2a. Bernoulli's Differential Equation

A differential equation of the form

$$y' + p(t)y = g(t)y^n \tag{1}$$

is called Bernoulli's differential equation.

If $n = 0$ or $n = 1$, this is linear. If $n \neq 0, 1$, we make the change of variables $v = y^{1-n}$. This transforms (1) into a linear equation.

Let us see this.

We have

$$v = y^{1-n}$$

$$v' = (1-n)y^{-n}y'$$

$$y' = \frac{1}{1-n}y^n v'$$

and

$$y = y^n v$$

Hence,

$$y' + py = gy^n$$

becomes

$$\frac{1}{1-n}y^n v' + py^n v = gy^n$$

Dividing y^n through and multiplying by $1-n$ gives

$$v' + (1-n)pv = (1-n)g.$$

We can then find v and, hence, $y = v^{\frac{1}{1-n}}$.
Example.
Find the general solution to

$$y' + ty = ty^3.$$

We put $v = y^{-2}$
We get

$$v' = (-2)y^{-3}y',\ y = y^3 v$$

So,

$$y' + ty = ty^3$$

$$(-1/2)y^3 v' + ty^3 v = ty^3$$

$$v' - 2tv = -2t$$

$$\mu = e^{-t^2}$$

$$\begin{aligned} v &= e^{t^2}\left(\int^t e^{-t^2}(-2t)dt + c\right) \\ &= e^{t^2}\left(e^{-t^2} + c\right) \\ &= 1 + ce^{t^2}, \end{aligned}$$

and,

$$y = v^{-\frac{1}{2}} = \left[1 + ce^{t^2}\right]^{-\frac{1}{2}}.$$

a Bernoulli IVP

Solve $x\frac{dy}{dx} + 5y = 2x^2 y^4$, $y(1) = 3$

or $y' + \frac{5}{x} y = 2x y^4$

Bernoulli with $n = 4$

Let $v = y^{1-n} = y^{-3}$

Get $v' - \frac{15}{x} v = -6x$ linear

$p = -\frac{15}{x}$

$u = e^{\int p\,dx} = e^{-15\log(x)} = x^{-15}$

$v = x^{15}\left[\int -6x^{-14}\,dx + C\right]$

$= x^{15}\left(-6\frac{x^{-13}}{-13} + C\right)$

$= \frac{6}{13}x^2 + Cx^{15}$, $v(1) = 3^{-3} = \frac{1}{27}$

Use $v(1) = \frac{1}{27}$: $\frac{6}{13} + C = \frac{1}{27}$, $C = \frac{1}{27} - \frac{6}{13}$

$y = \left[\frac{6}{13}x^2 + \left(\frac{1}{27} - \frac{6}{13}\right)x^{15}\right]^{-\frac{1}{3}}$