Regular Singular Points

It is of interest to solve the differential equation

$$N(x)y'' + P(x)y' + Q(x)y = 0, \tag{1}$$

or, in standard form,

$$y'' + p(x)y' + q(x)y = 0. \tag{2}$$

in the neighborhood of a singular point, as the behavior of the solutions there may be among their most important features. When the singularities are not too wild, a modification of the technique of power series can be used to calculate the solutions there.

To simplify the discussion, we shall restrict attention to equations of the form (1) where N, P, and Q are polynomials, which we may assume to have no common factors. (This situation includes the most important examples.) The singular points of the equation are then the points where $N(x) = 0$. Suppose x_0 is a singular point. Multiplying through by $(x - x_0)^2/N(x)$, we may rewrite (1) as

$$(x - x_0)^2 y'' + (x - x_0)u(x)y' + v(x)y = 0, \tag{3}$$

where

$$u(x) = \frac{(x - x_0)P(x)}{N(x)}, \qquad v(x) = \frac{(x - x_0)^2 Q(x)}{N(x)}. \tag{4}$$

We say that x_0 is a **regular** singular point if the rational functions $u(x)$ and $v(x)$ have no singularity at x_0—that is, if the factors of $x - x_0$ in $N(x)$ that cause $N(x)$ to vanish at x_0 are canceled by such factors in $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$. Otherwise x_0 is an **irregular** singular point.

Example 1. The singular points of the equation

$$x^2(x - 2)^2 y'' + (x - 2)y' + 3x^2 y = 0$$

are 0 and -2. At $x_0 = 0$ we have

$$u(x) = x(x - 2)/x^2(x - 2)^2 = 1/x(x - 2)$$

and

$$v(x) = (x^2)(3x^2)/x^2(x - 2)^2 = 3x^2/(x - 2)^2;$$

$v(x)$ is nonsingular at $x = 0$ but $u(x)$ blows up, so 0 is an irregular singular point. At $x_0 = 2$ we have

$$u(x) = (x - 2)(x - 2)/x^2(x - 2)^2 = 1/x^2$$

and

$$v(x) = (x - 2)^2(3x^2)/x^2(x - 2)^2 = 3;$$

these are both nonsingular at $x = 2$, so 2 is a regular singular point.

Henceforth we consider a fixed regular singular point x_0, and by the usual change of variable we assume that $x_0 = 0$.

The simplest examples of equations with a regular singular point at $x_0 = 0$ are the Euler equations

$$x^2y'' + axy' + by = 0, \tag{5}$$

which are of the form (3) with $u = a$ and $v = b$. In the previous set of notes we saw that if r_1 and r_2 are the roots of the equation

$$r(r-1) + ar + b - 0,$$

then the solutions of (5) are linear combinations of x^{r_1} and x^{r_2}, or x^{r_1} and $x^{r_1}\log|x|$ when $r_2 = r_1$—with suitable interpretation if r_1 and r_2 are complex numbers or if they are nonintegers and $x < 0$. Now, if $u(x)$ and $v(x)$ are continuous at 0, the general equation (3) (with $x_0 = 0$) looks very much like the Euler equation

$$x^2y'' + u(0)xy' + v(0)y = 0 \tag{6}$$

near $x = 0$, so we would expect its solutions to resemble linear combinations of x^{r_1} and x^{r_2} near $x = 0$, for suitable r_1 and r_2. This suggests that we should look for solutions of the form

$$y = x^r[a_0 + a_1x + a_2x^2 + \cdots] = \sum_0^\infty a_kx^{k+r}, \qquad a_0 \neq 0. \tag{7}$$

We require that $a_0 \neq 0$ because we want the leading term of the series to be x^r and not some higher power of x.

We proceed just as in the construction of series solutions about an ordinary (nonsingular) point. That is, we plug (7) into the differential equation—usually in the original form (1) rather than (3)—and obtain a sequence of equations for the coefficients a_k that can be solved recursively. *The main difference occurs at the initial step. In the previous situation, a_0 and a_1 could be chosen arbitrarily, and we got two independent solutions by making different choices of a_0 and a_1. In the present situation, a_1 is usually determined by a_0, and we get two independent solutions by using two different values of r.*

In more detail: When we plug (7) into the left side of (1) or (3) and set the coefficients of the various powers of x equal to zero, we get a sequence of equations involving the a_k's that look like this:

$$F(r)a_0 = 0, \tag{8.0}$$

$$\text{for } k > 0, F(k+r)a_k = \text{ terms involving } a_0, \dots, a_{k-1}, \tag{8.k}$$

where F is a certain quadratic polynomial. (Up to a constant factor, it is the polynomial corresponding to the Euler equation (6).) Since we require $a_0 \neq 0$, (8.0) is equivalent to $F(r) = 0$. This is called the **indicial equation** for the singular points, and its two roots r_1 and r_2 are called the **characteristic exponents**. We then obtain two distinct solutions by taking $r = r_1$ or $r = r_2$ and solving the equations (8) recursively for the a_k's. As in the case of ordinary points, it can be shown that the radius of convergence of the resulting series is at least the distance to the nearest *other* singular point.

There are two situations in which this procedure fails to yield the general solution of the differential equation. First, if $r_2 = r_1$, we clearly get only one solution this way. The other peculiar case is

when r_1 and r_2 differ by an integer—say, $r_2 = r_1 - N$, taking r_1 to be the larger one. Here our procedure always yields a solution with $r = r_1$; but when $r = r_2$ the coefficient of a_N in $(8.N)$ is $F(r_2 + N) = F(r_1) = 0$. Usually this means that we cannot solve for a_N and our method fails to yield a second solution. However, occasionally the other terms in $(8.N)$ will also cancel out, so that $(8.N)$ collapses to the triviality $0 \cdot a_N = 0$; in this case we can take $a_N = 0$ and proceed.
When our procedure yields only one solution, the second solution will involve $\ln x$ as well as powers of x. We shall say no more about it here. The full story can be found, for example, in *Ordinary Differential Equations* by G. Birkhoff and G. C. Rota.

Example 2. Let us solve the equation

$$2x^2y'' - xy' + (1 + x)y = 0, \tag{9}$$

which has a regular singular point at $x = 0$. Substituting $y = \sum a_k x^{k+r}$ in the left side of (9) yields

$$2\sum_0^\infty (k + r)(k + r - 1)a_k x^{k+r} - \sum_0^\infty (k + r)a_k x^{k+r} + \sum_0^\infty a_k x^{k+r} + \sum_1^\infty a_{k-1}x^{k+r}. \tag{10}$$

To obtain the last sum, we have taken the series $\sum_0^\infty a_k x^{k+1+r}$ for xy and shifted the index of summation to make the exponent of x match up with that in the other sums. The total coefficient of x^r in (10) is

$$[2r(r - 1) - r + 1]a_0 = (2r - 1)(r - 1)a_0.$$

(Note that the last sum in (10) does not contribute here.) Since we assume $a_0 \neq 0$, we must have $r = 1$ or $r = \frac{1}{2}$. These are the characteristic exponents.

For $k \geq 1$, the coefficient of x^{k+r} in (9) is

$$[2(k + r)(k + r - 1) - (k + r) + 1]a_k + a_{k-1} = (2k + 2r - 1)(k + r - 1)a_k + a_{k-1}.$$

Setting this equal to zero, we obtain the recursion formula

$$a_k = -\frac{a_{k-1}}{(2k + 2r - 1)(k + r - 1)}. \tag{11}$$

If we take $r = 1$, (11) becomes $a_k = a_{k-1}/(2k + 1)k$, which gives

$$a_1 = -\frac{a_0}{3 \cdot 1}, \quad a_2 = -\frac{a_1}{5 \cdot 2} = \frac{a_0}{[3 \cdot 5]2!}, \quad \cdots, \quad a_k = \frac{(-1)^k a_0}{[3 \cdot 5 \cdots (2k + 1)]k!}.$$

On the other hand, if we take $r = \frac{1}{2}$, (11) becomes $a_k = -a_{k-1}/k(2k - 1)$, so

$$a_1 = -\frac{a_0}{1 \cdot 1}, \quad a_2 = -\frac{a_1}{2 \cdot 3} = \frac{a_0}{2![1 \cdot 3]}, \quad \cdots, \quad a_k = \frac{(-1)^k a_0}{k![1 \cdot 3 \cdots (2k - 1)]}.$$

Thus the general solution is $c_1y_1 + c_2y_2$ where

$$y_1 = \sum_0^\infty \frac{(-1)^k x^{k+1}}{[1 \cdot 3 \cdot 5 \cdots (2k + 1)]k!}, \qquad y_2 = \sum_0^\infty \frac{(-1)^k x^{k+(1/2)}}{k![1 \cdot 3 \cdots (2k - 1)]}.$$

Example 3. Consider the equation $xy'' + 3y' - xy = 0$. Setting $y = \sum_0^\infty a_k x^{k+r}$, we get

$$\sum_0^\infty (k+r)(k+r-1)a_k x^{k+r-1} + 3\sum_0^\infty (k+r)a_k x^{k+r-1} + \sum_0^\infty a_{k-2}x^{k+r-1} = 0, \tag{12}$$

where, for the last term, we have put $xy = \sum_0^\infty a_k x^{k+r+1}$ and then shifted the index of summation so that the exponent of x is $k+r-1$ throughout. Setting the coefficient of x^{k+r-1} equal to 0, we obtain:

$$k = 0 : [r(r-1) + 3r]a_0 = 0, \tag{13.0}$$
$$k = 1 : [(1+r)r + 3(1+r)]a_1 = 0, \tag{13.1}$$
$$k \geq 2 : [(k+r)(k+r-1) + 3(k+r)]a_k - a_{k-2} = (k+r)(k+r+2)a_k - a_{k-2} = 0. \tag{13.k}$$

(The last sum on the left of (13) contributes only when $k \geq 2$.) Since $a_0 \neq 0$, (13.0) gives the indicial equation $r^2 + 2r = 0$, so the characteristic exponents are 0 and -2.
First take $r = 0$. Then (13.1) becomes $3a_1 = 0$, so $a_1 = 0$. Also, (13.k) becomes

$$a_k = a_{k-2}/k(k+2).$$

Hence:

$$a_3 = \frac{a_1}{3 \cdot 5} = 0, \quad a_5 = \frac{a_3}{5 \cdot 7} = 0, \quad \cdots, \quad a_{2n+1} = 0;$$

and

$$a_2 = \frac{a_0}{2 \cdot 4}, \quad a_4 = \frac{a_2}{4 \cdot 6} = \frac{a_0}{2 \cdot 4^2 \cdot 6}, \quad \cdots, \quad a_{2n} = \frac{a_0}{2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2(2n+2)}.$$

Thus one solution is

$$y_1 = \sum_0^\infty \frac{x^{2n}}{2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2(2n+2)} = \sum_0^\infty \frac{x^{2n}}{2^{2n} n!(n+1)!}.$$

Now take $r = -2$. Here (13.1) becomes $-a_1 = 0$ so $a_1 = 0$. But (13.k) becomes $(k-2)ka_k - a_{k-2} = 0$, and for $k = 2$ this says $0 \cdot a_2 - a_0 = 0$. Since $a_0 \neq 0$ this is impossible, and there is no solution of the form $\sum_0^\infty a_k x^{k-2}$.

Example 4. Let us modify the previous example slightly: $xy'' + 4y' - xy = 0$. The analogue of equations (13) here is

$$[r(r-1) + 4r]a_0 = 0, \tag{14.0}$$
$$[(1+r)r + 4(1+r)]a_1 = 0, \tag{14.1}$$
$$(k+r)(k+r+3)a_k - a_{k-2} = 0 \qquad (k \geq 2). \tag{14.k}$$

The indicial equation (14.0) is $r(r+3) = 0$, so $r = 0$ or $r = -3$. For either of these values, (14.1) implies that $a_1 = 0$. Taking $r = 0$, we can then use (14.k) to solve for all the a_k as in Example 3 to get a solution y_1. On the other hand, if we take $r = -3$, (14.k) becomes $k(k-3)a_k = a_{k-2}$. When $k = 3$, this says $0 \cdot a_3 = a_1$, which is automatically true since $a_1 = 0$. Thus, in this case, we can choose a_3 at will and then use (14.k) to determine all the other a_k. The simplest choice is $a_3 = 0$, which leads to a solution y_2 in which $a_k = 0$ for all odd k. (If we chose another a_3, we would get $y_2 + a_3 y_1$ instead.) We leave it as an exercise to verify that y_1 and y_2 are given by

$$y_1 = 1 + \sum_1^\infty \frac{x^{2n}}{2^n n! [5 \cdot 7 \cdots (2n+3)]}, \qquad y_2 = x^{-3} - \sum_1^\infty \frac{x^{2n-3}}{2^n n! [1 \cdot 3 \cdots (2n-3)]}.$$