if

$$z_0 > z_\alpha \tag{9-12}$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.1 Hypothesis Tests on the Mean (Continued)

as shown in Figure 9-7(b). Similarly, to test

$$H_0: \mu = \mu_0$$
$$H_1: \mu < \mu_0 \quad (9\text{-}13)$$

we would calculate the test statistic Z_0 and reject H_0 if the value of z_0 is too small. That is, the critical region is in the **lower tail** of the standard normal distribution as shown in Figure 9-7(c), and we reject H_0 if

$$z_0 < -z_\alpha \quad (9\text{-}14)$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

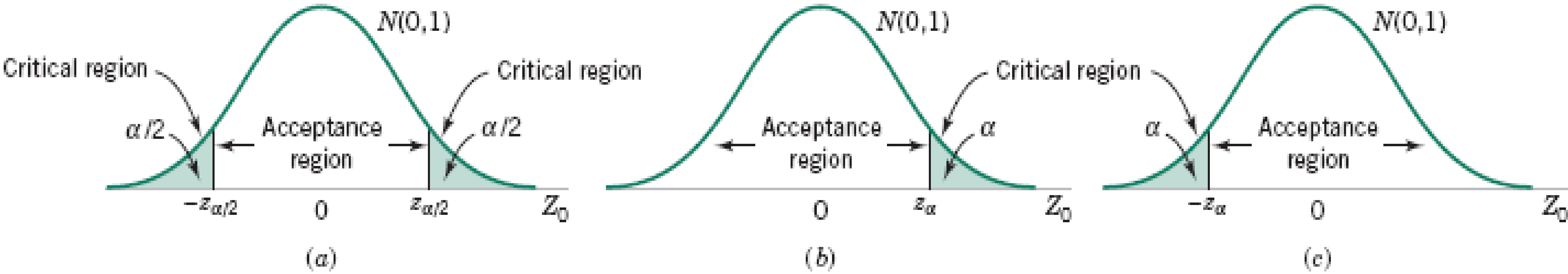


Figure 9-7 The distribution of Z_0 when H_0: $\mu = \mu_0$ is true, with critical region for (a) the two-sided alternative $H_1 : \mu \neq \mu_0$, (b) the one-sided alternative $H_1 : \mu > \mu_0$, and (c) the one-sided alternative $H_1 : \mu < \mu_0$.

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.1 Hypothesis Tests on the Mean (Continued)

Null hypothesis: $H_0\colon \mu = \mu_0$

Test statistic: $$Z_0 = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$$

Alternative hypothesis	Rejection criteria
$H_1\colon \mu \neq \mu_0$	$z_0 > z_{\alpha/2,}$ or $z_0 < -z_{\alpha/2}$
$H_1\colon \mu > \mu_0$	$z_0 > z_{\alpha,}$
$H_1\colon \mu < \mu_0$	$z_0 < -z_{\alpha,}$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

P-Values in Hypothesis Tests

The ***P*-value** is the smallest level of significance that would lead to rejection of the null hypothesis H_0 with the given data.

$$P = \begin{cases} 2[1 - \Phi(|z_0|)] & \text{for a two-tailed test: } H_0\text{: } \mu = \mu_0 \quad H_1\text{: } \mu \neq \mu_0 \\ 1 - \Phi(z_0) & \text{for a upper-tailed test: } H_0\text{: } \mu = \mu_0 \quad H_1\text{: } \mu > \mu_0 \\ \Phi(z_0) & \text{for a lower-tailed test: } H_0\text{: } \mu = \mu_0 \quad H_1\text{: } \mu < \mu_0 \end{cases} \tag{9-15}$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.2 Type II Error and Choice of Sample Size

Finding the Probability of Type II Error β

Consider the two-sided hypothesis

$$H_0\colon \mu = \mu_0$$
$$H_1\colon \mu \neq \mu_0$$

Suppose that the null hypothesis is false and that the true value of the mean is $\mu = \mu_0 + \delta$, say, where $\delta > 0$. The test statistic Z_0 is

$$Z_0 = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{\overline{X} - (\mu_0 + \delta)}{\sigma/\sqrt{n}} + \frac{\delta\sqrt{n}}{\sigma}$$

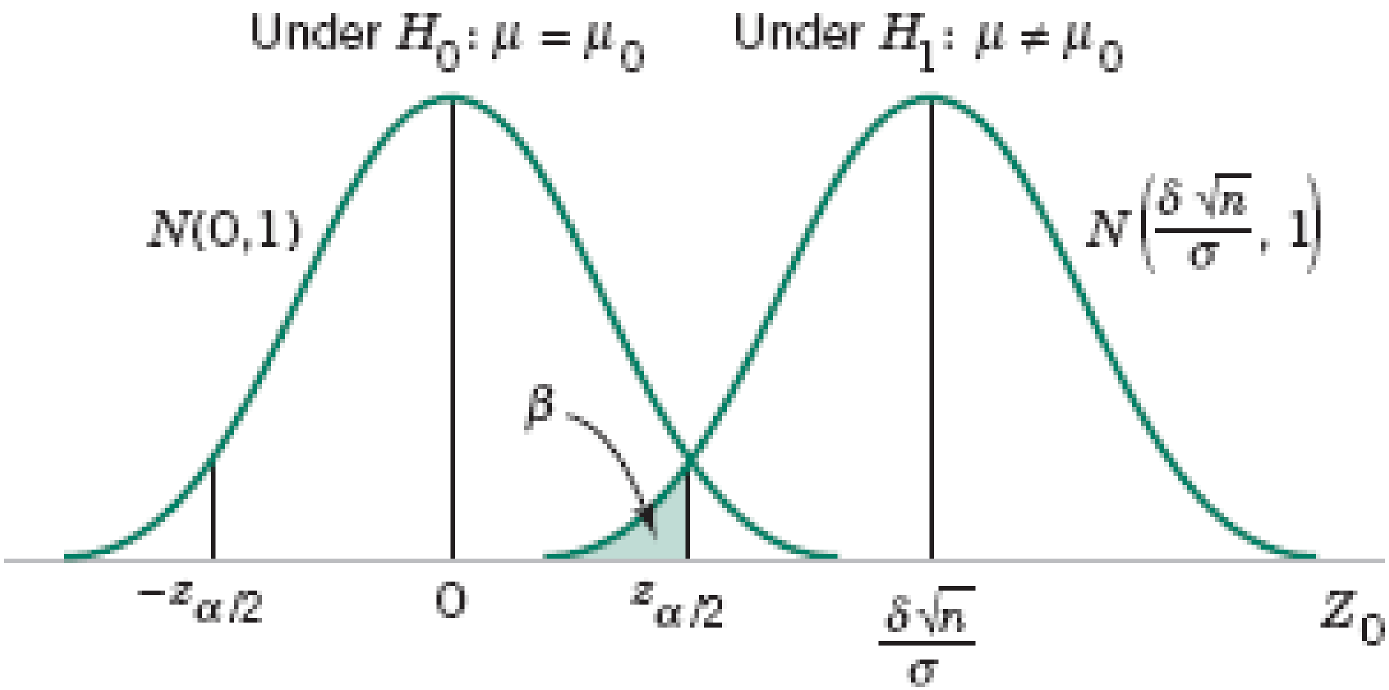
Therefore, the distribution of Z_0 when H_1 is true is

$$Z_0 \sim N\left(\frac{\delta\sqrt{n}}{\sigma}, 1\right) \qquad (9\text{-}16)$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.2 Type II Error and Choice of Sample Size

Finding the Probability of Type II Error β



The distribution of Z_0 under H_0 and H_1

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.2 Type II Error and Choice of Sample Size

Finding the Probability of Type II Error β

$$\beta = \Phi\left(z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right) - \Phi\left(-z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right) \qquad (9\text{-}17)$$

$$\underbrace{\Phi\left(-z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right)}_{0 \quad \text{if } \delta>0}$$

Let z_β be the 100β upper percentile of the standard normal distr.

$\beta = \Phi(-z_\beta)$

$$-z_\beta \simeq z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.2 Type II Error and Choice of Sample Size

Sample Size Formulas

For a two-sided alternative hypothesis:

$$n \simeq \frac{(z_{\alpha/2} + z_{\beta})^2 \sigma^2}{\delta^2} \quad \text{where} \quad \delta = \mu - \mu_0 \qquad (9\text{-}19)$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.2 Type II Error and Choice of Sample Size

Sample Size Formulas

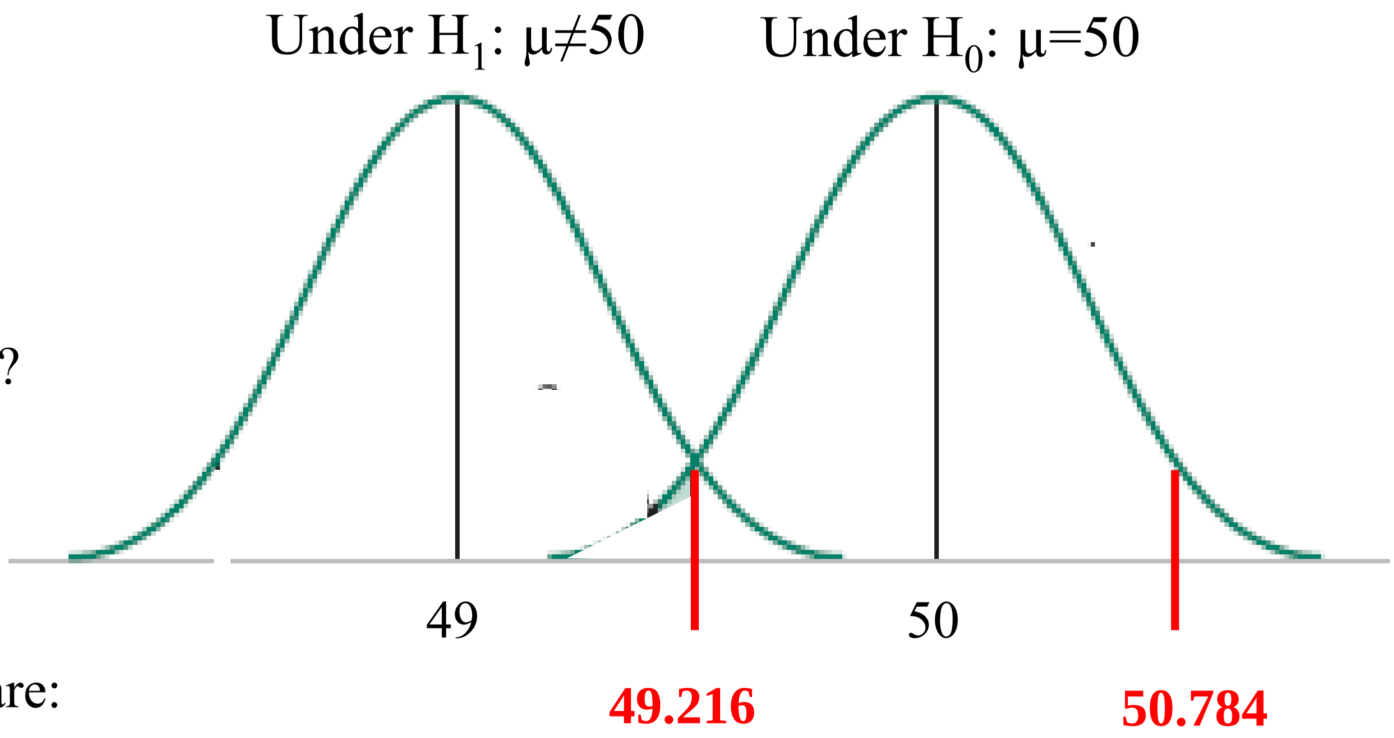
For a one-sided alternative hypothesis:

$$n = \frac{(z_\alpha + z_\beta)^2 \sigma^2}{\delta^2} \quad \text{where} \quad \delta = \mu - \mu_0 \qquad (9\text{-}20)$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

Example 9-3

H_0: $\mu=50$
H_1: $\mu\neq50$
$\sigma=2$
$\alpha=0.05$
$n=25$
If true $\mu=49$, $\beta=?$



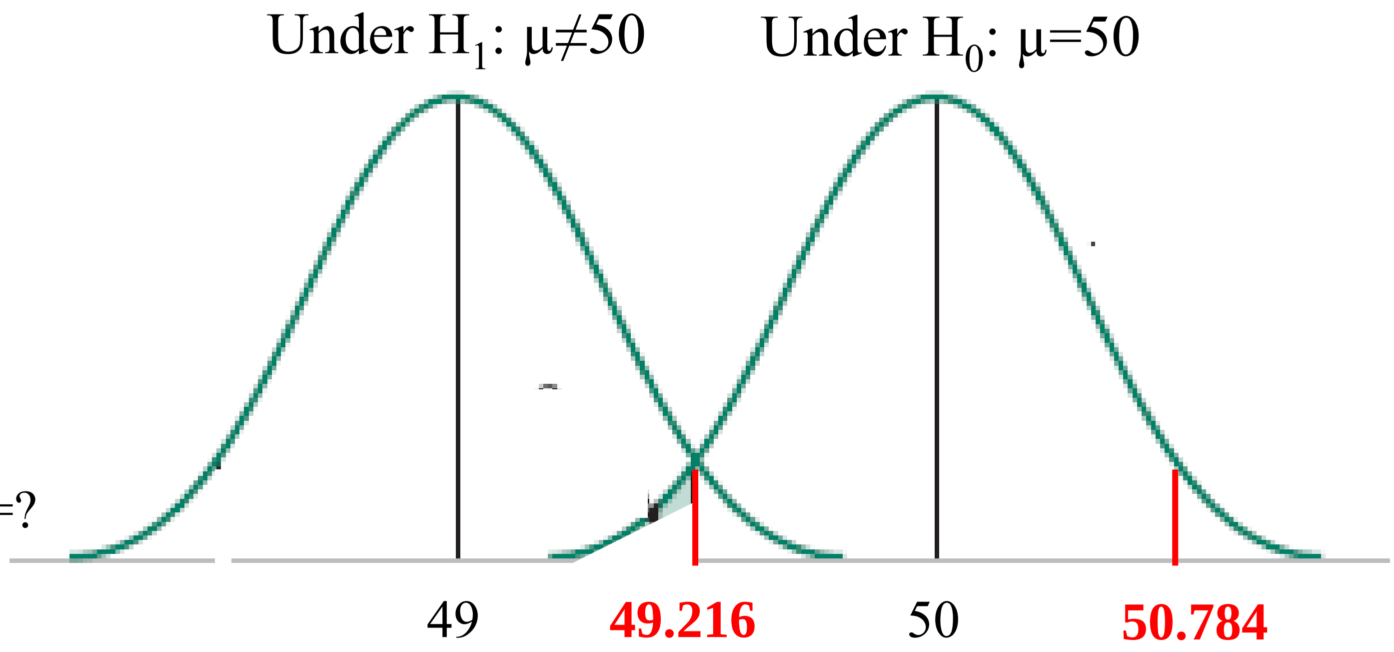
$z_{0.025}=1.96$

Critical points are:
$50 \pm z_{0.025}\sigma/\sqrt{n}$
$= 50 \pm 1.96*2/5$
49.216 and 50.784

9-2 Tests on the Mean of a Normal Distribution, Variance Known

Example 9-3

H_0: $\mu=50$
H_1: $\mu\neq50$
$\sigma=2$
$\alpha=0.05$
$n=25$
If true $\mu=49$, $\beta=?$



$$\beta = P(49.216 \le \bar{X} \le 50.784 \quad \text{when } \mu=49)$$

$$\beta = P\left(\frac{49.216-49}{2/\sqrt{25}} \le Z \le \frac{50.784-49}{2/\sqrt{25}}\right)$$

$$\beta = P\left(0.54 \le Z \le 4.46\right) = 0.295$$

$\Rightarrow$ $z_\beta = z_{0.295} = 0.54$

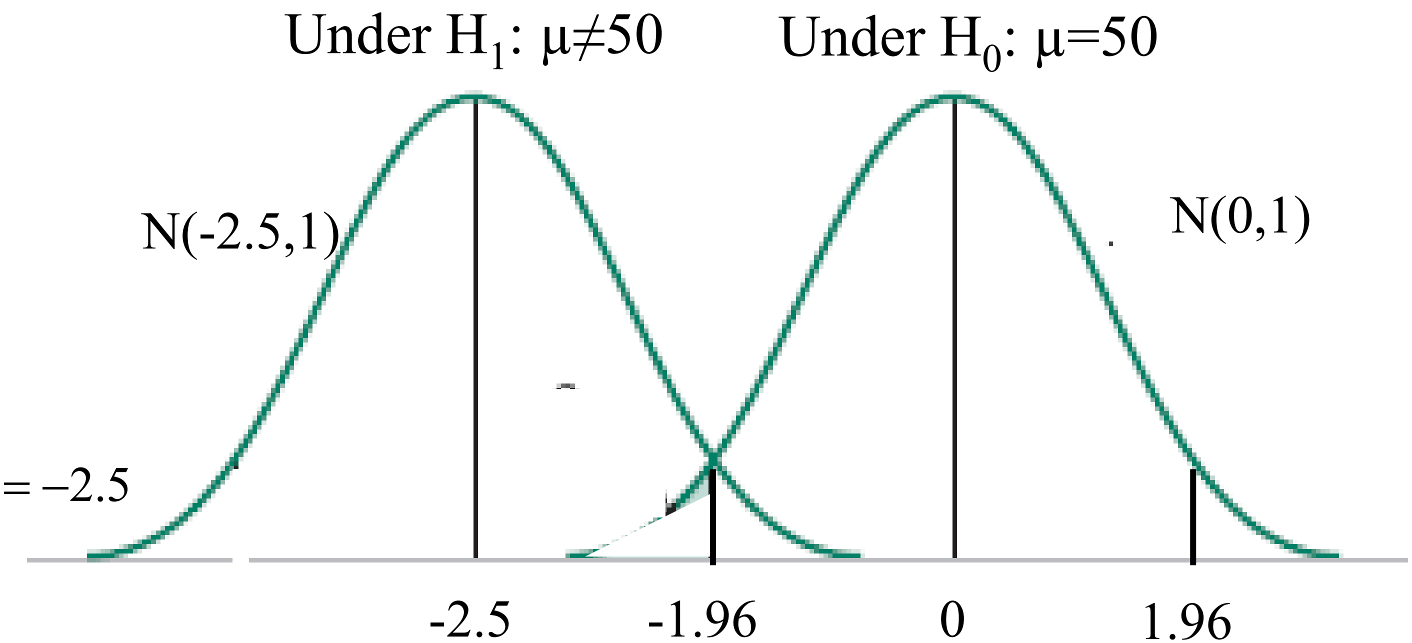
9-2 Tests on the Mean of a Normal Distribution, Variance Known

Example 9-3

Standardize the normal graphs

$\delta = -1$

$$\frac{\delta\sqrt{n}}{\sigma} = \frac{-1*\sqrt{25}}{2} = -2.5$$



With standard normal graph:

$$\beta = P(-1.96 \le \bar{X} \le 1.96 \text{ when } \mu = \text{-}2.5)$$

$$\beta = P\left(\frac{-1.96+2.5}{1} \le Z \le \frac{1.96+2.5}{1}\right)$$

$$\beta = P(0.54 \le Z \le 4.46) = 0.295$$

⟺

With the formula:

$$\beta = \Phi\left(1.96 - \frac{-1*\sqrt{25}}{2}\right) - \Phi\left(-1.96 - \frac{-1*\sqrt{25}}{2}\right)$$

$$\beta = \Phi(4.46) - \Phi(0.54) = 0.295$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

Example 9-3

Consider the rocket propellant problem of Example 9-2. Suppose that the analyst wishes to design the test so that if the true mean burning rate differs from 50 centimeters per second by as much as 1 centimeter per second, the test will detect this (i.e., reject H_0: $\mu = 50$) with a high probability, say 0.90. Now, we note that $\sigma = 2$, $\delta = 51 - 50 = 1$, $\alpha = 0.05$, and $\beta = 0.10$. Since $z_{\alpha/2} = z_{0.025} = 1.96$ and $z_\beta = z_{0.10} = 1.28$, the sample size required to detect this departure from H_0: $\mu = 50$ is found by Equation 9-19 as

$$n \simeq \frac{(z_{\alpha/2} + z_\beta)^2 \sigma^2}{\delta^2} = \frac{(1.96 + 1.28)^2 2^2}{(1)^2} \simeq 42$$

The approximation is good here, since $\Phi(-z_{\alpha/2} - \delta\sqrt{n}/\sigma) = \Phi(-1.96 - (1)\sqrt{42}/2) = \Phi(-5.20) \simeq 0$, which is small relative to β.

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.2 Type II Error and Choice of Sample Size

Using Operating Characteristic Curves

When performing sample size or type II error calculations, it is sometimes more convenient to use the **operating characteristic (OC) curves** in Appendix Charts VII*a* and VII*b*. These curves plot β as calculated from Equation 9-17 against a parameter d for various sample sizes n. Curves are provided for both $\alpha = 0.05$ and $\alpha = 0.01$. The parameter d is defined as

$$d = \frac{|\mu - \mu_0|}{\sigma} = \frac{|\delta|}{\sigma} \tag{9-21}$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

Example 9-4

Consider the propellant problem in Example 9-2. Suppose that the analyst is concerned about the probability of type II error if the true mean burning rate is $\mu = 51$ centimeters per second. We may use the operating characteristic curves to find β. Note that $\delta = 51 - 50 = 1$, $n = 25$, $\sigma = 2$, and $\alpha = 0.05$. Then using Equation 9-21 gives

$$d = \frac{|\mu - \mu_0|}{\sigma} = \frac{|\delta|}{\sigma} = \frac{1}{2}$$

and from Appendix Chart VIIa, with $n = 25$, we find that $\beta = 0.30$. That is, if the true mean burning rate is $\mu = 51$ centimeters per second, there is approximately a 30% chance that this will not be detected by the test with $n = 25$.

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.3 Large Sample Test

If the distribution of the population is not known, but **n>40** sample standard deviation “s” can be substituted for “σ” and test procedures in Section 9.2 are valid.

9-3 Tests on the Mean of a Normal Distribution, Variance Unknown

9-3.1 Hypothesis Tests on the Mean

One-Sample *t*-Test

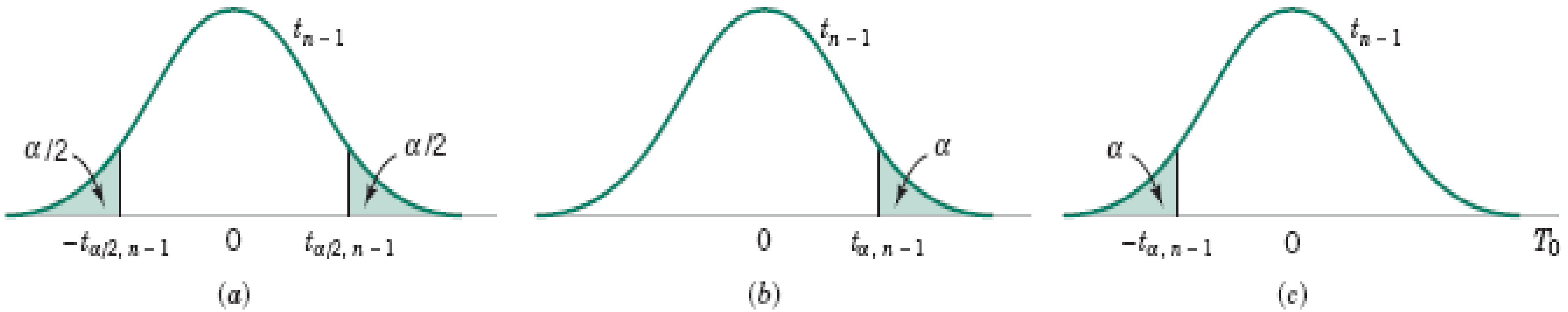
Null hypothesis: $H_0\colon \mu = \mu_0$

Test statistic: $$T_0 = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$$

Alternative hypothesis	Rejection criteria
$H_1\colon \mu \neq \mu_0$	$t_0 > t_{\alpha/2,n-1}$ or $t_0 < -t_{\alpha/2,n-1}$
$H_1\colon \mu > \mu_0$	$t_0 > t_{\alpha,n-1}$
$H_1\colon \mu < \mu_0$	$t_0 < -t_{\alpha,n-1}$

9-3 Tests on the Mean of a Normal Distribution, Variance Unknown

9-3.1 Hypothesis Tests on the Mean



The reference distribution for H_0: $\mu = \mu_0$ with critical region for (a) H_1: $\mu \neq \mu_0$, (b) H_1: $\mu > \mu_0$, and (c) H_1: $\mu < \mu_0$.

9-3 Tests on the Mean of a Normal Distribution, Variance Unknown

Example 9-6

The increased availability of light materials with high strength has revolutionized the design and manufacture of golf clubs, particularly drivers. Clubs with hollow heads and very thin faces can result in much longer tee shots, especially for players of modest skills. This is due partly to the "spring-like effect" that the thin face imparts to the ball. Firing a golf ball at the head of the club and measuring the ratio of the outgoing velocity of the ball to the incoming velocity can quantify this spring-like effect. The ratio of velocities is called the coefficient of restitution of the club. An experiment was performed in which 15 drivers produced by a particular club maker were selected at random and their coefficients of restitution measured. In the experiment the golf balls were fired from an air cannon so that the incoming velocity and spin rate of the ball could be precisely controlled. It is of interest to determine if there is evidence (with $\alpha = 0.05$) to support a claim that the mean coefficient of restitution exceeds 0.82. The observations follow:

0.8411	0.8191	0.8182	0.8125	0.8750
0.8580	0.8532	0.8483	0.8276	0.7983
0.8042	0.8730	0.8282	0.8359	0.8660

9-3 Tests on the Mean of a Normal Distribution, Variance Unknown

Example 9-6

The sample mean and sample standard deviation are $\bar{x} = 0.83725$ and $s = 0.02456$. The normal probability plot of the data in Fig. 9-9 supports the assumption that the coefficient of restitution is normally distributed. Since the objective of the experimenter is to demonstrate that the mean coefficient of restitution exceeds 0.82, a one-sided alternative hypothesis is appropriate.

The solution using the eight-step procedure for hypothesis testing is as follows:

1. The parameter of interest is the mean coefficient of restitution, μ.
2. H_0: $\mu = 0.82$
3. H_1: $\mu > 0.82$. We want to reject H_0 if the mean coefficient of restitution exceeds 0.82.
4. $\alpha = 0.05$
5. The test statistic is

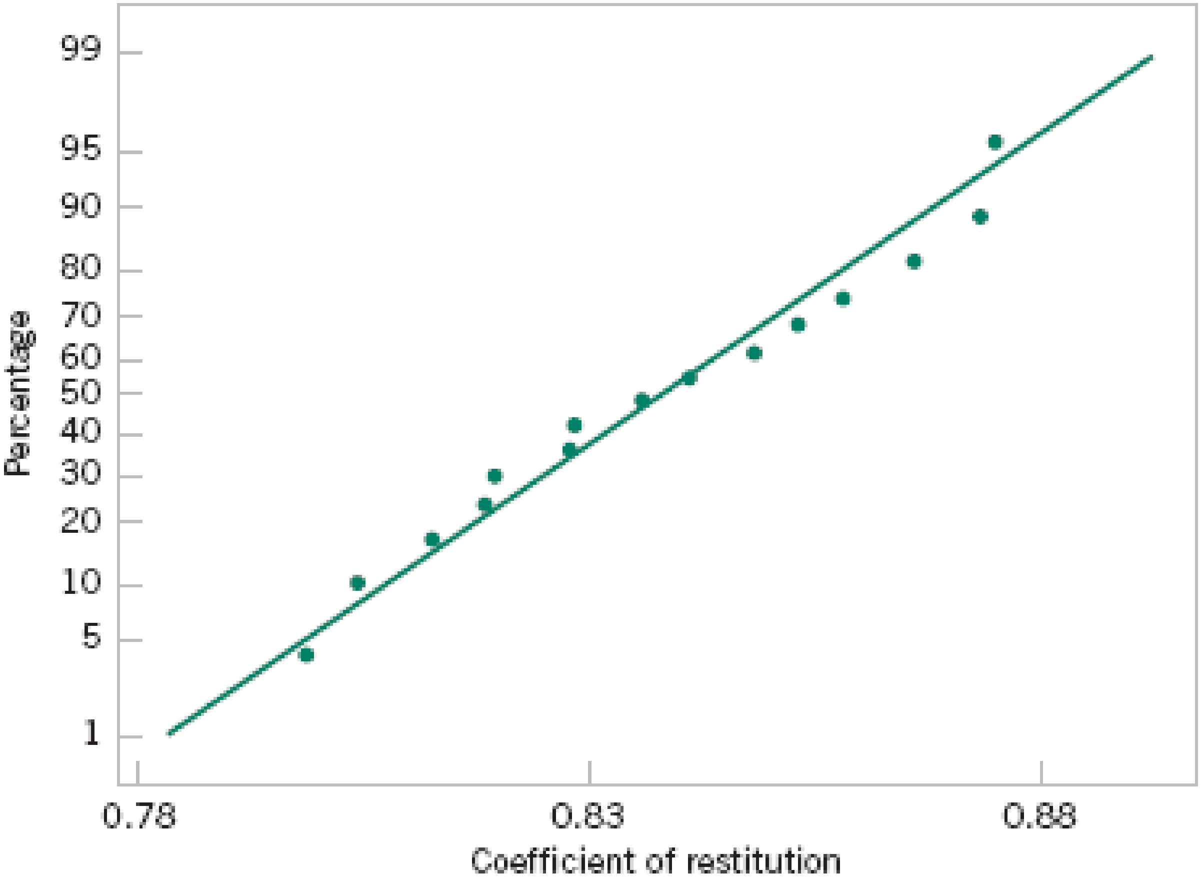
$$t_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

6. Reject H_0 if $t_0 > t_{0.05,14} = 1.761$

9-3 Tests on the Mean of a Normal Distribution, Variance Unknown

Example 9-6

Normal probability plot of the coefficient of restitution data from Example 9-6.



9-3 Tests on the Mean of a Normal Distribution, Variance Unknown

Example 9-6

7. Computations: Since $\bar{x} = 0.83725$, $s = 0.02456$, $\mu_0 = 0.82$, and $n = 15$, we have

$$t_0 = \frac{0.83725 - 0.82}{0.02456/\sqrt{15}} = 2.72$$

8. Conclusions: Since $t_0 = 2.72 > 1.761$, we reject H_0 and conclude at the 0.05 level of significance that the mean coefficient of restitution exceeds 0.82.

9-3 Tests on the Mean of a Normal Distribution, Variance Unknown

9-3.2 *P*-value for a *t*-Test

The *P*-value for a *t*-test is just the smallest level of significance at which the null hypothesis would be rejected.

To illustrate, consider the *t*-test based on 14 degrees of freedom in Example 9-6. The relevant critical values from Appendix Table IV are as follows:

Critical Value:	0.258	0.692	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
Tail Area:	0.40	0.25	0.10	0.05	0.025	0.01	0.005	0.0025	0.001	0.0005

t_0 = 2.72, this is between two tabulated values, 2.624 and 2.977. Therefore, the *P*-value must be between 0.01 and 0.005.

$0.005 < P\text{-value} < 0.01.$

Suppose t_0 = 2.72 for a two-sided test, then

$0.005*2 < P\text{-value} < 0.01*2$ ➔ $0.01 < P\text{-value} < 0.02$

9-3 Tests on the Mean of a Normal Distribution, Variance Unknown

9-3.3 Type II Error and Choice of Sample Size

The type II error of the two-sided alternative would be

$$\begin{aligned}\beta &= P\{-t_{\alpha/2,n-1} \le T_0 \le t_{\alpha/2,n-1} \mid \delta \ne 0\} \\ &= P\{-t_{\alpha/2,n-1} \le T_0' \le t_{\alpha/2,n-1}\}\end{aligned}$$

Where T_0' denotes the noncentral t random variable.

9-3 Tests on the Mean of a Normal Distribution, Variance Unknown

Example 9-7

Consider the golf club testing problem from Example 9-6. If the mean coefficient of restitution exceeds 0.82 by as much as 0.02, is the sample size $n = 15$ adequate to ensure that H_0: $\mu = 0.82$ will be rejected with probability at least 0.8?

To solve this problem, we will use the sample standard deviation $s = 0.02456$ to estimate σ. Then $d = |\delta|/\sigma = 0.02/0.02456 = 0.81$. By referring to the operating characteristic curves in Appendix Chart VII*g* (for $\alpha = 0.05$) with $d = 0.81$ and $n = 15$, we find that $\beta = 0.10$, approximately. Thus, the probability of rejecting H_0: $\mu = 0.82$ if the true mean exceeds this by 0.02 is approximately $1 - \beta = 1 - 0.10 = 0.90$, and we conclude that a sample size of $n = 15$ is adequate to provide the desired sensitivity.

9-4 Hypothesis Tests on the Variance and Standard Deviation of a Normal Distribution

9-4.1 Hypothesis Test on the Variance

Suppose that we wish to test the hypothesis that the variance of a normal population σ^2 equals a specified value, say σ_0^2, or equivalently, that the standard deviation σ is equal to σ_0. Let X_1, $X_2, \ldots, X_n$ be a random sample of n observations from this population. To test

$$H_0\colon \sigma^2 = \sigma_0^2$$

$$H_1\colon \sigma^2 \neq \sigma_0^2$$

we will use the **test statistic:**

$$X_0^2 = \frac{(n-1)S^2}{\sigma_0^2}$$

the null hypothesis $H_0\colon \sigma^2 = \sigma_0^2$ would be rejected if

$$\chi_0^2 > \chi^2_{\alpha/2,n-1} \quad \text{or if} \quad \chi_0^2 < \chi^2_{1-\alpha/2,n-1}$$

9-4 Hypothesis Tests on the Variance and Standard Deviation of a Normal Distribution

9-4.1 Hypothesis Test on the Variance

The same test statistic is used for one-sided alternative hypotheses.
For the one-sided hypothesis

$$H_0: \sigma^2 = \sigma_0^2$$
$$H_1: \sigma^2 > \sigma_0^2$$

we would reject H_0 if $\chi_0^2 > \chi_{\alpha,n-1}^2$, whereas for the other one-sided hypothesis

$$H_0: \sigma^2 = \sigma_0^2$$
$$H_1: \sigma^2 < \sigma_0^2$$

we would reject H_0 if $\chi_0^2 < \chi_{1-\alpha,n-1}^2$.

9-4 Hypothesis Tests on the Variance and Standard Deviation of a Normal Distribution

9-4.1 Hypothesis Test on the Variance

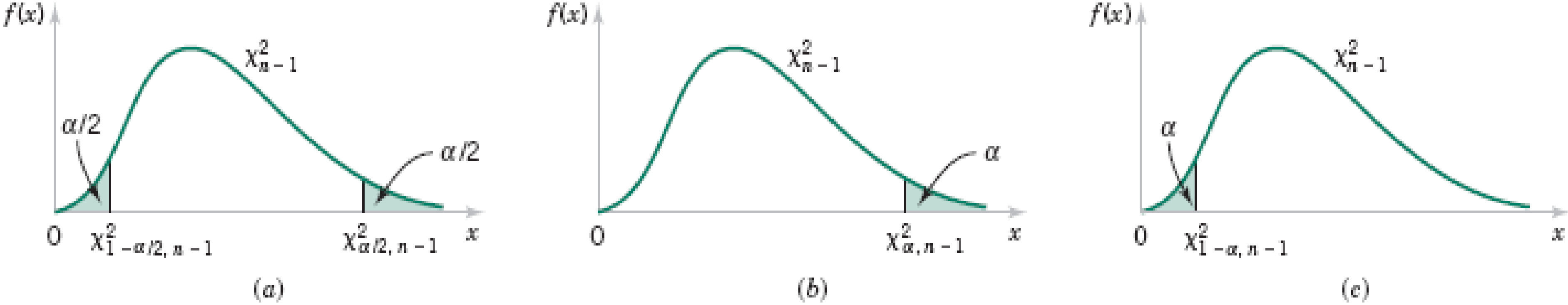


Figure 9-11 Reference distribution for the test of H_0: $\sigma^2 = \sigma_0^2$ with critical region values for (a) H_1: $\sigma^2 \neq \sigma_0^2$, (b) H_1: $\sigma^2 > \sigma_0^2$, and (c) H_1: $\sigma^2 < \sigma_0^2$.

9-4 Hypothesis Tests on the Variance and Standard Deviation of a Normal Distribution

Example 9-8

An automatic filling machine is used to fill bottles with liquid detergent. A random sample of 20 bottles results in a sample variance of fill volume of $s^2 = 0.0153$ (fluid ounces)^2. If the variance of fill volume exceeds 0.01 (fluid ounces)^2, an unacceptable proportion of bottles will be underfilled or overfilled. Is there evidence in the sample data to suggest that the manufacturer has a problem with underfilled or overfilled bottles? Use $\alpha = 0.05$, and assume that fill volume has a normal distribution.

Using the eight-step procedure results in the following:

1. The parameter of interest is the population variance σ^2.
2. H_0: $\sigma^2 = 0.01$
3. H_1: $\sigma^2 > 0.01$
4. $\alpha = 0.05$
5. The test statistic is

$$\chi_0^2 = \frac{(n-1)s^2}{\sigma_0^2}$$

9-4 Hypothesis Tests on the Variance and Standard Deviation of a Normal Distribution

Example 9-8

6. Reject H_0 if $\chi_0^2 > \chi_{0.05,19}^2 = 30.14$.

7. Computations:

$$\chi_0^2 = \frac{19(0.0153)}{0.01} = 29.07$$

8. Conclusions: Since $\chi_0^2 = 29.07 < \chi_{0.05,19}^2 = 30.14$, we conclude that there is no strong evidence that the variance of fill volume exceeds 0.01 (fluid ounces)^2.

9-4 Hypothesis Tests on the Variance and Standard Deviation of a Normal Distribution

9-4.2 Type II Error and Choice of Sample Size

For the two-sided alternative hypothesis:

$$\lambda = \frac{\sigma}{\sigma_0}$$

Operating characteristic curves are provided in Charts VII*i* and VII*j*.

9-4 Hypothesis Tests on the Variance and Standard Deviation of a Normal Distribution

Example 9-9

Consider the bottle-filling problem from Example 9-8. If the variance of the filling process exceeds 0.01 (fluid ounces)^2, too many bottles will be underfilled. Thus, the hypothesized value of the standard deviation is $\sigma_0 = 0.10$. Suppose that if the true standard deviation of the filling process exceeds this value by 25%, we would like to detect this with probability at least 0.8. Is the sample size of $n = 20$ adequate?

To solve this problem, note that we require

$$\lambda = \frac{\sigma}{\sigma_0} = \frac{0.125}{0.10} = 1.25$$

This is the abscissa parameter for Chart VII*k*. From this chart, with $n = 20$ and $\lambda = 1.25$, we find that $\beta \simeq 0.6$. Therefore, there is only about a 40% chance that the null hypothesis will be rejected if the true standard deviation is really as large as $\sigma = 0.125$ fluid ounce.

To reduce the β-error, a larger sample size must be used. From the operating characteristic curve with $\beta = 0.20$ and $\lambda = 1.25$, we find that $n = 75$, approximately. Thus, if we want the test to perform as required above, the sample size must be at least 75 bottles.

9-5 Tests on a Population Proportion

9-5.1 Large-Sample Tests on a Proportion

Many engineering decision problems include hypothesis testing about *p*.

$$H_0: p = p_0$$
$$H_1: p \neq p_0$$

An appropriate **test statistic** is

$$Z_0 = \frac{X - np_0}{\sqrt{np_0(1 - p_0)}} \quad (9\text{-}32)$$

and reject $H_0: p = p_0$ if

$$z_0 > z_{\alpha/2} \quad \text{or} \quad z_0 < -z_{\alpha/2}$$

9-5 Tests on a Population Proportion

Example 9-10

A semiconductor manufacturer produces controllers used in automobile engine applications. The customer requires that the process fallout or fraction defective at a critical manufacturing step not exceed 0.05 and that the manufacturer demonstrate process capability at this level of quality using $\alpha = 0.05$. The semiconductor manufacturer takes a random sample of 200 devices and finds that four of them are defective. Can the manufacturer demonstrate process capability for the customer?

We may solve this problem using the eight-step hypothesis-testing procedure as follows:

1. The parameter of interest is the process fraction defective p.
2. $H_0: p = 0.05$
3. $H_1: p < 0.05$

   This formulation of the problem will allow the manufacturer to make a strong claim about process capability if the null hypothesis $H_0: p = 0.05$ is rejected.
4. $\alpha = 0.05$

9-5 Tests on a Population Proportion

Example 9-10

5. The test statistic is (from Equation 9-32)

$$z_0 = \frac{x - np_0}{\sqrt{np_0(1 - p_0)}}$$

where $x = 4$, $n = 200$, and $p_0 = 0.05$.

6. Reject H_0: $p = 0.05$ if $z_0 < -z_{0.05} = -1.645$

7. Computations: The test statistic is

$$z_0 = \frac{4 - 200(0.05)}{\sqrt{200(0.05)(0.95)}} = -1.95$$

8. Conclusions: Since $z_0 = -1.95 < -z_{0.05} = -1.645$, we reject H_0 and conclude that the process fraction defective p is less than 0.05. The P-value for this value of the test statistic z_0 is $P = 0.0256$, which is less than $\alpha = 0.05$. We conclude that the process is capable.

9-5 Tests on a Population Proportion

Another form of the test statistic Z_0 is obtained by dividing the numerator and denominator by n

$$Z_0 = \frac{X/n - p_0}{\sqrt{p_0(1 - p_0)/n}} \quad \text{or} \quad Z_0 = \frac{\hat{P} - p_0}{\sqrt{p_0(1 - p_0)/n}}$$

9-5 Tests on a Population Proportion

9-5.2 Type II Error and Choice of Sample Size

For a two-sided alternative

$$\beta = \Phi\left(\frac{p_0 - p + z_{\alpha/2}\sqrt{p_0(1-p_0)/n}}{\sqrt{p(1-p)/n}}\right) - \Phi\left(\frac{p_0 - p - z_{\alpha/2}\sqrt{p_0(1-p_0)/n}}{\sqrt{p(1-p)/n}}\right)$$

If the alternative is $p < p_0$

$$\beta = 1 - \Phi\left(\frac{p_0 - p - z_{\alpha}\sqrt{p_0(1-p_0)/n}}{\sqrt{p(1-p)/n}}\right)$$

If the alternative is $p > p_0$

where p is true value of the population proportion

$$\beta = \Phi\left(\frac{p_0 - p + z_{\alpha}\sqrt{p_0(1-p_0)/n}}{\sqrt{p(1-p)/n}}\right)$$

9-5 Tests on a Population Proportion

9-5.2 Type II Error and Choice of Sample Size

Less complex representation:

$$\sigma_{H_1} = \sqrt{\frac{p(1-p)}{n}} \qquad \sigma_{H_0} = \sqrt{\frac{p_0(1-p_0)}{n}}$$

Critical values (C_1 and C_2) are:

$$C_1 = p_o - z_{\alpha/2}\sigma_{H_0} \qquad C_2 = p_o + z_{\alpha/2}\sigma_{H_0}$$

Type II error is:

$$\beta = P(C_1 < \hat{P} < C_2)$$

$$= P\left(\frac{C_1 - \mu_{H_1}}{\sigma_{H_1}} < Z < \frac{C_2 - \mu_{H_1}}{\sigma_{H_1}}\right)$$

$$= \Phi\left(\frac{C_2 - \mu_{H_1}}{\sigma_{H_1}}\right) - \Phi\left(\frac{C_1 - \mu_{H_1}}{\sigma_{H_1}}\right)$$

9-5 Tests on a Population Proportion

9-5.3 Type II Error and Choice of Sample Size

For a two-sided alternative

$$n = \left[\frac{z_{\alpha/2}\sqrt{p_0(1 - p_0)} + z_\beta \sqrt{p(1 - p)}}{p - p_0} \right]^2 \qquad (9\text{-}37)$$

For a one-sided alternative

$$n = \left[\frac{z_{\alpha}\sqrt{p_0(1 - p_0)} + z_\beta \sqrt{p(1 - p)}}{p - p_0} \right]^2 \qquad (9\text{-}38)$$

9-5 Tests on a Population Proportion

Example 9-11

Consider the semiconductor manufacturer from Example 9-10. Suppose that its process fall-out is really $p = 0.03$. What is the β-error for a test of process capability that uses $n = 200$ and $\alpha = 0.05$?

$$\sigma_{H_1} = \sqrt{\frac{0.03(0.97)}{200}} = 0.012 \qquad \sigma_{H_0} = \sqrt{\frac{0.05(0.95)}{200}} = 0.0154$$

Critical value (C_1) is:

$$C_1 = p_o - z_\alpha \sigma_{H_0} = 0.05 - 1.645(0.0154) = 0.0247$$

$$\beta = P(C_1 < \hat{P})$$

$$= P\left(\frac{0.0247 - 0.03}{0.012} < Z\right)$$

$$= 1 - \Phi(-0.44) = 0.67$$

9-5 Tests on a Population Proportion

Example 9-11

Suppose that the semiconductor manufacturer was willing to accept a β-error as large as 0.10 if the true value of the process fraction defective was $p = 0.03$. If the manufacturer continues to use $\alpha = 0.05$, what sample size would be required?

The required sample size can be computed from Equation 9-38 as follows:

$$n = \left[\frac{1.645\sqrt{0.05(0.95)} + 1.28\sqrt{0.03(0.97)}}{0.03 - 0.05}\right]^2$$

$$\simeq 832$$

where we have used $p = 0.03$ in Equation 9-38. Note that $n = 832$ is a very large sample size. However, we are trying to detect a fairly small deviation from the null value $p_0 = 0.05$.

9-7 Testing for Goodness of Fit

- The test is based on the chi-square distribution.
- Assume there is a sample of size n from a population whose probability distribution is unknown.
- Let O_i be the observed frequency in the ith class interval.
- Let E_i be the expected frequency in the ith class interval.

The test statistic

$$X_0^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

has approximately chi-square distribution with k-p-1 degrees of freedom.

p: number of parameters of the hypothesized distribution estimated by sample statistics.

9-7 Testing for Goodness of Fit

Example 9-12

A Poisson Distribution

The number of defects in printed circuit boards is hypothesized to follow a Poisson distribution. A random sample of $n = 60$ printed boards has been collected, and the following number of defects observed.

Number of Defects	Observed Frequency
0	32
1	15
2	9
3	4

9-7 Testing for Goodness of Fit

Example 9-12

The mean of the assumed Poisson distribution in this example is unknown and must be estimated from the sample data. The estimate of the mean number of defects per board is the sample average, that is, $(32 \cdot 0 + 15 \cdot 1 + 9 \cdot 2 + 4 \cdot 3)/60 = 0.75$. From the Poisson distribution with parameter 0.75, we may compute p_i, the theoretical, hypothesized probability associated with the ith class interval. Since each class interval corresponds to a particular number of defects, we may find the p_i as follows:

$$p_1 = P(X = 0) = \frac{e^{-0.75}(0.75)^0}{0!} = 0.472$$

$$p_2 = P(X = 1) = \frac{e^{-0.75}(0.75)^1}{1!} = 0.354$$

$$p_3 = P(X = 2) = \frac{e^{-0.75}(0.75)^2}{2!} = 0.133$$

$$p_4 = P(X \geq 3) = 1 - (p_1 + p_2 + p_3) = 0.041$$

9-7 Testing for Goodness of Fit

Example 9-12

The expected frequencies are computed by multiplying the sample size $n = 60$ times the probabilities p_i. That is, $E_i = np_i$. The expected frequencies follow:

Number of Defects	Probability	Expected Frequency
0	0.472	28.32
1	0.354	21.24
2	0.133	7.98
3 (or more)	0.041	2.46

9-7 Testing for Goodness of Fit

Example 9-12

Since the expected frequency in the last cell is less than 3, we combine the last two cells:

Number of Defects	Observed Frequency	Expected Frequency
0	32	28.32
1	15	21.24
2 (or more)	13	10.44

The chi-square test statistic in Equation 9-39 will have $k - p - 1 = 3 - 1 - 1 = 1$ degree of freedom, because the mean of the Poisson distribution was estimated from the data.

9-7 Testing for Goodness of Fit

Example 9-12

The eight-step hypothesis-testing procedure may now be applied, using $\alpha = 0.05$, as follows:

1. The variable of interest is the form of the distribution of defects in printed circuit boards.
2. H_0: The form of the distribution of defects is Poisson.
3. H_1: The form of the distribution of defects is not Poisson.
4. $\alpha = 0.05$
5. The test statistic is

$$\chi_0^2 = \sum_{i=1}^{k} \frac{(o_i - E_i)^2}{E_i}$$

9-7 Testing for Goodness of Fit

Example 9-12

6. Reject H_0 if $\chi_0^2 > \chi_{0.05,1}^2 = 3.84$.
7. Computations:

$$\chi_0^2 = \frac{(32 - 28.32)^2}{28.32} + \frac{(15 - 21.24)^2}{21.24} + \frac{(13 - 10.44)^2}{10.44} = 2.94$$

8. Conclusions: Since $\chi_0^2 = 2.94 < \chi_{0.05,1}^2 = 3.84$, we are unable to reject the null hypothesis that the distribution of defects in printed circuit boards is Poisson. The P-value for the test is $P = 0.0864$. (This value was computed using an HP-48 calculator.)

9-7 Testing for Goodness of Fit

Example 9-13

A Continuous Distribution

A manufacturing engineer is testing a power supply used in a notebook computer and, using $\alpha = 0.05$, wishes to determine whether output voltage is adequately described by a normal distribution. Sample estimates of the mean and standard deviation of $\bar{x} = 5.04$ V and $s = 0.08$ V are obtained from a random sample of $n = 100$ units.

A common practice in constructing the class intervals for the frequency distribution used in the chi-square goodness-of-fit test is to choose the cell boundaries so that the expected frequencies $E_i = np_i$ are equal for all cells. To use this method, we want to choose the cell boundaries $a_0, a_1, \ldots, a_k$ for the k cells so that all the probabilities

$$p_i = P(a_{i-1} \leq X \leq a_i) = \int_{a_{i-1}}^{a_i} f(x)\, dx$$

are equal. Suppose we decide to use $k = 8$ cells. For the standard normal distribution, the intervals that divide the scale into eight equally likely segments are [0, 0.32), [0.32, 0.675) [0.675, 1.15), [1.15, ∞) and their four "mirror image" intervals on the other side of zero. For each interval $p_i = 1/8 = 0.125$, so the expected cell frequencies are $E_i = np_i = 100(0.125) = 12.5$. The complete table of observed and expected frequencies is as follows:

9-7 Testing for Goodness of Fit

Example 9-13

X=5.04+Z*0.08

Z	X
-1.15	4.95
-0.68	4.99
-0.32	5.01
0	5.04
0.32	5.07
0.68	5.09
1.15	5.13

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Example 9-13

Class Interval	Observed Frequency o_i	Expected Frequency E_i
$x < 4.948$	12	12.5
$4.948 \leq x < 4.986$	14	12.5
$4.986 \leq x < 5.014$	12	12.5
$5.014 \leq x < 5.040$	13	12.5
$5.040 \leq x < 5.066$	12	12.5
$5.066 \leq x < 5.094$	11	12.5
$5.094 \leq x < 5.132$	12	12.5
$5.132 \leq x$	14	12.5
Totals	100	100

The boundary of the first class interval is $\bar{x} - 1.15s = 4.948$. The second class interval is $[\bar{x} - 1.15s, \bar{x} - 0.675s)$ and so forth. We may apply the eight-step hypothesis-testing procedure to this problem.

9-7 Testing for Goodness of Fit

1. The variable of interest is the form of the distribution of power supply voltage.

2. H_0: The form of the distribution is normal.

3. H_1: The form of the distribution is nonnormal.

4. $\alpha = 0.05$

5. The test statistic is

$$\chi_0^2 = \sum_{i=1}^{k} \frac{(o_i - E_i)^2}{E_i}$$

6. Since two parameters in the normal distribution have been estimated, the chi-square statistic above will have $k - p - 1 = 8 - 2 - 1 = 5$ degrees of freedom. Therefore, we will reject H_0 if $\chi_0^2 > \chi_{0.05,5}^2 = 11.07$.

7. Computations:

$$\begin{aligned}\chi_0^2 &= \sum_{i=1}^{8} \frac{(o_i - E_i)^2}{E_i} \\ &= \frac{(12 - 12.5)^2}{12.5} + \frac{(14 - 12.5)^2}{12.5} + \cdots + \frac{(14 - 12.5)^2}{12.5} \\ &= 0.64\end{aligned}$$

8. Conclusions: Since $\chi_0^2 = 0.64 < \chi_{0.05,5}^2 = 11.07$, we are unable to reject H_0, and there is no strong evidence to indicate that output voltage is not normally distributed. The P-value for the chi-square statistic $\chi_0^2 = 0.64$ is $P = 0.9861$.