Written Assignment #3: Riccati Equations (Solutions)

1. Equations of the form $\dfrac{dy}{dx} = A(x)y^2 + B(x)y + C(x)$ are called Riccati equations. If $y_1(x)$ is a known particular solution to a Ricatti equation, then the substitution $v = y - y_1$ will transform the Riccati equation into a Bernoulli equation.

   (a) If $v(x) = y(x) - y_1(x)$, then what do $y(x)$ and $y'(x)$ equal (in terms of v and y_1)?

   Solution

   Since $v(x) = y(x) - y_1(x)$, we have

   $$y(x) = v(x) + y_1(x)$$

   and

   $$y'(x) = v'(x) + y_1'(x).$$

   (b) Suppose that $y_1(x)$ is a solution to the Riccati equation

   $$\frac{dy}{dx} = A(x)y^2 + B(x)y + C(x).$$

   Make the change of variable $v = y - y_1$ to transform this equation into a Bernoulli equation.

   Solution

   Since $y_1(x)$ solves the Riccati equation, it must be that

   $$y_1' = A(x)y_1^2 + B(x)y_1 + C(x).$$

   Plugging in our substitutions yields

   $$\begin{aligned} \underbrace{v' + y_1'}_{y'(x)} &= A(x)[\underbrace{v + y_1}_{y(x)}]^2 + B(x)[\underbrace{v + y_1}_{y(x)}] + C(x) \\ \Rightarrow v' + [\underbrace{A(x)y_1^2 + B(x)y_1 + C(x)}_{y_1'(x)}] &= A(x)v^2 + 2A(x)y_1v + A(x)y_1^2 \\ &\quad + B(x)v + B(x)y_1 + C(x) \\ \Rightarrow v' &= A(x)v^2 + 2A(x)y_1v + B(x)v \\ \Rightarrow v' + [\underbrace{-2A(x)y_1(x) - B(x)}_{p(x)}]v &= \underbrace{A(x)}_{q(x)}v^2. \end{aligned}$$

   This is in the form of a Bernoulli equation.

2. In each of the following problems is a Riccati equation, a function y_1 and an initial condition. Verify that the function given is a particular solution to the Riccati equation, make the change of variable $v = y - y_1$ to reduce the Ricatti equation to a Bernoulli equation, and solve the resulting Bernoulli equation to obtain all solutions $v = v(x)$. Then return to the original variable and express the solutions as functions $y = y(x)$ and find the particular solution satisfying the initial condition given.

(a) $y' = (y - x)^2 + 1; \quad y_1(x) = x; \quad y(0) = \frac{1}{2}.$

Solution

First, we verify that $y_1 = x$ is a solution to this equation. Computing, we find that

$$\left.\begin{aligned} y_1' &= 1; \\ (y_1 - x)^2 + 1 = (x - x)^2 + 1 &= 1. \end{aligned}\right\} \text{so } \; y_1' = (y_1 - x)^2 + 1,$$

so y_1 is a solution to the differential equation.

Now we solve the equation:

Step 1: Make the change of variables:
substituting $y = v + x$ and $y' = v' + 1$ yields
$v' + 1 = ((v + x) - x)^2 + 1.$

Step 2: Simplify to a Bernoulli equation:
$$\underbrace{v' = v^2}_{\text{Bernoulli equation}} .$$
(Note that this is also a separable equation and could be solved as such.)

Step 3: Solve the Bernoulli equation for v.

substep 1: $v = w^{-1}$ and $v' = -w^{-2}w'$, so
$$-w^{-2}w' = \left(w^{-1}\right)^2$$

substep 2: $w' = -1.$

substep 3: $w = -x + C.$

substep 4: $v = (C - x)^{-1} = \underbrace{\dfrac{1}{C - x}}_{\text{General Solution}} .$

substep 5: Yes, $v = 0$ is a solution, and it is singular (not represented in the general solution).

The solutions to the Bernoulli equation are
$$v = \frac{1}{C - x} \text{ and } v = 0.$$

Step 4: Reverse the substitution: $y = v + x$
$$y = \frac{1}{C - x} + x \text{ and } y = x.$$

So the solutions are $y = \dfrac{1}{C-x} + x$ and $y = x$.

Finally, we use the initial condition. The solution $y = x$ can not satisfy the initial condition $y(0) = \dfrac{1}{2}$, so we use the general solution.

$$y(x) = \frac{1}{C-x} + x \Rightarrow y(0) = \frac{1}{C-0} + 0 = \frac{1}{C} = \frac{1}{2} \Rightarrow C = 2.$$

$$\boxed{y = \frac{1}{2-x} + x.}$$

(b) $y' = y^2 - \dfrac{y}{x} - \dfrac{1}{x^2}$, $x > 0$; $\quad y_1(x) = \dfrac{1}{x}$; $\quad y(1) = 2$.

Solution

First, we verify that $y_1 = \dfrac{1}{x}$ is a solution to this equation. Computing, we see that

$$\left.\begin{aligned} y_1' &= -\frac{1}{x^2}; \\ y_1^2 - \frac{y_1}{x} - \frac{1}{x^2} = \left(\frac{1}{x}\right)^2 - \left(\frac{1}{x}\right)^2 - \frac{1}{x^2} &= -\frac{1}{x^2}. \end{aligned}\right\} \text{so } y_1' = y_1^2 - \frac{y_1}{x} - \frac{1}{x^2}$$

so y_1 is a solution to the differential equation.

Now we solve the equation:

Step 1: Make the change of variables:
substituting $y = v + \dfrac{1}{x}$ and $y' = v' - \dfrac{1}{x^2}$ yields

$$v' - \frac{1}{x^2} = \left(v + \frac{1}{x}\right)^2 - \frac{1}{x}\left(v + \frac{1}{x}\right) - \frac{1}{x^2}.$$

Step 2: Simplify to a Bernoulli equation:

$$v' = v^2 + \frac{2}{x}v - \frac{1}{x}v \Rightarrow \underbrace{v' - \frac{1}{x}v = v^2}_{\text{Bernoulli equation}}.$$

Step 3: Solve the Bernoulli equation for v.

substep 1: $v = w^{-1}$ and $v' = -w^{-2}w'$, so

$$-w^{-2}w' - \frac{1}{x}w^{-1} = \left(w^{-1}\right)^2$$

substep 2: $w' + \dfrac{1}{x}w = -1.$

substep 3: Solve this linear equation for w

$$\mu(x) = e^{\int \frac{1}{x}\,dx} = e^{\ln|x|} = e^{\ln x} = x$$

$$xw' + w = -x \Rightarrow \frac{d}{dx}[xw] = -x$$

$$\Rightarrow xw = -\frac{1}{2}x^2 + C \Rightarrow w = -\frac{1}{2}x + \frac{C}{x}.$$

$$\Rightarrow w = \frac{C - x^2}{2x}$$

substep 4: $v = \left(\dfrac{C - x^2}{2x}\right)^{-1} = \underbrace{\dfrac{2x}{C - x^2}}_{\text{General Solution}}.$

substep 5: Yes, $v = 0$ is a solution to the Bernoulli equation, and it is singular (not represented in the general solution).

The solutions to the Bernoulli equation are

$$v = \frac{2x}{C - x^2} \text{ and } v = 0.$$

Step 4: Reverse the substitution: $y = v + \dfrac{1}{x}$

$$y = \frac{2x}{C - x^2} + \frac{1}{x} \text{ and } y = \frac{1}{x}.$$

So the solutions are $y = \dfrac{2x}{C - x^2} + \dfrac{1}{x}$ and $y = \dfrac{1}{x}$.

Finally, we use the initial condition. The solution $y = \dfrac{1}{x}$ can not satisfy the initial condition $y(1) = 2$, so we use the general solution.

$$y(x) = \frac{2x}{C - x^2} + \frac{1}{x} \Rightarrow y(1) = \frac{2 \cdot 1}{C - 1^2} + \frac{1}{1} = \frac{2}{C - 1} + 1 = 2 \Rightarrow C = 3.$$

$$\boxed{y = \frac{2x}{3 - x^2} + \frac{1}{x}.}$$