SECTION 15.1 Exact First-Order Equations

Exact Differential Equations • Integrating Factors

Exact Differential Equations

In Section 5.6, you studied applications of differential equations to growth and decay problems. In Section 5.7, you learned more about the basic ideas of differential equations and studied the solution technique known as separation of variables. In this chapter, you will learn more about solving differential equations and using them in real-life applications. This section introduces you to a method for solving the first-order differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

for the special case in which this equation represents the exact differential of a function $z = f(x, y)$.

Definition of an Exact Differential Equation

The equation $M(x, y)\,dx + N(x, y)\,dy = 0$ is an **exact differential equation** if there exists a function f of two variables x and y having continuous partial derivatives such that

$$f_x(x, y) = M(x, y) \qquad \text{and} \qquad f_y(x, y) = N(x, y).$$

The general solution of the equation is $f(x, y) = C$.

From Section 12.3, you know that if f has continuous second partials, then

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial N}{\partial x}.$$

This suggests the following test for exactness.

THEOREM 15.1 Test for Exactness

Let M and N have continuous partial derivatives on an open disc R. The differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Exactness is a fragile condition in the sense that seemingly minor alterations in an exact equation can destroy its exactness. This is demonstrated in the following example.

EXAMPLE 1 Testing for Exactness

NOTE Every differential equation of the form

$$M(x)\,dx + N(y)\,dy = 0$$

is exact. In other words, a separable variables equation is actually a special type of an exact equation.

a. The differential equation $(xy^2 + x)\,dx + yx^2\,dy = 0$ is exact because

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[xy^2 + x] = 2xy \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[yx^2] = 2xy.$$

But the equation $(y^2 + 1)\,dx + xy\,dy = 0$ is not exact, even though it is obtained by dividing both sides of the first equation by x.

b. The differential equation $\cos y\,dx + (y^2 - x\sin y)\,dy = 0$ is exact because

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[\cos y] = -\sin y \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[y^2 - x\sin y] = -\sin y.$$

But the equation $\cos y\,dx + (y^2 + x\sin y)\,dy = 0$ is not exact, even though it differs from the first equation only by a single sign.

Note that the test for exactness of $M(x, y)\,dx + N(x, y)\,dy = 0$ is the same as the test for determining whether $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ is the gradient of a potential function (Theorem 14.1). This means that a general solution $f(x, y) = C$ to an exact differential equation can be found by the method used to find a potential function for a conservative vector field.

EXAMPLE 2 Solving an Exact Differential Equation

Solve the differential equation $(2xy - 3x^2)\,dx + (x^2 - 2y)\,dy = 0$.

Solution The given differential equation is exact because

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[2xy - 3x^2] = 2x = \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[x^2 - 2y].$$

The general solution, $f(x, y) = C$, is given by

$$\begin{aligned} f(x, y) &= \int M(x, y)\,dx \\ &= \int (2xy - 3x^2)\,dx = x^2y - x^3 + g(y). \end{aligned}$$

In Section 14.1, you determined $g(y)$ by integrating $N(x, y)$ with respect to y and reconciling the two expressions for $f(x, y)$. An alternative method is to partially differentiate this version of $f(x, y)$ with respect to y and compare the result with $N(x, y)$. In other words,

$$f_y(x, y) = \frac{\partial}{\partial y}[x^2y - x^3 + g(y)] = x^2 + g'(y) = \overbrace{x^2 - 2y}^{N(x,\,y)}.$$

$g'(y) = -2y$

Thus, $g'(y) = -2y$, and it follows that $g(y) = -y^2 + C_1$. Therefore,

$$f(x, y) = x^2y - x^3 - y^2 + C_1$$

and the general solution is $x^2y - x^3 - y^2 = C$. Figure 15.1 shows the solution curves that correspond to $C = 1, 10, 100$, and 1000.

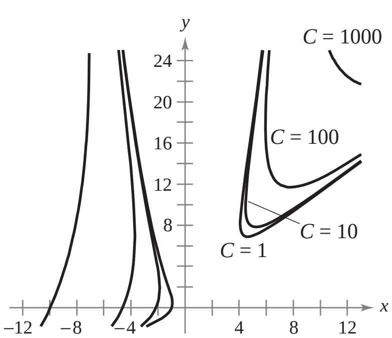


Figure 15.1

EXAMPLE 3 Solving an Exact Differential Equation

Find the particular solution of

$$(\cos x - x \sin x + y^2)\,dx + 2xy\,dy = 0$$

that satisfies the initial condition $y = 1$ when $x = \pi$.

Solution The differential equation is exact because

$$\overbrace{\frac{\partial}{\partial y}[\cos x - x \sin x + y^2] = 2y}^{\frac{\partial M}{\partial y}} = \overbrace{\frac{\partial}{\partial x}[2xy]}^{\frac{\partial N}{\partial x}}.$$

Because $N(x, y)$ is simpler than $M(x, y)$, it is better to begin by integrating $N(x, y)$.

$$f(x, y) = \int N(x, y)\,dy = \int 2xy\,dy = xy^2 + g(x)$$

$$f_x(x, y) = \frac{\partial}{\partial x}[xy^2 + g(x)] = y^2 + g'(x) = \overbrace{\cos x - x \sin x + y^2}^{M(x,\,y)}$$

$g'(x) = \cos x - x \sin x$

Thus, $g'(x) = \cos x - x \sin x$ and

$$g(x) = \int (\cos x - x \sin x)\,dx$$
$$= x \cos x + C_1$$

which implies that $f(x, y) = xy^2 + x \cos x + C_1$, and the general solution is

$$xy^2 + x \cos x = C.$$ General solution

Applying the given initial condition produces

$$\pi(1)^2 + \pi \cos \pi = C$$

which implies that $C = 0$. Hence, the particular solution is

$$xy^2 + x \cos x = 0.$$ Particular solution

The graph of the particular solution is shown in Figure 15.3. Notice that the graph consists of two parts: the ovals are given by $y^2 + \cos x = 0$, and the y-axis is given by $x = 0$.

TECHNOLOGY You can use a graphing utility to graph a particular solution that satisfies the initial condition of a differential equation. In Example 3, the differential equation and initial conditions are satisfied when $xy^2 + x \cos x = 0$, which implies that the particular solution can be written as $x = 0$ or $y = \pm\sqrt{-\cos x}$. On a graphing calculator screen, the solution would be represented by Figure 15.2 together with the y-axis.

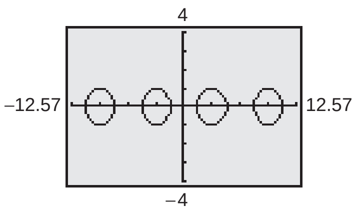


Figure 15.2

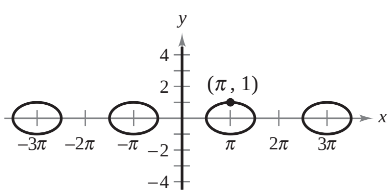


Figure 15.3

In Example 3, note that if $z = f(x, y) = xy^2 + x \cos x$, the total differential of z is given by

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy$$
$$= (\cos x - x \sin x + y^2)\,dx + 2xy\,dy$$
$$= M(x, y)\,dx + N(x, y)\,dy.$$

In other words, $M\,dx + N\,dy = 0$ is called an *exact* differential equation because $M\,dx + N\,dy$ is exactly the differential of $f(x, y)$.

Integrating Factors

If the differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ is not exact, it may be possible to make it exact by multiplying by an appropriate factor $u(x, y)$, which is called an **integrating factor** for the differential equation.

EXAMPLE 4 Multiplying by an Integrating Factor

a. If the differential equation

$$2y\,dx + x\,dy = 0 \qquad \text{Not an exact equation}$$

is multiplied by the integrating factor $u(x, y) = x$, the resulting equation

$$2xy\,dx + x^2\,dy = 0 \qquad \text{Exact equation}$$

is exact—the left side is the total differential of x^2y.

b. If the equation

$$y\,dx - x\,dy = 0 \qquad \text{Not an exact equation}$$

is multiplied by the integrating factor $u(x, y) = 1/y^2$, the resulting equation

$$\frac{1}{y}\,dx - \frac{x}{y^2}\,dy = 0 \qquad \text{Exact equation}$$

is exact—the left side is the total differential of x/y.

Finding an integrating factor can be difficult. However, there are two classes of differential equations whose integrating factors can be found routinely—namely, those that possess integrating factors that are functions of either x alone or y alone. The following theorem, which we present without proof, outlines a procedure for finding these two special categories of integrating factors.

THEOREM 15.2 Integrating Factors

Consider the differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$.

1. If

$$\frac{1}{N(x, y)}\left[M_y(x, y) - N_x(x, y)\right] = h(x)$$

is a function of x alone, then $e^{\int h(x)\,dx}$ is an integrating factor.

2. If

$$\frac{1}{M(x, y)}\left[N_x(x, y) - M_y(x, y)\right] = k(y)$$

is a function of y alone, then $e^{\int k(y)\,dy}$ is an integrating factor.

STUDY TIP If either $h(x)$ or $k(y)$ is constant, Theorem 15.2 still applies. As an aid to remembering these formulas, note that the subtracted partial derivative identifies both the denominator and the variable for the integrating factor.

EXAMPLE 5 Finding an Integrating Factor

Solve the differential equation $(y^2 - x)\,dx + 2y\,dy = 0$.

Solution The given equation is not exact because $M_y(x, y) = 2y$ and $N_x(x, y) = 0$. However, because

$$\frac{M_y(x, y) - N_x(x, y)}{N(x, y)} = \frac{2y - 0}{2y} = 1 = h(x)$$

it follows that $e^{\int h(x)\,dx} = e^{\int dx} = e^x$ is an integrating factor. Multiplying the given differential equation by e^x produces the exact differential equation

$$(y^2e^x - xe^x)\,dx + 2ye^x\,dy = 0$$

whose solution is obtained as follows.

$$f(x, y) = \int N(x, y)\,dy = \int 2ye^x\,dy = y^2e^x + g(x)$$

$$f_x(x, y) = y^2e^x + g'(x) = \overbrace{y^2e^x - xe^x}^{M(x,\,y)}$$

$g'(x) = -xe^x$

Therefore, $g'(x) = -xe^x$ and $g(x) = -xe^x + e^x + C_1$, which implies that

$$f(x, y) = y^2e^x - xe^x + e^x + C_1.$$

The general solution is $y^2e^x - xe^x + e^x = C$, or $y^2 - x + 1 = Ce^{-x}$.

In the next example, we show how a differential equation can help in sketching a force field given by $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$.

EXAMPLE 6 An Application to Force Fields

Sketch the force field given by

$$\mathbf{F}(x, y) = \frac{2y}{\sqrt{x^2 + y^2}}\mathbf{i} - \frac{y^2 - x}{\sqrt{x^2 + y^2}}\mathbf{j}$$

by finding and sketching the family of curves tangent to **F**.

Solution At the point (x, y) in the plane, the vector $\mathbf{F}(x, y)$ has a slope of

$$\frac{dy}{dx} = \frac{-(y^2 - x)/\sqrt{x^2 + y^2}}{2y/\sqrt{x^2 + y^2}} = \frac{-(y^2 - x)}{2y}$$

which, in differential form, is

$$2y\,dy = -(y^2 - x)\,dx$$

$$(y^2 - x)\,dx + 2y\,dy = 0.$$

From Example 5, we know that the general solution of this differential equation is $y^2 - x + 1 = Ce^{-x}$, or $y^2 = x - 1 + Ce^{-x}$. Figure 15.4 shows several representative curves from this family. Note that the force vector at (x, y) is tangent to the curve passing through (x, y).

Force field:
$\mathbf{F}(x, y) = \frac{2y}{\sqrt{x^2 + y^2}}\mathbf{i} - \frac{y^2 - x}{\sqrt{x^2 + y^2}}\mathbf{j}$
Family of tangent curves to **F**:
$y^2 = x - 1 + Ce^{-x}$

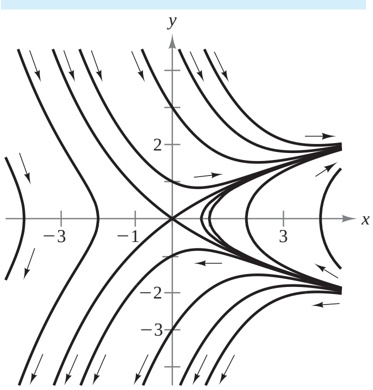


Figure 15.4

LAB SERIES Lab 20

EXERCISES FOR SECTION 15.1

In Exercises 1–10, determine whether the differential equation is exact. If it is, find the general solution.

1. $(2x - 3y)\,dx + (2y - 3x)\,dy = 0$
2. $ye^x\,dx + e^x\,dy = 0$
3. $(3y^2 + 10xy^2)\,dx + (6xy - 2 + 10x^2y)\,dy = 0$
4. $2\cos(2x - y)\,dx - \cos(2x - y)\,dy = 0$
5. $(4x^3 - 6xy^2)\,dx + (4y^3 - 6xy)\,dy = 0$
6. $2y^2e^{xy^2}\,dx + 2xye^{xy^2}\,dy = 0$
7. $\dfrac{1}{x^2 + y^2}(x\,dy - y\,dx) = 0$
8. $e^{-(x^2+y^2)}(x\,dx + y\,dy) = 0$
9. $\dfrac{1}{(x - y)^2}(y^2\,dx + x^2\,dy) = 0$
10. $e^y\cos xy\,[y\,dx + (x + \tan xy)\,dy] = 0$

In Exercises 11 and 12, (a) sketch an approximate solution of the differential equation satisfying the initial condition by hand on the direction field, (b) find the particular solution that satisfies the initial condition, and (c) use a graphing utility to graph the particular solution. Compare the graph with the hand-drawn graph of part (a).

	Differential Equation	*Initial Condition*
11.	$(2x\tan y + 5)\,dx + (x^2\sec^2 y)\,dy = 0$	$y\left(\frac{1}{2}\right) = \pi/4$
12.	$\dfrac{1}{\sqrt{x^2 + y^2}}(x\,dx + y\,dy) = 0$	$y(4) = 3$

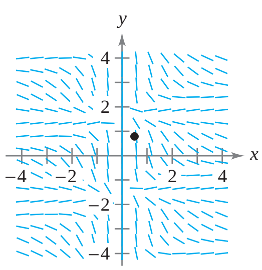


Figure for 11

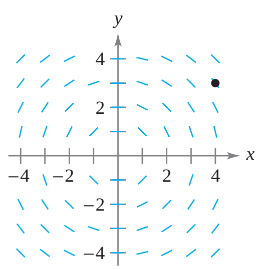


Figure for 12

In Exercises 13–16, find the particular solution that satisfies the initial condition.

	Differential Equation	*Initial Condition*
13.	$\dfrac{y}{x - 1}\,dx + [\ln(x - 1) + 2y]\,dy = 0$	$y(2) = 4$
14.	$\dfrac{1}{x^2 + y^2}(x\,dx + y\,dy) = 0$	$y(0) = 4$
15.	$e^{3x}(\sin 3y\,dx + \cos 3y\,dy) = 0$	$y(0) = \pi$
16.	$(x^2 + y^2)\,dx + 2xy\,dy = 0$	$y(3) = 1$

In Exercises 17–26, find the integrating factor that is a function of x or y alone and use it to find the general solution of the differential equation.

17. $y\,dx - (x + 6y^2)\,dy = 0$
18. $(2x^3 + y)\,dx - x\,dy = 0$
19. $(5x^2 - y)\,dx + x\,dy = 0$
20. $(5x^2 - y^2)\,dx + 2y\,dy = 0$
21. $(x + y)\,dx + \tan x\,dy = 0$
22. $(2x^2y - 1)\,dx + x^3\,dy = 0$
23. $y^2\,dx + (xy - 1)\,dy = 0$
24. $(x^2 + 2x + y)\,dx + 2\,dy = 0$
25. $2y\,dx + \left(x - \sin\sqrt{y}\right)dy = 0$
26. $(-2y^3 + 1)\,dx + (3xy^2 + x^3)\,dy = 0$

In Exercises 27–30, use the integrating factor to find the general solution of the differential equation.

27. $(4x^2y + 2y^2)\,dx + (3x^3 + 4xy)\,dy = 0$
    $u(x, y) = xy^2$
28. $(3y^2 + 5x^2y)\,dx + (3xy + 2x^3)\,dy = 0$
    $u(x, y) = x^2y$
29. $(-y^5 + x^2y)\,dx + (2xy^4 - 2x^3)\,dy = 0$
    $u(x, y) = x^{-2}y^{-3}$
30. $-y^3\,dx + (xy^2 - x^2)\,dy = 0$
    $u(x, y) = x^{-2}y^{-2}$

31. Show that each of the following is an integrating factor for the differential equation

    $y\,dx - x\,dy = 0.$

    (a) $\dfrac{1}{x^2}$ (b) $\dfrac{1}{y^2}$ (c) $\dfrac{1}{xy}$ (d) $\dfrac{1}{x^2 + y^2}$

32. Show that the differential equation

    $(axy^2 + by)\,dx + (bx^2y + ax)\,dy = 0$

    is exact only if $a = b$. If $a \neq b$, show that x^my^n is an integrating factor, where

    $m = -\dfrac{2b + a}{a + b}, \quad n = -\dfrac{2a + b}{a + b}.$

In Exercises 33–36, use a graphing utility to graph the family of tangent curves to the given force field.

33. $\mathbf{F}(x, y) = \dfrac{y}{\sqrt{x^2 + y^2}}\mathbf{i} - \dfrac{x}{\sqrt{x^2 + y^2}}\mathbf{j}$
34. $\mathbf{F}(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}}\mathbf{i} - \dfrac{y}{\sqrt{x^2 + y^2}}\mathbf{j}$
35. $\mathbf{F}(x, y) = 4x^2y\mathbf{i} - \left(2xy^2 + \dfrac{x}{y^2}\right)\mathbf{j}$
36. $\mathbf{F}(x, y) = (1 + x^2)\,\mathbf{i} - 2xy\,\mathbf{j}$

In Exercises 37 and 38, find an equation for the curve with the specified slope passing through the given point.

	Slope	*Point*
37.	$\dfrac{dy}{dx} = \dfrac{y - x}{3y - x}$	$(2, 1)$
38.	$\dfrac{dy}{dx} = \dfrac{-2xy}{x^2 + y^2}$	$(0, 2)$

39. ***Cost*** If $y = C(x)$ represents the cost of producing x units in a manufacturing process, the **elasticity of cost** is defined as

$$E(x) = \frac{\text{marginal cost}}{\text{average cost}} = \frac{C'(x)}{C(x)/x} = \frac{x}{y}\frac{dy}{dx}.$$

Find the cost function if the elasticity function is

$$E(x) = \frac{20x - y}{2y - 10x}$$

where $C(100) = 500$ and $x \geq 100$.

40. ***Euler's Method*** Consider the differential equation $y' = F(x, y)$ with the initial condition $y(x_0) = y_0$. At any point (x_k, y_k) in the domain of F, $F(x_k, y_k)$ yields the slope of the solution at that point. Euler's Method gives a discrete set of estimates of the y values of a solution of the differential equation using the iterative formula

$$y_{k+1} = y_k + F(x_k, y_k)\,\Delta x$$

where $\Delta x = x_{k+1} - x_k$.

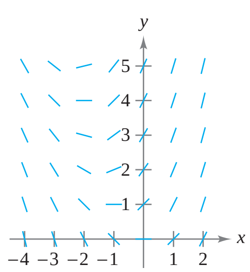
(a) Write a short paragraph describing the general idea of how Euler's Method works.

(b) How will decreasing the magnitude of Δx affect the accuracy of Euler's Method?

41. ***Euler's Method*** Use Euler's Method (see Exercise 40) to approximate $y(1)$ for the values of Δx given in the table if $y' = x + \sqrt{y}$ and $y(0) = 2$. (Note that the number of iterations increases as Δx decreases.) Sketch a graph of the approximate solution on the direction field in the figure.

Δx	0.50	0.25	0.10
Estimate of $y(1)$			

The value of $y(1)$, accurate to three decimal places, is 4.213.



42. ***Programming*** Write a program for a graphing utility or computer that will perform the calculations of Euler's Method for a specified differential equation, interval, Δx, and initial condition. The output should be a graph of the discrete points approximating the solution.

Euler's Method **In Exercises 43–46, (a) use the program of Exercise 42 to approximate the solution of the differential equation over the indicated interval with the specified value of Δx and the initial condition, (b) solve the differential equation analytically, and (c) use a graphing utility to graph the particular solution and compare the result with the graph of part (a).**

	Differential Equation	*Interval*	Δx	*Initial Condition*
43.	$y' = x\sqrt[3]{y}$	$[1, 2]$	0.01	$y(1) = 1$
44.	$y' = \dfrac{\pi}{4}(y^2 + 1)$	$[-1, 1]$	0.1	$y(-1) = -1$
45.	$y' = \dfrac{-xy}{x^2 + y^2}$	$[2, 4]$	0.05	$y(2) = 1$
46.	$y' = \dfrac{6x + y^2}{y(3y - 2x)}$	$[0, 5]$	0.2	$y(0) = 1$

47. ***Euler's Method*** Repeat Exercise 45 for $\Delta x = 1$ and discuss how the accuracy of the result changes.

48. ***Euler's Method*** Repeat Exercise 46 for $\Delta x = 0.5$ and discuss how the accuracy of the result changes.

True or False? **In Exercises 49–52, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

49. The differential equation $2xy\,dx + (y^2 - x^2)\,dy = 0$ is exact.

50. If $M\,dx + N\,dy = 0$ is exact, then $xM\,dx + xN\,dy = 0$ is also exact.

51. If $M\,dx + N\,dy = 0$ is exact, then $[f(x) + M]\,dx + [g(y) + N]\,dy = 0$ is also exact.

52. The differential equation $f(x)\,dx + g(y)\,dy = 0$ is exact.