Date: December 12, 2003 NAME, SURNAME: ____________

Time: 17:00-18:00

MATH 202 NUMBER: ____________

STUDENT ID: ____________

MATH 202 SECOND MIDTERM SOLUTION KEY

IMPORTANT

1. Write your name, surname on top of each page.
2. The exam consists of 4 questions some of which have more than one part.
3. Please read the questions carefully and write your answers neatly under the corresponding questions.
4. Show all your work. Correct answers without sufficient explanation might <u>not</u> get full credit.
5. Calculators are <u>not</u> allowed.

1	2	3	4	TOTAL
30 pts.	20 pts.	30 pts.	25 pts.	105 pts.

1.)[30] Find a power series solution of the initial value problem:

$$y'' - 2xy' - 2y = 0, \quad y(0) = 1,\, y'(0) = 0.$$

When does the series solution converge and what is its sum?

Solution:

Given the IC at $x = 0$ we need to solve the DE about this point. Clearly $x = 0$ is an ordinary point and hence a power series solution near this point has the form: $\sum_{n=0}^{\infty} a_n x^n$. Computing the derivatives:

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1} \quad \text{and} \quad \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}.$$

To find the recurrence relation equations we plug these into the equation:

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - 2\sum_{n=1}^{\infty} a_n n x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0.$$

Rewriting the first summation so that each summation has the same x factor by $n \mapsto n+2$ yields:

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - 2\sum_{n=1}^{\infty} a_n n x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Leftrightarrow \quad 2(a_2 - a_0) + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - 2(n+1)a_n\right] x^n = 0$$

$$\Leftrightarrow \quad a_2 = a_0 \quad \text{and} \quad a_{n+2} = \frac{2a_n}{n+2} \quad n \geq 1.$$

The last line above is the recurrence relation which lets us compute every single coefficient a_j. The nth coefficient depends only upon the $(n-2)$nd one. So it is natural to split the solution series into 2 parts: even powers and odd powers. This is:

$$\begin{aligned} a_{2n} &= \frac{2a_{n-2}}{2n} = \frac{2 \cdot 2a_{2n-4}}{2n(2n-2)} = \cdots = \frac{2^n a_0}{2^n n!} = \frac{a_0}{n!}\,, \\ a_{2n+1} &= \frac{2a_{2n-1}}{2n+1} = \frac{2 \cdot 2a_{2n-3}}{(2n+1)(2n-1)} = \cdots = \frac{2^n a_1}{(2n+1)(2n-1)\cdots 5 \cdot 3}\,. \end{aligned}$$

Hence the general solution becomes:

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} + a_1 \sum_{n=0}^{\infty} \frac{2^n}{3 \cdot 5 \cdots (2n+1)} x^{2n+1}\,.$$

Now using the IC: $y(0) = a_0 = 1$ and $y'(0) = a_1 = 0$ the solution of the IVP is:

$$\boxed{y(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}}$$

This series converges everywhere. Recalling $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ we substitute $x \mapsto x^2$ and show that:

$$\boxed{y(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = e^{x^2}}$$

♦

2.) A second order linear homogeneous differential equation with a *regular singular* point $x = 0$ has the indicial equation $F(r) = r^2 - 1 = 0$. Let the recurrence relation be:

$$a_1 = 0 \quad \text{and} \quad \left[1 - (n+r)^2\right] a_n = a_{n-2}, \quad n \geq 2.$$

(a)[14] How many solutions of the form $y = x^r \sum_{n=0}^{\infty} a_n x^n$ do there exist? **Explain fully**.

Solution:

Roots of the indicial equation are $r_1 = 1$ and $r_2 = -1$ (which are exponents at singularity). Since $r_1 - r_2 = 2$ is an integer it is not guaranteed that both solutions are of the desired form. Nevertheless, for the bigger root $r_1 = 1$ this type of a solution exists.

$\underline{r_1 = 1}$: Being the bigger root, a solution of the form:

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$

exits, where a_n are obtained from the given recurrence relation.

$\underline{r_2 = -1}$: We assume that $y_2(x) = \sum_{n=0}^{\infty} a_n x^{n-1}$ where a_j are to be computed. Letting $a_0 \neq 0$ we start calculating the coefficients:

$$a_1 = 0, \quad \left[1 - (n-1)^2\right] a_n = a_{n-2}, \quad n \geq 2.$$

For $n = 2$, we get:

$$\left[1 - (2-1)^2\right] a_2 = a_0 \Leftrightarrow 0 \cdot a_2 = a_0 \Leftrightarrow a_0 = 0$$

which is clearly contradictory to our assumption that $a_0 \neq 0$. Hence, we conclude that $y_2(x)$ contains a logarithmic term.

There is only one solution of the required form, for $r_1 = 1$.

♦

(b)[6] Write down first *three* nonzero terms of the solution(s) of the form $y = x^r \sum_{n=0}^{\infty} a_n x^n$.

Solution:

First remark is only terms like x^{2n+1} survive, i.e. $a_1 = 0 \Rightarrow a_{2n+1} = 0$ for all $n \in \mathbb{Z}$. We need to compute the coefficients of the terms x^{0+1}, x^{2+1} and x^{4+1}: a_0 is free and:

$$a_2 = \frac{a_0}{[1-(2+1)^2]} = -\frac{a_0}{8}$$

$$a_4 = \frac{a_2}{[1-(4+1)^2]} = -\frac{a_2}{24} = \frac{a_0}{24 \cdot 8} = \frac{a_0}{192}$$

Hence we have obtained:

$$\boxed{y_1(x) = a_0 \left(x - \frac{x^3}{8} + \frac{x^5}{192} - \cdots \right)}$$

♦

3.)[30] Solve the initial value problem:

$$y'' + 2y' + 2y = \begin{cases} 0 & 0 \le t < 2 \\ \delta(t-4) & 2 \le t \end{cases} \qquad y(0) = 1,\ y'(0) = 0,$$

where $\delta(t-4)$ is a Dirac delta function. [Hint: You might need to use the definition of the Laplace transform]. Show also that $y(t)$ is continuous.

Solution:

Taking the Laplace transform of both sides:

$$\mathcal{L}\{y'' + 2y' + 2y\} = \mathcal{L}\{u_2(t)\delta(t-4)\},$$

which is equivalent to:

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + 2Y(s) &= \int_0^\infty e^{-st}u_2(t)\delta(t-4)dt = \\ &= \int_2^\infty e^{-st}\delta(t-4)dt = e^{-4s} \end{aligned}$$

as $4 \in (2, \infty)$ followed by the integration properties of Dirac delta function. Substituting the initial conditions:

$$(s^2 + 2s + 2)Y(s) - (s+2) = e^{-4s} \Leftrightarrow (s^2 + 2s + 2)Y(s) = e^{-4s} + s + 2.$$

Now leaving $Y(s)$ alone and using $s^2 + 2s + 2 = (s+1)^2 + 1$ we receive:

$$Y(s) = \frac{e^{-4s}}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}.$$

We now take the inverse Laplace transform by using the shift properties and $\mathcal{L}^{-1}\{Y(s)\} = y(t)$. One useful shift theorem is:

$$\mathcal{L}^{-1}\left\{e^{-cs}F(s)\right\} = u_c(t)f(t-c), \quad \text{where} \quad \mathcal{L}^{-1}\{F(s)\} = f(t).$$

First noting that $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = e^{-t}\sin t$ we see that the solution $y(t)$ is:

$$\boxed{y(t) = u_4(t)e^{-(t-4)}\sin(t-4) + e^{-t}\sin t + e^{-t}\cos t}$$

$y(t)$ is trivially continuous everywhere except possibly at $t = 4$. So check only limits from right and left as t goes to 4:

$$\begin{aligned} \lim_{t\to 4^-} y(t) &= e^{-4}(\sin 4 + \cos 4) \\ \lim_{t\to 4^+} y(t) &= \sin 0 + e^{-4}(\sin 4 + \cos 4) = e^{-4}(\sin 4 + \cos 4) = y(4) \end{aligned}$$

which are equal. Hence $y(t)$ is continuous.

♦

4.)[25] Find a continuous function $f(t)$ which satisfies the following integro-differential equation and the initial condition:

$$f'(t) - \int_0^t f(\xi)d\xi = u_2(t) + t, \quad f(0) = 1,$$

where $u_2(t)$ denotes a unit step function (Heaviside function).

Solution:

We observe that $\int_0^t f(\xi)d\xi = f * 1$. Hence the convolution theorem implies: $\mathfrak{L}\{f * 1\} = \mathfrak{L}\{f\}\mathfrak{L}\{1\} = F(s)/s$. Furthermore, $\mathfrak{L}\{t\} = (-1)^1 d(\mathfrak{L}\{1\})/ds = 1/s^2$. Combining these and taking the Laplace transform of the equation:

$$sF(s) - f(0) - \frac{F(s)}{s} = \frac{e^{-2s}}{s} + \frac{1}{s^2}.$$

We use the initial condition as well to find:

$$\frac{s^2-1}{s}F(s) = \frac{e^{-2s}}{s} + \frac{1}{s^2} + 1 \Leftrightarrow F(s) = \frac{e^{-2s}}{s^2-1} + \frac{1}{s(s^2-1)} + \frac{s}{s^2-1}.$$

With the help of partial fractions we immediately see that:

$$\frac{1}{s(s^2-1)} = \frac{s}{s^2-1} - \frac{1}{s}.$$

So $F(s)$ has become:

$$F(s) = \frac{e^{-2s}}{s^2-1} + 2\frac{s}{s^2-1} - \frac{1}{s}.$$

Inverse Laplace transform finds $f(t)$ to be:

$$\boxed{f(t) = u_2(t)\sinh(t-2) + 2\cosh t - 1}$$

Note that each hyperbolic sine or cosine function is an exponential, hence your answer could be: $f(t) = u_2(t)\sinh(t-2) + e^t + e^{-t} - 1$.

♦

Some basic Laplace transforms you might need

$\mathfrak{L}\{1\} = \frac{1}{s}$, $s > 0$	$\mathfrak{L}\{e^{at}\} = \frac{1}{s-a}$, $s > a$	$\mathfrak{L}\{\sin at\} = \frac{a}{s^2+a^2}$, $s > 0$
$\mathfrak{L}\{\cos at\} = \frac{s}{s^2+a^2}$, $s > 0$	$\mathfrak{L}\{\sinh at\} = \frac{a}{s^2-a^2}$, $s > \|a\|$	$\mathfrak{L}\{\cosh at\} = \frac{s}{s^2-a^2}$, $s > \|a\|$

B U Department of Mathematics
Math 202 Fall 2004 Second Midterm

1. Consider the vectors $\mathbf{x}^{(1)}(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$ and $\mathbf{x}^{(2)}(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$

a)Discuss their linear dependency and independency.

Solution:

$$c_1 \begin{pmatrix} t^2 \\ 2t \end{pmatrix} + c_2 \begin{pmatrix} e^t \\ e^t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} t^2 & e^t \\ 2t & e^t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{vmatrix} t^2 & e^t \\ 2t & e^t \end{vmatrix} = t(t-2)e^t$$

$\Rightarrow$ they are linearly independent on any subset of $\mathbb{R}$ not containing $t = 0$ or $t = 2$, i.e. on $(-\infty, 0), (0, 2)$, and $(2, \infty)$.

b)Can they be solutions to a system $\mathbf{x}' = A\mathbf{x}$ with A consisting of **constant numbers**?

Solution:

They can not as such a system with numerical A will have a Wronskian which can never be zero.

c)Find a system of equations that they satisfy.

Solution:

$$\text{Solve } \begin{pmatrix} t^2 & e^t \\ 2t & e^t \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t^2 & e^t \\ 2t & e^t \end{pmatrix} \text{ for } a, b, c, d:$$

$$\Rightarrow A = \begin{pmatrix} 0 & 1 \\ \dfrac{2(1-t)}{t(t-2)} & \dfrac{t^2-2}{t(t-2)} \end{pmatrix}$$

2. Find the general solution of the given system of equations.

$$\mathbf{x}' = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} t^{-1} \\ 2t^{-1} + 4 \end{pmatrix}, \qquad t > 0$$

Solution:

$x' = Px + g(t)$

Eigenvalues and eigenvectors:

$|A - rI| = 0 \Rightarrow \begin{vmatrix} 4-r & 2 \\ 2 & -1-r \end{vmatrix} = 0$

$r_1 = 0, r_2 = -5$ and $\xi^1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\xi^2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$x_h = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^0 + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-5t}$

$T = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$ then $T^{-1} = \begin{pmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{pmatrix}$

$y' = Dy + T^{-1}g(t)$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{pmatrix} \begin{pmatrix} t^{-1} \\ 2t^{-1} + 4 \end{pmatrix}$$

Then, $y_1 = \ln t + 8/5t + c_1$ and $y_2 = 4/25 + c_2 e^{-5t}$

$x = Ty$ then,

$$x = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-5t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \ln t + 8/5 \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + 4/25 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

3. For the differential equation $2xy'' + 3y' + xy = 0$, show that it has a solution in the form $\sum_{n=0}^{\infty} a_n x^n$. Find the **domain of convergence** of this solution.

Solution:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Put these into the equation then;

$$\sum_{n=1}^{\infty} 2n(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} 3(n+1)a_{n+1}x^n + \sum_{n=1}^{\infty} a_{n-1}x^n = 0$$

$3a_1 = 0 \Rightarrow a_1 = 0$

$(2n+3)(n+1)a_{n+1} + a_{n-1} = 0$

$a_3 = a_5 = a_7 = ... = 0$

$a_2 = -\dfrac{a_0}{2.5} \quad a_4 = \dfrac{a_0}{2.4.5.9} \quad a_6 = \dfrac{a_0}{2.4.5.9.6.13}$... then

$$y = a_0 \left(1 - \frac{x^2}{2.5} + \frac{x^4}{2.4.5.9} - ...\right)$$

Apply ratio test:

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}x^{2n+2}}{2.4...2(n+1).5.9.13...(4n+5)} . \frac{2.4.6...2n.5.9.13.(4n+1)}{(-1)^n x^{2n}} \right|$$

$$\lim_{n\to\infty} \left| \frac{x^2}{(2n+2)(4n+5)} \right| = 0 < 1 \quad \forall x.$$

Therefore the domain of convergence is $(-\infty, \infty)$

4)Given the same differential equation $2xy'' + 3y' + xy = 0$ as above, **show that** $x = 0$ is a **regular singular point**. Find a second (linearly independent) solution of this equation near $x = 0$ by the series method. Also discuss its domain of convergence.

Solution:

$$\lim_{n\to 0} x.\frac{3}{2x} = \frac{3}{2} < \infty \qquad \text{and} \lim_{n\to 0} x^2 \frac{x}{2x} = 0 < \infty \text{ i.e. } x = 0 \text{ is a regular singular point.}$$

Put $y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$

Put these into the equation,

$$(2r^2 + r)a_0 x^{r-1} + (1+r)(2r+3)a_1 x^r + \sum_{n=2}^{\infty}((n+r)(2n+2r+1)a_n + a_{n-2})x^{n+r-1}$$

$2r^2 + r = 0 \Rightarrow r_1 = 0, \quad r_2 = -1/2, \quad a_1 = 0$ and a_0 is arbitrary.

For $r_1 = 0 \Rightarrow a_n = -\frac{a_{n-2}}{n(2n+1)}$

$$y_1 = a_0(1 - \frac{x^2}{2.5} + \frac{x^4}{2.5.4.9} - \frac{x^6}{2.4.5.9.6.13}...)$$

For $r_2 = -1/2 \Rightarrow a_n = -\frac{a_{n-2}}{n(2n-1)}$

$$y_2 = a_0(1 - \frac{x^2}{2.3} + \frac{x^4}{2.3.4.7} - \frac{x^6}{2.3.4.7.6.11}...)$$

B U Department of Mathematics

Math 202 Differential Equations

Fall 2005 Second Midterm

1. By using the Method of Undetermined Coefficients find the solutions of the given initial value problem.

$$\mathbf{y''+2y'+2y}=\sin\mathbf{t};\ \ \mathbf{y(0)=1},\ \ \mathbf{y'(0)=0}.$$

Solution:

$r^2+2r+2=0 \Rightarrow r_{1,2}=-1\pm i$

Hence $y_h=e^{-t}(c_1\cos t+c_2\sin t)\quad c_1,c_2\in\mathbb{R}$

By the method of undetermined coefficients $y_p=A\sin t+B\cos t$.

Using diff. eqn., $A=\frac{1}{5},\ B=-\frac{2}{5}$.

So, $y_{gen}=e^{-t}(c_1\cos t+c_2\sin t)+\frac{1}{5}\sin t-\frac{2}{5}\cos t$

For $y(0)=c_1-\frac{2}{5}=1\Rightarrow c_1=\frac{7}{5}$

$y'_{gen}=-e^{-t}(c_1\cos t+c_2\sin t)+e^{-t}(-c_1\sin t+c_2\cos t)+\frac{1}{5}\cos t+\frac{2}{5}\sin t$

$y'(0)=-(c_1)+c_2+\frac{1}{5}=0\Rightarrow c_2=\frac{6}{5}$

Hence the solution

$$y=e^{-t}(\frac{7}{5}\cos t+\frac{6}{5}\sin t)-\frac{1}{5}\sin t-\frac{2}{5}\cos t$$

2. Find all singular points of $\mathbf{3x^2y'' + (x - x^2)y' - y = 0}$ and determine whether each one is regular or irregular. If possible, find the series solutions corresponding to $\mathbf{x_0 = 0}$, by finding the indicial equation, its roots and the recurrence relation. Also find the radius of convergence of these series solutions.

Solution:

$P(x) = 3x^2$ and $P(0) = 0$

$$\left.\begin{array}{lll} \lim_{x\to 0} x\dfrac{(x-x^2)}{3x^2} = \dfrac{1}{3}, & & xp(x) = \dfrac{1-x}{3} \\ & \& & \\ \lim_{x\to 0} x^2\dfrac{(-1)}{3x^2} = \dfrac{-1}{3}, & & x^2q(x) = \dfrac{-1}{3} \end{array}\right\} \text{analytic at } x_0 = 0.$$

Hence $x_0 = 0$ is a regular singular point.

No other singular point.

Let $y = \sum_{n=0}^{\infty} a_n x^{r+n}$ be a serial solution about $x_0 = 0$.

Then $y' = \sum_{n=0}^{\infty}(r+n)a_n x^{r+n-1}$ and $y'' = \sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-2}$

Using diff. eqn.

$$\sum_{n=0}^{\infty} 3(r+n)(r+n-1)a_n x^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n x^{r+n} - \sum_{n=1}^{\infty}(r+n-1)a_{n-1}x^{r+n} - \sum_{n=0}^{\infty} a_n x^{r+n} = 0$$

$\Rightarrow [3(r)(r-1) + r - 1]a_0 x^r = 0$ and

$$\sum_{n=1}^{\infty}\{[3(r+n)(r+n-1) + (r+n) - 1]a_n - (r+n-1)a_{n-1}\}x^{r+n} = 0$$

So, $I(r) = 3r^2 - 2r - 1 = 0$ (indicial equation)

and $r_1 = 1$, $r_2 = -\frac{1}{3}$

Recurrence relation: $a_n = \dfrac{r+n-1}{(r+n)(3r+3n-2)-1}a_{n-1}, \quad n \geq 1$

For $\underline{r_1 = 1}$: $a_n = \dfrac{n}{(n+1)(3n+1)-1}a_{n-1} = \dfrac{1}{3n+4}a_{n-1}, \quad n \geq 1$

$a_1 = \frac{1}{7}a_0, \quad a_2 = \frac{1}{10}a_1, \quad a_3 = \frac{1}{13}a_2, \quad \ldots$

$\Rightarrow a_1 \cdot a_2 \cdot \ldots \cdot a_n = \dfrac{1}{7}a_0 \dfrac{1}{10}a_1 \dfrac{1}{13}a_2 \dfrac{1}{3n+4}a_{n-1}$

$\Rightarrow a_n = \dfrac{1}{7 \cdot 10 \cdot 13 \cdot \ldots \cdot (3n+4)}a_0, \quad n \geq 1$

Hence $y_1 = x + x\sum_{1}^{\infty} \dfrac{1}{7 \cdot 10 \cdot 13 \cdot ... \cdot (3n+4)} x^n$ by taking $a_0 = 1$.

For $\underline{r_2 = -\frac{1}{3}}$: $a_n = \dfrac{1}{3n} a_{n-1}, \quad n \geq 1$

$a_1 = \frac{1}{3}a_0, \quad a_2 = \frac{1}{3 \cdot 2}a_1, \quad a_3 = \frac{1}{3 \cdot 3}a_2, \quad ... \ , \ a_n = \frac{1}{3n}a_{n-1}$

$\Rightarrow a_1 \cdot a_2 \cdot ... \cdot a_n = \dfrac{1}{3 \cdot 1}a_0 \dfrac{1}{3 \cdot 2}a_1 \dfrac{1}{3 \cdot 3}a_2 ... \dfrac{1}{3 \cdot n}a_{n-1}$

$\Rightarrow a_n = \frac{1}{3^n \cdot n!}a_0, \quad n \geq 1$ (true also for n=0)

Hence $y_2 = x^{\frac{-}{1}3} \sum_{0}^{\infty} \dfrac{1}{3^n \cdot n!} x^n$

Radius of convergence for both series solutions is ∞ by the Ratio Test.

(as $\rho = \lim\limits_{x \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = 0 < 1$)

3. If $\mathbf{m}$ is a positive constant, find that particular solution $\mathbf{y = f(x)}$ of the differential equation $\mathbf{y''' - my'' + m^2y' - m^3y = 0}$, which satisfies conditions $\mathbf{f(0) = f'(0) = 0, \ \ f''(0) = 1}$.

   Solution:

   $(D^3 - mD^2 + m^2D - m^3)y = 0$

   $[D^2(D - m) + m^2(D - m)]y = 0$

   $(D^2 + m^2)(D - m)y = 0$

   $\Rightarrow \{\cos mx, \sin mx, e^{mx}\}$ form a fundamental set of solutions.

   Hence the general solution:

   $y = ae^{mx} + b\cos mx + c\sin mx \Rightarrow y(0) = a + b = 0 \Rightarrow a + b = 0,$

   $y' = mae^{mx} - mb\sin mx + mc\cos mx \Rightarrow y'(0) = ma + mc = 0 \Rightarrow a + c = 0,$

   So, $b = c = -a$.

   Then $y' = ma(e^{mx} + \sin mx - \cos mx)$, $y'' = m^2a(e^{mx} + \cos mx + \sin mx)$.

   $y''(0) = 1$ and $y''(0) = m^2a(1 + 1) = 1 \Rightarrow a = \dfrac{1}{2m^2}$.

   Hence $y = f(x) = \dfrac{1}{2m^2}(e^{mx} - \cos mx - \sin mx)$ is the solution.

4. Solve the initial value problem,

$$\mathbf{y'' + 4y' + 4y} = \begin{cases} \mathbf{0}, & \mathbf{if\ t < 2,} \\ \mathbf{e^{t-2}}, & \mathbf{if\ t \geq 2} \end{cases}, \ \mathbf{y(0) = 1,\ y'(0) = 0}$$

Solution:

i.e. solve $y'' + 4y' + 4y = u_2(t)e^{t-2}$ satisfying $y(0) = 1$, $y'(0) = 0$ conditions.

By Laplace transform,

$s^2\mathfrak{L}[y] - sy(0) - y'(0) + 4[s\mathfrak{L}[y] - y(0)] + 4\mathfrak{L}[y] = \mathfrak{L}[u_2(t)e^{t-2}]$

For $g(t-2) = e^{t-2}$ and $g(t) = e^t$, by 2^{nd} Shifting Theorem, $\mathfrak{L}[u_2(t)e^{t-2}] = e^{-2s}\mathfrak{L}[e^t]$

$\Rightarrow (s^2 + 4s + 4)\mathfrak{L}[y] - s - 4 = e^{-2s}\mathfrak{L}[e^t]$

$$(s+2)^2\mathfrak{L}[y] = s + 4 + \frac{e^{-2s}}{s-1}$$

$$\mathfrak{L}[y] = \frac{1}{s+2} + \frac{2}{(s+2)^2} + \frac{e^{-2s}}{(s-1)(s+2)^2}$$

$$\mathfrak{L}[y] = \frac{1}{s+2} + \frac{2}{(s+2)^2} + e^{-2s}\left[\frac{1/9}{s-1} - \frac{1/9}{s+2} - \frac{1/3}{(s+2)^2}\right]$$

$$\Rightarrow y = e^{-2t} + 2te^{-2t} + u_2(t)\left[\frac{1}{9}e^t - \frac{1}{9}e^{-2t} - \frac{1}{3}te^{-2t}\right]$$

5. Find the inverse Laplace transform of $\ln(\mathbf{1} + \frac{\mathbf{4}}{\mathbf{s^2}})$.

Solution:

$$\frac{d}{ds}\ln\left(1 + \frac{4}{s^2}\right) = \frac{d}{ds}\left[\ln\left(s^2+4\right) - 2\ln s\right] = \frac{2s}{s^2+4} - \frac{2}{s}$$

$$\Rightarrow \mathfrak{L}^{-1}\left[\frac{d}{ds}\ln\left(1 + \frac{4}{s^2}\right)\right] = 2\cos 2t - 2 = (-1)t\mathfrak{L}^{-1}\left[\ln\left(1 + \frac{4}{s^2}\right)\right]$$

$$\text{So, } \mathfrak{L}^{-1}\left[\ln\left(1 + \frac{4}{s^2}\right)\right] = \frac{2}{t}(1 - \cos 2t)$$

Recall $\mathfrak{L}[tf(t)] = (-1)\frac{d}{ds}\mathfrak{L}[f(t)]$ or $\mathfrak{L}^{-1}\left[(-1)\frac{d}{ds}\varphi(s)\right] = t\mathfrak{L}^{-1}[\varphi(s)]$

B U Department of Mathematics

Math 202 Differential Equations

Date: May 14, 2004	Full Name	:
Time: 18:10-19:30	Math 202 Number	:
	Student ID	:

Spring 2004 Second Midterm – Solution Key

IMPORTANT

1. Write your name, surname on top of each page. **2.** The exam consists of 4 questions some of which have more than one part. **3.** Read the questions carefully and write your answers neatly under the corresponding questions. **4.** Show all your work. Correct answers without sufficient explanation might <u>not</u> get full credit. **5.** Calculators are <u>not</u> allowed.

Q1	Q2	Q3	Q4	TOTAL
25 pts	25 pts	30 pts	25 pts	105 pts

Some basic Laplace transforms you might need

$\mathcal{L}\{1\} = \dfrac{1}{s},\ s > 0$	$\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a},\ s > a$	$\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2},\ s > 0$
$\mathcal{L}\{\cos at\} = \dfrac{s}{s^2+a^2},\ s > 0$	$\mathcal{L}\{\sinh at\} = \dfrac{a}{s^2-a^2},\ s > \lvert a\rvert$	$\mathcal{L}\{\cosh at\} = \dfrac{s}{s^2-a^2},\ s > \lvert a\rvert$

1.)[25] Use the Laplace Transform to solve the initial value problem:

$$y'' - 2y' + 5y = g(t) \quad \text{where} \quad g(t) = \begin{cases} 0 & 0 \le t < 1 \\ e^t & 1 \le t \end{cases}$$

subject to the initial conditions $y(0) = y'(0) = 0$. Is the solution continuous at $t = 1$?

Solution:

Using the step function representation:

$$y'' - 2y' + 5y = u_1(t)e^t.$$

Taking the Laplace Transform of both sides we get:

$$\begin{aligned} &\mathcal{L}\{y'' - 2y' + 5y\} = \mathcal{L}\{u_1(t)e^t\} \\ \Rightarrow\ & s^2Y(s) - sy(0) - y'(0) - 2sY(s) + 2y(0) + 5Y(s) = e\mathcal{L}\{u_1(t)e^{t-1}\} = e\frac{e^{-s}}{s-1}. \end{aligned}$$

Now using the initial conditions and leaving $Y(s)$ alone on the left:

$$Y(s) = e\frac{e^{-s}}{(s-1)(s^2-2s+5)} = e\frac{e^{-s}}{(s-1)((s-1)^2+4)}.$$

We now expand this product into partial fractions so that we obtain a sum: let $s-1=p$ then:

$$\frac{1}{p(p^2+4)} = \frac{A}{p} + \frac{Bp+C}{p^2+4} \Rightarrow Ap^2+4A+Bp^2+Cp=1 \Rightarrow A=1/4,\ B=-1/4,\ C=0.$$

Then we have to invert the equality in s-domain:

$$Y(s) = e\frac{e^{-s}}{4}\underbrace{\left(\frac{1}{s-1} - \frac{s-1}{(s-1)^2+4}\right)}_{H(s)}.$$

Let us denote by $h(t)$ the inverse Laplace of $H(s)$. Then:

$$h(t) = e^t - e^t\cos 2t.$$

Hence $y(t) = (e/4)\mathcal{L}^{-1}\{e^{-s}H(s)\} = (e/4)u_1(t)h(t-1)$. Namely:

$$y(t) = \frac{e}{4}u_1(t)[e^{t-1} - e^{t-1}\cos 2(t-1)] = \frac{u_1(t)e^t}{4}[1-\cos 2(t-1)]$$

is the unique solution of the problem.

This solution is definitely continuous everywhere, by construction. Let us show this at $t=1$ (at other points there is no problem at all!):

(a) $\lim\limits_{t\to 1^-} y(t) = 0$ since $y(t)=0$ for $t<1$.

(b) $\lim\limits_{t\to 1^+} y(t) = \lim\limits_{t\to 1^+} e^t[1-\cos 2(t-1)]/4 = 0$ since $u_1(t)=1$ for $t\geq 1$.

(c) We conclude $\lim\limits_{t\to 1} y(t) = 0$.

(d) Finally $y(0) = 0 = \lim\limits_{t\to 1} y(t)$, hence $y(t)$ is continuous at $t=1$.

2.)[25] Find the solution to the following initial value problem:

$$y'' + 4y' + 4y = \delta(t - n\pi) + \sum_{k=1}^{10} \delta(t - k\pi)\sin t$$

where $y(0) = 0$ and $y'(0) = 2$. Find also $\lim_{n\to\infty} y(t)$.

Solution:

We simply take the Laplace transform of both sides:

$$s^2Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 4Y(s) = e^{-n\pi s} + \sum_{k=1}^{10} e^{-k\pi s}\sin k\pi.$$

Using the fact that $\sin k\pi = 0$ for every integer k and the given initial values we receive:

$$(s^2 + 4s + 4)Y(s) - 2 = e^{-n\pi s} \Leftrightarrow Y(s) = \frac{e^{-n\pi s}}{(s+2)^2} + \frac{2}{(s+2)^2}.$$

We now recall the rule: $\mathfrak{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$ and $\dfrac{1}{(s+2)^2} = -\left(\dfrac{1}{s+2}\right)'$. If we set $h(t) = \mathfrak{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} = te^{-2t}$ then easily:

$$y(t) = u_{n\pi}(t)h(t - n\pi) + 2h(t) = u_{n\pi}(t)(t - n\pi)e^{-2(t-n\pi)} + 2te^{-2t}$$

becomes the unique solution of the problem.

Now since we are interested in the case as $n \to \infty$ we fix t. Since t is fixed and n grows to infinity, obviously t will become less than $n\pi$ eventually. This is, for large enough n:

$$t < n\pi \Rightarrow u_{n\pi}(t) \equiv 0.$$

But this is equivalent to saying:

$$\lim_{n\to\infty} y(t) = 2te^{-2t}.$$

3.) Consider the Chebyshev equation: $(1-x^2)y'' - xy' + \alpha^2 y = 0$, where α is a constant.
(a)[15] Determine two linearly independent solutions in powers of x and the region of validity of these solutions.

Solution:

In powers of x means we are required to expand the solution as a series about $x = 0$. Clearly $x = 0$ is an ordinary point since $1 - x^2 = 0 \Rightarrow x = \pm 1$ are the only singularities. Furthermore we can without finding the solution decide that the solutions are valid (at least) in the region which enlarges starting from $x = 0$ to the singularities. That is for $|x| < 1$ the solutions will be valid. Now we find the fundamental solutions.

We set $y = \sum_{n=0}^{\infty} a_n x^n$ and write its consequences:

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

and substitute them in the DE:

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + \alpha^2 \sum_{n=0}^{\infty} a_n x^n = 0.$$

Expanding the parentheses:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + \alpha^2 \sum_{n=0}^{\infty} a_n x^n = 0.$$

Transforming the first sum into a sum of x^n terms:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n,$$

and rewriting by separating $n = 0$ and $n = 1$ terms:

$$(2\cdot 1 a_2 + \alpha^2 a_0)x^0 + (3\cdot 2 a_3 - a_1 + \alpha^2 a_1)x^1 + \sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2} - (n(n-1) + n - \alpha^2)a_n]x^n = 0.$$

Thus we obtain the recurrence relation to be:

$$a_2 = -\frac{\alpha^2}{2}a_0, \quad a_3 = \frac{1-\alpha^2}{6}a_1$$

$$\text{for } n \geq 2: \quad a_{n+2} = \frac{n^2 - \alpha^2}{(n+2)(n+1)}a_n.$$

We can naturally separate even and odd indexed coefficients:

$$\begin{aligned} a_{2n} &= \frac{(2n-2)^2 - \alpha^2}{(2n(2n-1)}a_{2n-2} = \frac{[(2n-2)^2 - \alpha^2][(2n-4)^2 - \alpha^2]}{2n(2n-1)(2n-2)(2n-3)}a_{2n-4} = \cdots \\ &= \frac{[(2n-2)^2 - \alpha^2][(2n-4)^2 - \alpha^2]\cdots[2-\alpha^2][-\alpha^2]}{(2n)!}a_0. \end{aligned}$$

Likewise (without even recomputing):

$$a_{2n+1} = \frac{[(2n-1)^2 - \alpha^2][(2n-3)^2 - \alpha^2]\cdots[3-\alpha^2][1-\alpha^2]}{(2n+1)!} a_1.$$

We are now ready to write down the fundamental solutions y_1 obtained by $a_0 = 1, a_1 = 0$ and y_2 obtained by $a_0 = 0, a_1 = 1$:

$$\begin{aligned} y_1 &= 1 + \sum_{n=1}^{\infty} \frac{[(2n-2)^2 - \alpha^2][(2n-4)^2 - \alpha^2]\cdots[2-\alpha^2][-\alpha^2]}{(2n)!} x^{2n} \\ y_2 &= x + \sum_{n=1}^{\infty} \frac{[(2n-1)^2 - \alpha^2][(2n-3)^2 - \alpha^2]\cdots[3-\alpha^2][1-\alpha^2]}{(2n+1)!} x^{2n+1}. \end{aligned}$$

(b)[15] Show that if α is a non-negative integer k, then one of the fundamental solutions found in part **(a)** becomes a polynomial of degree k. These are called *Chebyshev polynomials*. Find Chebyshev polynomials for $k = 1, 2$.

Solution:

From the recurrence relation we observe that certain values of α make the product zero.

Let $\alpha = k = 2m$, i.e. an even non-negative integer. Then:

$$a_{2m+2} = \frac{(2m)^2 - \alpha^2}{(2m+2)(2m+1)} a_{2m} = 0.$$

But this entails immediately $a_{2m+4} = a_{2m+6} = \cdots = 0$ meaning that y_1 for $\alpha = k = 2m$ contains terms up to $(2m)$th degree terms. Thus y_1 is a polynomial of degree $k = 2m$.

Let now $\alpha = k = 2m+1$, i.e. a odd non-negative integer. Then:

$$a_{2m+3} = \frac{(2m+1)^2 - \alpha^2}{(2m+3)(2m+2)} a_{2m+1} = 0.$$

Similarly we get $a_{2m+5} = a_{2m+7} = \cdots = 0$ meaning that y_2 for $\alpha = k = 2m+1$ contains terms up to $(2m+1)$st degree terms. Thus y_2 is a polynomial of degree $k = 2m+1$.

If $k = 1$ then $y_2 = x$. Hence the first Chebyshev polynomial is $C_1 = x$.

If $k = 2$ then $y_1 = 1 - \frac{2^2}{2!}x^2$. Hence the second Chebyshev polynomial is $C_2 = 1 - 2x^2$.

4.)[25] Consider the following differential equation for $\alpha \in \mathbb{R}$:

$$xu'' + (x-1)u' - \alpha u = 0.$$

Determine <u>the two</u> values of α so that the solutions to this differential equation around $x = 0$ have no logarithmic part [Write everything explicitly that leads you to find these values. Full justification is very important].

Solution:

First we notice that $x = 0$ is a singular point. To determine whether regular or irregular we check the limits:

$$\lim_{x\to 0} x\frac{x-1}{x} = -1 \quad \text{and} \quad \lim_{x\to 0} -x^2\frac{\alpha}{x} = 0,$$

both being finite we conclude that $x = 0$ is a regular singular point.

We can write a series solution about $x = 0$ in the form: $u = \sum_{n=0}^{\infty} a_n x^{n+r}$ and its derivatives:

$$u' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}, \quad u'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}.$$

Plugging them into the DE we get:

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - \alpha\sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

Making each summand a sum in terms of x^{n+r}:

$$\sum_{n=-1}^{\infty}(n+r+1)(n+r)a_{n+1}x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} - \sum_{n=-1}^{\infty}(n+r+1)a_{n+1}x^{n+r} - \alpha\sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

This quadruple sum can be merged into a single sum starting from $n = 0$ as:

$$a_0[r(r-1) - r]x^{r-1} + \sum_{n=0}^{\infty}[a_{n+1}(n+r+1)(n+r-1) + a_n(n+r-\alpha)]x^{n+r} = 0.$$

At the coefficient of x^{r-1} we obtain the indicial equation:

$$r(r-1) - r = 0 \Rightarrow r(r-2) = 0 \Rightarrow r = 0, 2.$$

Since the roots of the indicial equation differ by an integer it is not guaranteed that both solutions are free of the logarithmic term.

Still we know that for the bigger root $r = 2$ the solution is regular, i.e. does not contain a logarithm. Hence we have to deal with the smaller root $r = 0$.

The recurrence relation gives:

$$(n+r+1)(n+r-1)a_{n+1} = -(n+r-\alpha)a_n.$$

Setting here $r = 0$ we expect the inconsistency may occur when calculating $a_{0+2} = a_2$ (index is simply the greater root 2). Now:

$$n = 0 \Rightarrow 1 \cdot (-1)a_1 = \alpha a_0 \Rightarrow a_1 = -\alpha a_0.$$

Next we compute a_2:

$$n = 1 \Rightarrow 2 \cdot 0 a_2 = -(1-\alpha)a_1 = (1-\alpha)\alpha a_0.$$

This is obviously a consistency condition. Since $a_0 \neq 0$:

$$\alpha(1-\alpha)a_0 = 0 \Rightarrow \alpha = 0 \quad \text{or} \quad \alpha = 1.$$

If α is chosen to be either 0 or 1 then the solution for $r = 0$ does not contain a logarithm either.

B U Department of Mathematics

Math 202 Differential Equations

Spring 2005 Second Midterm

1. Use variation of parameters to find the **general** solution to

$$x^2y'' + 3xy' + y = \frac{1}{x} \qquad \text{for } x > 0.$$

Solution:

The corresponding homogenous equation $x^2y'' + 3xy' + y = 0$ is Euler's equation. The indicial equation is

$$r^2 + (3-1)r + 1 = r^2 + 2r + 1 = (r+1)^2 = 0.$$

So we have equal roots $r_1 = r_2 = -1$. Hence,

$$y_1(x) = x^{-1}, y_2(x) = x^{-1}\ln x$$

form a fundamental set of solutions.

Now by variation of parameters, we assume

$$y(x) = c_1(x)y_1 + c_2(x)y_2$$

is a particular solution. We insert y in the given differential equation. Assuming, as usual, that

$$c_1'y_1 + c_2'y_2 = 0$$

we obtain as the second identity:

$$x^2(c_1'y_1' + c_2'y_2') = \frac{1}{x};$$

that is, we get:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ x^{-3} \end{bmatrix}.$$

Therefore,

$$c_1' = -\frac{y_2/x^3}{W(y_1,y_2)} = -x^{-3}x^{-1}\ln x\cdot\begin{vmatrix} x^{-1} & x^{-1}\ln x \\ -x^{-2} & -x^{-2}\ln x + x^{-2} \end{vmatrix}^{-1} = -x^{-3}x^{-1}\ln x\cdot x^3 = -\frac{\ln x}{x}.$$

$$c_2' = \frac{y_1/x^3}{W(y_1,y_2)} = x^{-3}x^{-1}x^3 = x^{-1}.$$

Finally,

$$c_1 = -\int \frac{\ln x}{x}dx = -\int \ln x\cdot(\ln x)'dx = -\frac{1}{2}(\ln x)^2 \text{ and}$$

$$c_2 = \int x^{-1}dx = \ln x.$$

Hence, $y_p = -\frac{1}{2}(\ln x)^2x^{-1} + \ln x(x^{-1}\ln x) = \dfrac{(\ln x)^2}{2x}.$

2. Consider the piecewise continuous function

$$f(t) = \begin{cases} t^2 & , \quad 0 \le t < 3 \\ 9 & , \quad t \ge 3 \end{cases}$$

(a) Show that f is of exponential order.
(b) Express f in terms of the unit step function.
(c) Find Laplace transform of f and determine the allowed values for s.

Solution:

(a) Since $f(t) \le 9e^{0 \cdot t} = 9$ for all t, $f(t)$ is of exponential order.

(b) Think off f in two parts: $t \ge 3$ and $t < 3$. In this way we see that

$$f(t) = t^2(1 - u_3(t)) + 9u_3(t) = t^2 + (9 - t^2)u_3(t).$$

(c) Use the fact that $\mathfrak{L}(u_3(t)g(t-3)) = e^{-3s}G(s)$. For this, we put the expression for f in an appropriate form:

$$\begin{aligned} f(t) = t^2 + (9 - t^2)u_3(t) &= t^2 - \big((t-3)^2 + 6t - 18\big)u_3(t) \\ &= t^2 - (t-3)^2 u_3(t) - 6(t-3)u_3(t). \end{aligned}$$

Then

$$\mathfrak{L}(f)(s) = \frac{2}{s^3} - \frac{2e^{-3s}}{s^3} - \frac{6e^{-3s}}{s^2}.$$

Here $s > 0$.

3. Find all singular points of

$$2xy'' + y' + y = 0$$

and determine whether each one is regular or irregular. If possible, find the series solution corresponding to these points by finding the indicial equation, its roots and the recurrence relation.

Solution:

Let $P(x) = 2x$, $Q(x) = R(x) = 1$. Observe $P(x) = 0$ only if $x = 0$. Since $\lim_{x \to 0} x\frac{1}{2x} = \frac{1}{2}$ and $\lim_{x \to 0} x^2 \frac{1}{2x} = 0$ (both finite) then $x = 0$ is a regular singular point of the differential equation. Assume a solution of the form $y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}$. Then

$$y' = \sum_{n=0}^{\infty} (r+n)a_n x^{r+n-1}; y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n-2}.$$

Insert these in the differential equation:

$$\begin{aligned} 0 = 2xy'' + y' + y &= 2\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-1} + \sum_{n=0}^{\infty}(r+n)a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} \\ &= 2r(r-1)a_0 x^{r-1} + 2\sum_{n=1}^{\infty}(r+n)(r+n-1)a_n x^{r+n-1} \\ &\quad + ra_0 x^{r-1} + \sum_{n=1}^{\infty}(r+n)a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} \\ &= (2r(r-1) + r)a_0 x^{r-1} \\ &\quad + \sum_{n=0}^{\infty}\Big(2(r+n)(r+n+1)a_{n+1} + (r+n+1)a_{n+1} + a_n\Big)x^{r+n} \end{aligned}$$

Requiring each coefficient to be 0, we get:

$$0 = 2r^2 - r = r(2r-1) \text{ so that } r_1 = 0, r_2 = \frac{1}{2}$$
$$a_{n+1} = -\Big((r+n+1)(2r+2n+1)\Big)^{-1} a_n.$$

Now first put $r = r_1 = 0$. The recurrence relation is

$$a_{n+1} = -\frac{a_n}{(n+1)(2n+1)}$$

and the general term becomes

$$a_n = (-1)^n \frac{a_0}{2\cdot 3 \cdots n \ \cdot \ 3\cdot 5 \cdots (2n-1)} = (-1)^n \frac{2^n n! a_0}{n!(2n)!} = (-1)^n \frac{2^n a_0}{(2n)!}.$$

Similarly, put $r = r_2 = \frac{1}{2}$. We have the recurrence relation

$$a_{n+1} = -\frac{a_n}{(n+\frac{3}{2})(2n+2)} = -\frac{a_n}{(2n+3)(n+1)}$$

and the general term becomes

$$a_n = (-1)^n \frac{a_0}{3\cdot 5\cdot 7 \cdots (2n+1) \ \cdot \ 2\cdot 3 \cdots n} = (-1)^n \frac{\cdot 2^n n! a_0}{(2n+1)!n!} = (-1)^n \frac{\cdot 2^n a_0}{(2n+1)!}.$$

As a result we have a series solution

$$y(t) = c_1 \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{(2n)!} x^n + c_2 x^{\frac{1}{2}} \sum_{n=0}^{\infty} (-1)^n \frac{\cdot 2^n}{(2n+1)!}.$$

4. Suppose that $f(t) = t^3$ is a solution of the differential equation

$$P(t)y'' + Q(t)y' + R(t)y = 0$$

where $P(t)$, $Q(t)$ and $R(t)$ are continuous functions that are defined everywhere. Show that $P(0) = 0$.

(*Hint: Note that $P(0) = 0$ means the differential equation has a singular point at $t = 0$. Now remember the uniqueness and existence theorem for some initial value problems and question 5(b) of Midterm 1.*)

Solution:

Observe that $f(t) = t^3$ is a solution for the initial value problem

$$P(t)y'' + Q(t)y' + R(t)y = 0, y(0) = 0, y'(0) = 0$$

i.e. $f(0) = 0$ and $f'(0) = 0$. But $g(t) = 0$ is also a solution for this initial value problem. By Theorem, the solution is unique for such a problem provided that $\dfrac{Q(t)}{P(t)}$ and $\dfrac{R(t)}{P(t)}$ are continuous at $t = 0$. Since the solution is not unique (t^3 and 0 are distinct functions), 0 must be a singular point of the differential equation; in other words $P(0) = 0$.

5. Find the inverse Laplace transform of $F(s) = \dfrac{e^{-2s}}{s^2+2s+2}$ and determine the allowed values for s.

Solution:

$$\mathfrak{L}^{-1}\left(\frac{e^{-2s}}{s^2+2s+2}\right) = u_2(t)f(t-2)$$

where

$$f(t) = \mathfrak{L}^{-1}\left(\frac{1}{s^2+2s+2}\right) = \mathfrak{L}^{-1}\left(\frac{1}{(s+1)^2+1}\right) = e^{-t}\sin t, \qquad s > -1.$$

Hence we get

$$\mathfrak{L}^{-1}(F(s)) = u_2(t)e^{-(t-2)}\sin(t-2), \qquad s > -1.$$

B U Department of Mathematics

Math 202 Differential Equations

Summer 2000 Second Midterm

1. Using the method of Laplace transforms solve the initial value problem:

$$y'' - 2y' = -4, \quad y(0) = 0, \quad y'(0) = 0$$

(No credit for other approaches)

Solution:

$$\pounds[y''] = s^2\pounds[y] - sy(0) - y'(0) = s^2\pounds[y] - 4$$
$$\pounds[y'] = s\pounds[y] - y(0) = s\pounds[y].$$
$$\text{DE} \Rightarrow (s^2 - 2s)\pounds[y] = 4\left(1 - \frac{1}{s}\right),$$
$$\pounds[y] = \frac{4(s-1)}{s^2(s-2)} = -\frac{1}{s} + \frac{2}{s^2} + \frac{1}{s-2}$$
$$y = \pounds^{-1}\left[-\frac{1}{s} + \frac{2}{s^2} + \frac{1}{s-2}\right]$$
$$y = -1 + 2t + e^{2t}.$$

2. **(a)** Locate and classify all singular points of the equation

$$2x^3(1-x)y'' - x(1+7x)y' + Y = 0$$

(Do not consider the point at infinity)

Solution:

$p(x) = \dfrac{-(1+7x)}{2x^2(1-x)}, \qquad q(x) = \dfrac{1}{2x^3(1-x)}$

$\Rightarrow \quad x = 0, \quad x = 1$ are singular points.

$xp(x), \;\; x^2q(x)$ are not analytic if $x = 0, \;\; (x-1)p(x), \;\; (x-1)^2q(x)$ are analytic.

$\Rightarrow \quad x = 0$ is an irregular singular point,

$x = 1$ is a regular singular point.

(b) Find the general solution of

$$y'' = \frac{6}{x^2}y$$

.

Solution:

This is a simple Euler equation.

Let $y = x^\alpha$

DE: $\quad \alpha(\alpha-1)x^{\alpha-2} = 6x^{\alpha-2}$

$\alpha^2 - \alpha - 6 = 0 \quad \Rightarrow \quad (\alpha-3)(\alpha+2) = 0$

$$\Rightarrow \alpha_1 = -2, \quad \alpha_2 = 3$$

$\Rightarrow \quad y_1 = x^{-2}, \;\; y_2 = x^3$ are solutions and $\{y_1, y_2\}$ is fundamental set.

General solution: $y = c_1x^{-2} + c_2x^3 \quad c_1, c_2 \in \mathbb{R}$.

3. **(a)** Find the Laplace transform of the function:

$$f(t) = e^t \cos^2 wt.$$

Solution:

$\cos^2 wt = \dfrac{1}{2}\,(1 + \cos 2wt)$

$F(s) = \pounds[\cos^2 wt] = \dfrac{1}{2s} + \dfrac{s}{2(s^2 + 4w^2)}$

By first shifting theorem,

$$\begin{aligned}\pounds[f(t)] &= F(s-1) \\ &= \frac{1}{2(s-1)} + \frac{s-1}{2[(s-1)^2 + 4w^2]}.\end{aligned}$$

(b) Find the Laplace transform of the function:

$$F(s) = \frac{1}{s^3 + 4s}.$$

Solution:

$F(s) = \dfrac{1}{s^3 + 4s} = \dfrac{1}{s}\dfrac{1}{s^2 + 4} = \dfrac{1}{2s}\pounds[\sin t] = \dfrac{1}{2}\pounds\left[\displaystyle\int_0^t \sin 2u du\right]$

$f(t) = \pounds^{-1}[F(s)] = \dfrac{1}{2}\displaystyle\int_0^t \sin 2u du = -\dfrac{1}{4}\cos 2u|_0^t = \dfrac{1}{4}(1 - \cos 2t).$

4. **(a)** Find a solution of the differential equation:

$$xy'' + (1-2x)y' + (x-1)y = 0,$$

about the point x = 0 by the method of power series. Determine the radius of convergence and identify the function represented by this power series. Discuss the nature of general solution but do not construct it. (Extra 5 points will be given if you determine the general solution.)

Solution:

$x = 0$ is a regular singular point.

Let $\quad y = \sum_{n=0}^{\infty} c_n x^{n+\alpha}$

DE: $\quad \alpha^2 c_0 x^{\alpha-1} + [(\alpha+1)^2 c_1 - (2\alpha+1)c_0]x^{\alpha}$

$$+\sum_{k=1}^{\infty}[(k+\alpha+1)^2 c_{k+1} - (2k+2\alpha+1)c_k + c_{k-1}]x^{k+\alpha} = 0$$

$\Rightarrow \quad$ Indicial equation $\alpha^2 = 0$ with exponents $\alpha_1 = \alpha_2 = 0$.

Also $c_1 = c_0$ and $(k+1)^2 c_{k+1} = (2k+1)c_k - c_{k-1}, \qquad k = 1, 2, 3, \ldots$

$\Rightarrow \quad 2^2 c_2 = 3c_1 - c_0 = 2c_0 \quad \Rightarrow \quad c_2 = \frac{1}{2}c_0$

& $\quad 3^2 c_3 = 5c_2 - c_1 \quad \Rightarrow \quad c_3 = \frac{1}{2.3}c_0$

& $\quad 4^2 c_4 = 7c_3 - c_2 \quad \Rightarrow \quad c_4 = \frac{1}{2.3.4}c_0$

$\Rightarrow \quad c_n = \frac{1}{n!}c_0 \qquad n = 1, 2, 3\ldots$

$y = c_0 \sum_{n=0}^{\infty} \frac{1}{n!}x^n = c_0 e^x, \qquad R = \infty.$

Since $\alpha_1 = \alpha_2$ a second solution will have the form

$y_2 = y_1 \ln x + \sum_{n=1}^{\infty} b_n x^n.$

In this problem $b_n = 0$.

Therefore, general solution is

$$y = a_1 e^x + a_2 e^x \ln x \qquad (a_1, a_2 \in \mathbb{R})$$

B U Department of Mathematics

Math 202 Differential Equations

Summer 2001 Second Midterm

1. Find the general solution of the forth order equation:

$$y^{iv} - 4y'' = 16x^2 - 6e^x.$$

Solution:

DE: $D^2(D^2-4)y = 16x^2 - 6e^x$

$\Rightarrow \; y_H = c_1 + c_2 x + c_3 e^{2x} + c_4 e^{-2x}$

DE $\Rightarrow \; D^5(D-1)(D^2-4)y = 0$
$\Rightarrow \; y_p = Ax^2 + Bx^3 + Cx^4 + De^x$ (method of undetermined coefficients)

$\Rightarrow y_p'' = 2A + 6Bx + 12Cx^2 + De^x$

$y_p^{iv} = 24C + De^x.$

DE $\Rightarrow 24C - 8A - 24Bx - 48Cx^2 - 3De^x = 16x^2 - 6e^x \quad \Rightarrow 3C = A, B = 0, C = -\frac{1}{3}, D = 2, A = -1.$

Therefore, $y_p = 2e^x - x^2 - \frac{1}{3}x^4$.

General solution $y = y_H + y_p$
$y = c_1 + c_2 x + c_3 e^{2x} + c_4 e^{-2x} + 2e^x - x^2 - \frac{1}{3}x^4.$ $(c_1, c_2, c_3, c_4 \in \mathbb{R})$

2. Find the two linearly independent power series solutions of

$$(1 + x^2)y'' + 2xy' - 2y = 0,$$

about the point $x = 0$. In each of the solutions determine the general term and the radius of convergence of the series.

Solution:

$x = 0$ is an arbitrary point: $y = \sum\limits_{n=0}^{\infty} c_n x^n$

DE $\Rightarrow$ $\sum\limits_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum\limits_{n=2}^{\infty} n(n-1)c_n x^n + \sum\limits_{n=1}^{\infty} 2nc_n x^n - \sum\limits_{n=0}^{\infty} 2c_n x^n = 0$

$\Rightarrow 2(c_2 - c_0) + 6c_3 x + \sum\limits_{k=2}^{\infty} (k+2)[(k+1)c_{k+2} + (k-1)c_k]x^k = 0$

$\Rightarrow$ $c_2 = c_0$, $c_3 = 0$, $(k+1)c_{k+2} = -(k-1)c_k$, $k = 2, 3, \ldots$, c_1 is arbitrary, $c_{k+2} = -\frac{(k-1)}{(k+1)}c_k$, $k = 2, 3, \ldots$.

Since $c_3 = 0, c_{2n+1} = 0$ for $n = 1, 2, 3, \ldots$

$c_4 = -\frac{1}{3}c_2 = -\frac{1}{3}c_0$, $c_6 = -\frac{3}{5}c_4 = \frac{1}{5}c_2 = \frac{1}{5}c_0$

$c_8 = -\frac{5}{7}c_6 = -\frac{1}{7}c_0$, $c_{10} = -\frac{7}{9}c_8 = \frac{1}{9}c_0$

$\Rightarrow$ $c_{2n} = \frac{(-1)^{n+1}}{2n-1}c_0$, $n = 0, 1, 2, \ldots$

Therefore, $y = c_0 \sum\limits_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n-1} x^{2n} + c_1 x$

$y_1 = \sum\limits_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n-1} x^{2n}$ converges for $|x| \leq 1$

$y_2 = x$, converges for $|x| < \infty$. $(R_1 = 1, R_2 = \infty)$

y_1 series can, in fact, summed $y_1 = 1 + xtan^{-1}x$.

3. **(a)** Let $f(t)$ be piecewise continuous and of exponential order and $F(s) = \mathfrak{L}[f(t)]$. Suppose that limit of $f(t)/t$ as $t \to 0^+$ exists. Show that:

$$\mathfrak{L}[f(t)/t] = \int_s^\infty F(r)dr.$$

Solution:

Let $F(s) = \mathfrak{L}[f(t)]$, $G(s) = \mathfrak{L}[g(t)]$ with $F = \dfrac{dG}{ds}$

$\Rightarrow G(s) = \displaystyle\int_\infty^{s_0} F(r)dr.\ \dfrac{dG}{ds} = \mathfrak{L}[f(t)] \Rightarrow g(t) = \dfrac{-f(t)}{t}$

Therefore, $\mathfrak{L}\left[\dfrac{1}{t}f(t)\right] = -\displaystyle\int_{s_0}^{s} F(r)dr = \int_s^{s_0} F(r)dr$

Choosing $s_0 = \infty$ gives

$$\mathfrak{L}[f(t)/t] = \int_s^\infty F(r)dr$$

(b) Find the Laplace transform of the function: $f(t) = \dfrac{\cos wt - 1}{t}$, where w is a positive real constant.

Solution:

According to part **(a)**, $\mathfrak{L}\left[\dfrac{\cos wt - 1}{t}\right] = \displaystyle\int_s^\infty F(r)dr$, where $F(s) = \mathfrak{L}[\cos wt - 1] = \dfrac{s}{s^2+w^2} - \dfrac{1}{s}$.

$\displaystyle\int_s^\infty F(r)dr = \int_s^\infty \left(\frac{r}{r^2+w^2} - \frac{1}{r}\right) dr = \frac{1}{2}\left[\ln r^2 + w^2 - \ln r^2\right]_0^\infty$

$= \dfrac{1}{2}\ln\left(\dfrac{r^2+w^2}{r^2}\right)\Big|_s^\infty$

$= -\dfrac{1}{2}\ln\left(\dfrac{s^2+w^2}{s^2}\right) = \dfrac{1}{2}\ln\left(\dfrac{s^2}{s^2+w^2}\right)$

Therefore

$$\mathfrak{L}\left[\frac{\cos wt - 1}{t}\right] = \frac{1}{2}\ln\left(\frac{s^2}{s^2+w^2}\right)$$

4. **a.** Find an inverse Laplace transform of the function:

$$F(s) = \frac{e^{-7s}}{s^2 - 4s + 5}.$$

b. Find the general solution of $x^2y'' - 5xy' + 9y = 0, x > 0$.

Solution:

a. $\mathcal{L}[F(s)] = U_7(t)f(t-7)$ where $\mathcal{L}[f(t] = \dfrac{1}{s^2 - 4s + 5}$, (second shifting theorem)
$\mathcal{L}[f(t)] = \dfrac{1}{(s-2)^2 + 1}$, (first shifting theorem)
$\Rightarrow\ f(t) = e^{2t} sint$. Therefore,

$$\mathcal{L}^{-1}[F(s)] = U_7(t)e^{2(t-7)} sin(t-7).$$

b. This is an Euler equation. Letting $u = \ln x$ gives

$$\frac{d^2y}{du^2} - 6\frac{dy}{du} + 9y = 0.$$

Characteristic equation: $r^2 - 6r + 9 = (r-3)^2 = 0$ implies
$\{e^{3u}, ue^{3u}\}$ is a fundamental set.
$y = c_1e^{3u} + c_2ue^{3u}$ is the general solution. Therefore,

$$y = c_1x^3 + c_3x^3 \ln x$$

is the general solution of the above Euler equation.

B U Department of Mathematics

Math 202 Differential Equations

Summer 2002 Second Midterm

1. Using the method of variation of parameters find the general solution of

$$x^2y'' - 2xy' + 2y = x^3 \ln x, \ (x > 0)$$

Solution:

Homogeneous solutions: $y_1 = x$, $y_2 = x^2$ (Euler)

Let $y = A(x)x + B(x)x^2$.

$A'y_1 + B'y_2 = xA' + x^2B' = 0$

$A'y_1' + B'y_2' = A' + 2xB' = x \ln x$

$\Rightarrow B' = \ln x$, $A' = -xB' = -x \ln x$.

Integrations by parts give:

$$\begin{aligned} B &= x\ln x - x + c_2 \\ A &= \frac{x^2}{4} - \frac{x^2}{2}\ln x + c_1, \ (c_1, c_2 \in \mathbb{R}) \end{aligned}$$

$y = c_1x + c_2x^2 + \dfrac{x^3}{2}\ln x - \dfrac{3x^3}{4}$. (general soln.)

2. Given the differential equation

$$xy'' + 2y' + xy = 0$$

(a) Show that $x = 0$ is a regular singular point and determine the exponents at $x = 0$.

(b) Find the Frobenius series solution about the point x=0 which obeys the initial conditions: $y(0) = 1$, $y'(0) = 0$. Determine the radius of convergence and identify the function which is represented by this series.

Solution:

$$p(x) = \frac{2}{x},\ q(x) = 1 \Rightarrow xp(x) = 2,\ x^2q(x) = x^2, \text{ both analytic at } x = 0.$$

Thus $x = 0$ is a regular singular point.

Let $y = \sum_{n=0}^{\infty} c_n x^{n+r}$. Then the differential equation becomes:

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r+1} = 0$$

$$\Rightarrow r(r+1)c_0x^{r-1} + (r+1)(r+2)c_1x^r + \sum_{k=1}^{\infty}[(k+r+1)(k+r+2)c_{k+1} + c_{k-1}]x^{k+r} = 0$$

$c_0 \neq 0$; $r(r+1) = 0 \Rightarrow$ Exponents $r_1 = 0$, $r_2 = -1$.

Clearly, $r_2 = -1$ does not satisfy the initial conditions.

Let $r = r_1 = 0$. Then $c_1 = 0$ and $c_{k+1} = \dfrac{-1}{(k+1)(k+2)}c_{k-1}$ for $k = 1, 2, 3...$

$c_2 = \dfrac{-1}{2(3)}c_0$, $c_3 = 0$, $c_n = 0$ for n odd.

$c_4 = \dfrac{+1}{2(3)(4)(5)}c_0$, etc. Initial condition $\Rightarrow c_0 = 1$.

$$y = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!} = \frac{sinx}{x},\quad R = \infty.$$

3. Using Laplace transforms solve the initial value problem:

$$y'' - 4y' + 4y = 4e^{2t}, \quad y(0) = -1, \quad y'(0) = -4.$$

(No credit will be given for other approaches.)

Solution:

Let $Y(s) = \mathcal{L}[y(t)]$.

$\mathcal{L}[y''] = s^2 Y + s + 4, \quad \mathcal{L}[y'] = sY + 1.$

DE $\Rightarrow (s^2 - 4s + 4)Y(s) + s + 4 - 4 = \dfrac{4}{s-2}$

$(s-2)^2 Y(s) = \dfrac{4}{s-2} - s$

$$\begin{aligned} Y(s) &= \frac{4}{(s-2)^3} - \frac{s}{(s-2)^2} \\ &= \frac{4}{(s-2)^3} - \frac{2}{(s-2)^2} - \frac{1}{s-2} \end{aligned}$$

$y(t) = \mathcal{L}^{-1}[Y(s)]$

$$y(t) = 2t^2 e^{2t} - 2te^{2t} - e^{2t}.$$

4. (a) Find the Laplace transform $F(s)$ of the piecewise continuous function:

$$f(x) = \begin{cases} 1, & 0 \leq t < 1 \\ -3e^{-t}, & t \geq 1 \end{cases}$$

and determine the interval on which $F(s)$ is defined.

Solution:

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^1 e^{-st}dt - \int_1^\infty 3e^{-(s+1)t}dt$$

$$\int_0^1 e^{-st}dt = -\frac{1}{2}e^{-st}\Big|_0^1 = \frac{1-e^{-s}}{s}, \text{ if } s \neq 1$$

$$\int_1^\infty e^{-(s+1)t}dt = -\frac{1}{s+1}e^{(s+1)t}\Big|_1^\infty = \frac{e^{-(s+1)}}{s+1}, \quad s > -1$$

Therefore,

$$F(s) = \begin{cases} \dfrac{1-e^{-s}}{s} - \dfrac{3e^{-(s+1)}}{s+1} & , \text{ if } s > -1, \quad s \neq 0 \\ 1 - \dfrac{3}{e} & , \text{ if } s = 0 \end{cases}$$

(b) Determine the dimension and a basis for the null space of the linear differential operator:

$$L = (D^2 - 9I)(D^2 + 4I)^2(D^2 - 5D + 6I)$$

Solution:

$L = (D - 3I)^2(D + 3I)(D - 2I)(D^2 + 4I)^2$

Let $N(L)$ be the null space of L. L is an 8^{th} order operator; $dimN(L) = 8$.

A basis for $N(L)$ is

$$\{e^{3t}, te^{3t}, e^{-3t}, e^{2t}, \cos 2t, t\cos 2t, \sin 2t, t\sin 2t\}$$

B U Department of Mathematics

Math 202 Differential Equations

Summer 2003 Second Midterm

1. Using the method of variation of parameters find the general solution of

$$y'' + 3y' + 2y = cos(e^t)$$

Solution:

$\{y_1 = e^{-t},\ y_2 = e^{-2t}\}$ is a fundamental set of solutions.

Let $y = A(t)y_1 + B(t)y_2$ with $A'y_1 + B'y_2 = 0$ and $A'y_1' + B'y_2' = cos(e^t)$

$A' + e^{-t}B' = 0$ and $A' + 2e^{-t}B' = -e^t cos(e^t)$ implies

$A' = e^t cos(e^t)$ and $B' = -e^{2t} cos(e^t)$

$\Rightarrow \quad A = sin(e^t) + c_1,\ (c_1 \in \mathbb{R})$ and $B = -e^t sin(e^t) - cos(e^t) + c_2,\ (c_2 \in \mathbb{R})$

$\Rightarrow \quad y(t) = c_1 e^{-t} + c_2 e^{-2t} - e^{-2t} cos(e^t)$ is the general solution.

2. Given the differential equation

$$xy'' + 2y' - xy = 0,$$

show that $x = 0$ is a regular singular point and determine the exponents at $x = 0$.Starting from the smaller exponent find the two linearly independent Frobenius series solutions about the point $x = 0$.Determine the radii of convergence and identify the functions which are represented by the series.

Solution:

$p(x) = \dfrac{2}{x}, \;\; q(x) = -1, \;\; xp(x) = 2, \;\; x^2q(x) = -x^2$are both analytic.

$\Rightarrow \quad x = 0$ is a regular singular point since p(x) is not analytic there.

Hence we let $y = \sum_{n=0}^{\infty} c_n x^{n+\alpha}$

$$\text{DE} \;\Rightarrow\; \sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)c_n x^{n+\alpha-1} + \sum_{n=0}^{\infty} 2(n+\alpha)c_n x^{n+\alpha-1} - \sum_{n=0}^{\infty} c_n x^{n+\alpha+1} = 0$$

$$\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha+1)c_n x^{n+\alpha-1} - \sum_{n=0}^{\infty} c_n x^{n+\alpha+1} = 0$$

$$\sum_{k=-1}^{\infty}(k+\alpha+1)(k+\alpha+2)c_{k+1}x^{k+\alpha} - \sum_{k=1}^{\infty} c_{k-1}x^{k+\alpha} = 0$$

$$\Rightarrow \;\; \alpha(\alpha+1)c_0x^{\alpha-1} + (\alpha+1)(\alpha+2)c_1x^{\alpha} + \sum_{k=1}^{\infty}[(k+\alpha+1)(k+\alpha+2)c_{k+1} - c_{k-1}]x^{k+\alpha} = 0$$

Indicial Equation: $\alpha(\alpha+1) \;\; \Rightarrow \;\; \alpha_1 = 0, \;\; \alpha_2 = -1$

Recurrence Relation: $(k+\alpha+1)(k+\alpha+2)c_{k+1} = c_{k-1} \quad (k = 1, 2, 3)$

Let $\alpha = \alpha_2 = -1$. Then c_0, c_1 are arbitrary, and $k(k+1)c_{k+1} = c_{k-1}$

$$\Rightarrow \quad c_2 = \frac{1}{2}c_0, \;\; c_3 = \frac{1}{2.3}c_1, \;\; c_4 = \frac{1}{2.3.4}c_0, \;\; c_5 = \frac{1}{5!}c_1$$

$$c_{2n} = \frac{1}{(2n)!}c_0, \;\; c_{2n+1} = \frac{1}{(2n+1)!}c_1; \quad n = 1, 2, 3...$$

$$\Rightarrow \quad y = c_0y_1(x) + c_1y_2(x)$$

$$y_1(x) = \sum_{n=0}^{\infty}\frac{x^{2n-1}}{(2n)!} = \frac{1}{x}\,coshx, \quad R_1 = \infty \text{ and } y_2(x) = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n+1)!} = \frac{1}{x}\,sinhx,$$

$R_2 = \infty$

$y(x) = (c_0\;coshx + c_1\;sinhx)/x, \quad$ for$x > 0$

3. Using Laplace transforms solve the initial value problem:

$$y'' - 6y' + 9y = t^2 e^{3t}, \quad y(0) = 2, \quad y'(0) = 6$$

(No credit will be given for other approaches.)

Solution:

Let $Y(s) = L[y(t)]$, $L[y''] = s^2 Y(s) - 2s - 6$, $L[y'] = sY(s) - 2$

Taking the Laplace transform of the DE gives

$$(s^2 - 6s + 9)Y(s) - 2s + 6 = \frac{2}{(s-3)^2} = L[t^2 e^{3t}]$$

Therefore, $\quad (s-3)^2 Y(s) = 2(s-3) + \dfrac{2}{(s-3)^3}$

$$Y(s) = \frac{2}{s-3} + \frac{2}{(s-3)^5}$$

$$\frac{1}{(s-3)^5} = \frac{1}{24}\frac{d^4}{ds^4}\left(\frac{1}{s-3}\right) = \frac{1}{24}L[t^4 e^{3t}]$$

Therefore,

$$y(t) = L^{-1}[Y(s)] = 2e^{3t} + \frac{1}{12}t^4 e^{3t}$$

4. **a)** Find the general solution of $(D+3I)^4(D^2-10D+29I)^2y=0$

Solution:

Characteristic equation: $(r+3)^4(r^2-10r+29)^2=0$

$\Rightarrow \quad r_1=r_2=r_3=r_4=-3, \quad r_5=r_6=5+2i, \quad r_7=r_8=\overline{r_5}$

$\Rightarrow \quad \{e^{-3t}, te^{-3t}, t^2e^{-3t}, e^{5t}cos2t, e^{5t}sin2t, te^{5t}cos2t, te^{5t}sin2t\}$

is a fundamental set of solutions. General solution is therefore

$y(t)=e^{-3t}(c_1+c_2t+c_3t^2+c_4t^3)+e^{5t}(c_5+c_6t)cos2t+e^{5t}(c_7+c_8t)sin2t.$

b) Assume that the Laplace transform of the Taylor series for $f(t)=e^{at}$ where $a\in\mathbb{R}$ can be computed term by term. Determine the Laplace transform of $f(t)=e^{at}$from this series.

Solution:

$$e^{at}=\sum_{n=0}^{\infty}\frac{a^nt^n}{n!}, \quad L[e^{at}]=\sum_{n=0}^{\infty}\frac{a^n}{n!}L[t^n], \quad L[t^n]=\frac{n!}{s^{n+1}}$$

$$\Rightarrow \quad L[e^{st}]=\sum_{n=0}^{\infty}\frac{a^n}{s^{n+1}}=\frac{1}{s}\sum_{n=0}^{\infty}(a/s)^n, \quad \sum_{n=0}^{\infty}(a/s)^n=\frac{1}{1-a/s}, \quad |a/s|<1$$

$$\text{Therefore,} \quad L[e^{st}]=\frac{1}{s(1-a/s)}=\frac{1}{s-a}, \quad (s>a)$$

B U Department of Mathematics

Math 202 Differential Equations

Summer 2005 Second Midterm

1. Find the general solution of the fourth order equation: $y^{(iv)} - 8y' = 0$

   Solution:

   $(D^4 - 8D)y = D(D^3 - 8I)y = 0$
   $D^3 - 8I = (D - 2I)(D^2 + 2D + 4I)$
   Hence the DE is
   $D(D - 2I)(D^2 + 2D + 4I)y = 0$
   The characteristic eqn. is $r(r-2)(r^2 + 2r + 4) = 0$
   and the roots are $r_1 = 0, \quad r_2 = 2, \quad r_3 = -1 + i\sqrt{3}, \quad r_4 = \overline{r_3}$
   $\Rightarrow \quad \{1, e^{2t}, e^{-t}\cos(\sqrt{3}t), e^{-t}\sin(\sqrt{3}t)\}$ is a fundamental set of solns.
   The general solution is therefore,
   $y = c_1 + c_2 e^{2t} + e^{-t}(c_3 \cos(\sqrt{3}t) + c_4 \sin(\sqrt{3}t)),\ (c_1, c_2, c_3, c_4 \in \mathbb{R})$

2. Using the method of power series solve the initial value problem:
   $(x^2 + 5)y'' - 6xy' + 12y = 0, \quad y(0) = 25, y'(0) = 0$

   Solution:

   $x = 0$ is an ordinary point.
   Let $y = \sum_{n=0}^{\infty} c_n x^n$ so DE becomes

$$\sum_{n=2}^{\infty}(n-1)nc_n x^n + \sum_{n=2}^{\infty} 5n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} 6nc_n x^n + \sum_{n=0}^{\infty} 12c_n x^n = 0$$

$$\sum_{k=2}^{\infty} k(k-1)c_k x^k + \sum_{k=0}^{\infty} 5(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} 6kc_k x^k + \sum_{k=0}^{\infty} 12c_k x^k = 0$$

$$10c_2 + 12c_0 + 6c_1 x + 30c_3 x + \sum_{k=2}^{\infty}[(k(k-1) + 12 - 6k)c_k + 5(k+2)(k+1)c_{k+2}]x^k = 0$$

   $\Rightarrow \quad 5c_2 + 6c_0 = 0, \quad c_1 + 5c_3 = 0$
   $5(k+2)(k+1)c_{k+2} + (k^2 - 7k + 12)c_k = 0, \quad k = 2, 3, ...$

$$\Rightarrow \quad c_{k+2} = \frac{-(k-3)(k-4)}{5(k+2)(k+1)}\ c_k$$

   Initial conditions:
   $y(0) = c_0 = 25 \quad \Rightarrow \quad c_2 = -30$
   $y'(0) = c_1 = 0 \quad \Rightarrow \quad c_3 = 0$
   $\Rightarrow \quad c_4 = \dfrac{-(-1)(-2)}{5(4)(3)}\ c_2 = -\dfrac{1}{30}\ c_2 = 1$
   Recurrence relation $\quad \Rightarrow \quad c_k = 0$ for $k \geq 5$
   Therefore $y = 25 - 30x^2 + x^4$

3. **a)** Determine the interval of convergence of the series: $\sum_{n=1}^{\infty} \frac{(-1)^n n^2 (x+2)^n}{3^n}$.

Solution:

Let $c_n = \frac{(-1)^n n^2}{3^n}$
The radius of convergence R is
$$R = \lim_{n\to\infty} \left[\left|\frac{c_n}{c_{n+1}}\right|\right] = \lim_{n\to\infty} \left[\left|\frac{n^2}{3^n} \frac{3^{n+1}}{(n+1)^2}\right|\right] = 3$$
Hence the interval of convergence is $|x+2| < 3$

b) The gamma function is defined by $\Gamma(x) = \int_0^\infty e^{-u} u^{x-1} du$ for $x > 0$.

Prove that $\Gamma(x+1) = \int_0^1 [\ln(1/s)]^x \, ds$

Solution:

$$\Gamma(x+1) = \int_0^\infty e^{-u} u^x \, du$$
Let $e^{-u} = s \quad \Rightarrow \quad u = \ln(1/s)$
$u = 0 \quad \Rightarrow \quad s = 1, \quad u \to \infty \quad \Rightarrow \quad s \to 0, \quad du = -\frac{1}{s} \, ds$
Therefore
$$\begin{aligned} \Gamma(x+1) &= \int_1^0 s[\ln(1/s)]^x (-\frac{1}{s} \, ds) \\ &= \int_0^1 [ln(1/s)]^x \, ds \end{aligned}$$

4. Using Laplace transforms solve the initial value problem:
$y'' + 4y = 12\sin(2t), \quad y(\pi), y'(\pi) = -3$
(No credit will be given for other approaches.)
Solution:

To handle the initial conditions, let $u = t - \pi$
$\sin(2t) = \sin 2(u+\pi) = \sin(2u), \quad \dfrac{dy}{dt} = \dfrac{dy}{du}$
Hence the DE becomes $\frac{d^2y}{du^2} + 4y = 12\sin(2u)$
and the initial conditions are $y(0) = 5, \quad y'(0) = -3$
Taking now the Laplace transform gives
$s^2Y(s) - 5s + 3 + 4Y(s) = \dfrac{24}{s^2+4}, \quad Y(s) = \mathfrak{L}\{y(n)\}$

$$\begin{aligned} Y(s) &= \frac{5s-3}{s^2+4} + \frac{24}{(s^2+4)^2} = \frac{5s}{s^2+4} - \frac{3(s^2-4)}{(s^2+4)^2} \\ &= \frac{5s}{s^2+4} + 3\frac{d}{ds}\left(\frac{s}{s^2+4}\right) \\ &= \mathfrak{L}\{5\cos(2u)\} - 3\mathfrak{L}\{u\cos(2u)\} \\ y(u) &= \mathfrak{L}^{-1}\{Y(s)\} = 5\cos 2u - 3u\cos 2u \end{aligned}$$

$\Rightarrow \quad y(t) = 5\cos 2(t-\pi) - 3(t-\pi)\cos 2(t-\pi)$
$\Rightarrow \quad y(t) = (5 + 3\pi - 3t)\cos 2t$