Differential Equations

In physics, engineering, chemistry, economics, and other sciences mathematical models are built that involve rates at which things happen. These models are equations and the rates are derivatives. Equations containing derivatives are called **differential equations**.

Examples:
1) 100 grams of cane sugar in water are being converted into dextrose at a rate that is proportional to the amount unconverted. Find the differential equation expressing the rate of conversion after t minutes.
<u>Solution</u>:
Let q be the grams of sugar converted in t minutes, then (100 – q) is the number of grams unconverted and the rate of conversion is given by $\frac{dq}{dt} = k(100 - q)$ where k is the constant of proportionality.

2) A curve is defined by the condition that at each of its points (x,y), is slope dy/dx is equal twice the sum of the coordinates of the point, find the differential equation that defines the curve.
<u>Solution</u>:

$$\frac{dy}{dx} = 2(x + y)$$

The following are more examples of differential equations:

1) $\frac{dy}{dx} = \cos x \quad \text{or} \quad y' = \cos x$

2) $\frac{d^2y}{dx^2} + k^3 y = 0 \quad \text{or} \quad y'' + k^3 y = 0$

3) $(x^2 + y^2)dx = 2xdy$

4) $\frac{\partial u}{\partial t} = h^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

5) $\frac{\partial^2 w}{\partial x^2} + \frac{\partial w}{\partial y} = 0$

6) $\left(\frac{d^2 s}{dt^2}\right)^6 - xy\frac{ds}{dt} + s = 0$

7) $\frac{d^3 x}{dy^3} + x\frac{dx}{dy} = 4xy$

8) $\frac{d^5y}{dx^5} + x\left(\frac{dy}{dx}\right)^3 - 8y = 0$

9) $\frac{d^2y}{dt^2} + x\frac{d^2x}{dt^2} = x$

10) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$

11) $L\frac{d^2u}{dt^2} + R\frac{du}{dt} + \frac{1}{c}u = E\cos t$

12) $cy''' + my'' - y = 6$

Definition: An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a **differential equation**.

Remark: When a differential equation involves one or more derivatives with respect to a particular variable, that variable is called the **independent variable**. A variable is called **dependent** if a derivative of that variable occurs.

Example:

1) $L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{c}i = E\ \cos(t)$

i is the dependent variable, t is the independent variable, L, R, c, E are constants

2) $\frac{\partial^2 W}{\partial u^2} + \frac{\partial^2 W}{\partial v^2} = 0$

W is the dependent variable, u and v are the independent variables

3) $(x^2 + y^2)\ dx = 2xy\ dy = 0$

we can express this equation either in the form

$(x^2 + y^2) - 2xy\frac{dy}{dx} = 0$, where y is the dependent variable and x is the independent variable,

or in the form

$(x^2 + y^2)\frac{dx}{dy} - 2xy = 0$, where x is the dependent variable and y is the independent variable.

Classification of Differential Equations

a) Ordinary or Partial Differential Equations
One of the most obvious classifications is based on whether the unknown function depends on a single independent variable or on several independent variables.

Definition: A differential equation involving ordinary derivatives of one or more dependent variables is called an **ordinary** differential equation (O.D.E.).

Examples:

1) $\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + e^x = 0$

2) $\cos x \; dy - x \; dx = 0$

3) $\frac{dy}{dt} - \frac{dx}{dt} = x$

Definition: A equation involving partial derivatives of one or more dependent variables with respect to more than one independent variable is called a **partial** differential equation (P.D.E.).

Example:

1) $\frac{\partial^2\Omega}{\partial u^2} + \frac{\partial^2\Omega}{\partial u\partial v} = \frac{\partial^2\Omega}{\partial v^2}$

2) $x\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial y} = k$

b) Order
Definition: The **order** of a differential equation is the order of the highest derivative appearing in the equation.

Example:

1) $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - y = 0$ is an equation of order 3 or third order

2) $\frac{d^2y}{dx^2} + 2b\left(\frac{dy}{dx}\right)^5 + y = 0$ is an equation of order 2 or second order

3) $\frac{\partial\Omega}{\partial v} + \frac{\partial^2\Omega}{\partial u\partial v} = 0$ is an equation of order 2 or second order

4) $(x + y^2)\, dy - xy\, dx = 0$ is an equation of order 1 or first order

Remark: The equation $F(x, u(x), u'(x), u''(x), ..., u^{(n)}(x)) = 0$ is an ordinary differential equation of order n. The equation expresses a relation between the independent variable x and the values of the function u and its first n derivatives.
It is convenient and customary in differential equations to write y for u(x), with y', y'', …, $y^{(n)}$ standing for u'(x), u''(x),…, $u^{(n)}(x)$, so the ODE can be written as

$$F(x, y, y', y'', ..., y^{(n)}) = 0 .$$

Example:
1) $F(x,y,y') = x(y')^2 + 5y - 6x^2 = 0$ is a first order equation
2) $F(x,y,y',y'') = xy'' - 4 \sin x = 0$ is a second order equation

We assume that it is always possible to solve a given ODE for the highest derivative, obtaining $y^{(n)} = f(x, y', y'', ..., y^{(n-1)})$

Example:
Given the equation $(y')^2 + xy' + 4y = 0$, to solve for y' we must complete the square

$$(y')^2 + xy' + \frac{x^2}{4} - \frac{x^2}{4} + 4y = 0$$

$$\left(y' + \frac{x}{2}\right)^2 = \frac{x^2 - 16y}{4}$$

$$y' + \frac{x}{2} = \frac{\pm\sqrt{x^2 - 16y}}{2}$$

$$y' = \frac{-x \pm \sqrt{x^2 - 16y}}{2}$$

Solving for y' we obtain two equations, so we want to avoid this kind of situations

c) Degree
Definition: The **degree** of a differential equation is given by the exponent that is raised the highest derivative that occurs in the equation.

Example:
1) $y''' + 2(y'')^5 + y = 0$ is an equation of degree 1

2) $(y'')^3 + (y')^7 = 0$ is an equation of degree 3

3) $\frac{\partial^2 \Omega}{\partial u^2} + \left(\frac{\partial^2 \Omega}{\partial u \partial v}\right)^2 + \left(\frac{\partial \Omega}{\partial v}\right)^6 = 0$ is an equation of degree 2

d) Linear or Nonlinear Equation
Definition: A **linear** ODE of order n, in the independent variable x and the dependent variable y is an equation that can be expressed in the form:

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + ... + a_{n-1}(x)y' + a_n(x)y = b(x)$$

where $a_0(x) \neq 0$.

Notice that a linear ODE satisfies the following conditions:

1- The dependent variable y and its derivatives occur to the first degree only.

2- No products of y and/or any of its derivatives appear in the equation.

3- No transcendental functions of y and/or its derivatives occur.

Definition: Equations that not satisfy the above conditions are called nonlinear.

Examples:

1) $\frac{d^3y}{dx^3} + 5x\frac{d^2y}{dx^2} + y\sin x = 0$ linear

2) $y' + e^y = x$ nonlinear

3) $x^2y'' + xy' + y = \cos x$ linear

4) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + 7y = 0$ nonlinea

5) $\cos x\ y'' + e^x\ y' - y = \sin x$ linear

6) $\frac{\partial^2\Omega}{\partial u^2} + \frac{\partial^2\Omega}{\partial v^2} = 0$ linear

7) $\frac{\partial^2\Omega}{\partial u^2}\frac{\partial^2\Omega}{\partial v^2} = \frac{\partial^2\Omega}{\partial u\partial v}$ nonlinear