Date: November 7, 2003
Time: 17:00-18:00

NAME, SURNAME: ______________________

MATH 202 NUMBER: ______________________

STUDENT ID: ______________________

MATH 202 FIRST MIDTERM SOLUTION KEY

IMPORTANT

1. Write your name, surname on top of each page.
2. The exam consists of 4 questions some of which have more than one part.
3. Please read the questions carefully and write your answers neatly under the corresponding questions.
4. Show all your work. Correct answers without sufficient explanation might not get full credit.
5. Calculators are not allowed.

1	2	3	4	TOTAL
25 pts.	25 pts.	25 pts.	25 pts.	100 pts.

1.) Consider the differential equation $(x^n + y^n)y' - x^{n-1}y = 0$ for $x, y > 0$.
(a)[07] For which value(s) of n is the equation exact?

Solution:

Rewriting the DE: $(x^n + y^n)dy - x^{n-1}ydx = 0$. We set:

$$M = -x^{n-1}y \quad \text{and} \quad N = x^n + y^n.$$

Exactness implies $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ which is equivalent to:

$$-x^{n-1} = nx^{n-1}$$
$$\Longrightarrow \quad \boxed{n = -1}$$

■

(b)[18] When the equation is *not* exact, find an integrating factor and determine the general solution.

Solution:

We let $n \neq -1$. An integrating factor μ is either a function of x or of y.

$\dfrac{M_y - N_x}{N} = -\dfrac{(n+1)x^{n-1}}{x^n + y^n}$ is not a function of x only.

$\dfrac{N_x - M_y}{M} = -\dfrac{(n+1)x^{n-1}}{x^{n-1}y} = -\dfrac{n+1}{y}$ is a function of y only, and hence an integrating factor $\mu = \mu(y)$ can be obtained via:

$$\mu(y) = \exp\left[-\int \frac{n+1}{y}dy\right] = \exp[-(n+1)\ln y] = y^{-(n+1)}.$$

Thus the DE when multiplied by $\mu(y)$:

$$(x^n y^{-(n+1)} + y^{-1})dy - x^{n-1}y^{-n}dx = 0$$

becomes exact. Now we seek a differentiable $F(x, y)$ such that:

$$\frac{\partial F}{\partial x} = -x^{n-1}y^{-n} \quad \text{and} \quad \frac{\partial F}{\partial y} = x^n y^{-(n+1)} + y^{-1},$$

so that our DE takes the form $dF = 0$ and $F(x, y) =$ constant is the solution. Integrating the first one:

$$F = -x^n y^{-n}/n + \phi(y)$$

provided $n \neq 0$. If $n = 0$ this integral is not correct because of the division by n. Now comparing $\partial F/\partial y$ obtained in two ways we find that:

$$-x^n y^{-n-1} + \phi' = -x^n y^{-n-1} + y^{-1}$$

which implies $\phi(y) = \ln y$. This is if $n \neq 0$.

If $n = 0$, then $F = -\ln x + \psi(y)$. Moreover:

$$\frac{\partial F}{\partial y} = 2y^{-1} = \psi' \Longrightarrow \psi(y) = 2\ln y.$$

Hence the general solution of the DE is:

if $n \neq 0$ then $\boxed{F(x, y) = -x^n y^{-n}/n + \ln y = c_1}$

if $n = 0$ then $F(x, y) = -\ln x + 2\ln y = c_2 \Longrightarrow F(x, y) = \ln y^2/x = c_2 \Longrightarrow \boxed{y = c\sqrt{x}}$

[Note that all the intermediate integration constants may be omitted as we correct this in the end of the process.]

■

2.) Let $y'' + (\alpha+2)y' + 2\alpha y = 0$ be a constant coefficient differential equation, where $\alpha \in \mathbb{R}$.
(a)[09] Find the general solution for any α.

Solution:

The characteristic equation is $r^2 + (\alpha+2)r + 2\alpha = 0$. Discriminant is easily calculated to be $(\alpha+2)^2 - 8\alpha = (\alpha-2)^2$. Hence roots are given by:

$$r_{1,2} = \frac{-(\alpha+2) \pm (\alpha-2)}{2} \Longrightarrow r_1 = -2, \quad r_2 = -\alpha.$$

Roots are both real but if $\alpha = 2$ repeated. Thus the general solution is:

$$y = \begin{cases} c_1 e^{-2t} + c_2 e^{-\alpha t} & \text{if } \alpha \neq 2 \\ (c_1 + c_2 t)e^{-2t} & \text{if } \alpha = 2 \end{cases}$$

■

(b)[07] Analyse the large time behaviour of the solution, i.e. as $t \to \infty$.

Solution:

We check the limits as $t \to \infty$:

$$\alpha \neq 2 \Longrightarrow \lim_{t\to\infty}(c_1 e^{-2t} + c_2 e^{-\alpha t}) = c_2 \lim_{t\to\infty} e^{-\alpha t} = \begin{cases} c_2 & \text{if } \alpha = 0 \\ \infty & \text{if } \alpha < 0 \\ 0 & \text{if } \alpha > 0 \end{cases}$$

$$\alpha = 2 \Longrightarrow \lim_{t\to\infty}(c_1 + c_2 t)e^{-2t} = 0.$$

Note that $e^{-2t} \to 0$ as $t \to \infty$.

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(c)[09] Find a particular solution of $y'' + 3y' + 2y = 1 + e^{-t}$. [This is the equation above with $\alpha = 1$.]

Solution:

As found in part **(a)** $y_c = c_1 e^{-2t} + c_2 e^{-t}$. Hence the correct form for a particular solution reads:

$$\begin{aligned} & y_p = Ate^{-t} + B \\ \Longrightarrow \quad & y_p' = Ae^{-t} - Ate^{-t} \\ \Longrightarrow \quad & y_p'' = -2Ae^{-t} + Ate^{-t} \end{aligned}$$

Plugging these into the DE and performing the cancellations we obtain:

$$Ae^{-t} + 2B = 1 + e^{-t},$$

in which we compare the coefficients to find $A = 1$ and $B = 1/2$. Hence a particular solution is $\boxed{y_p = te^{-t} + 1/2}$

■

3.) Consider the equation $y'' - \frac{2}{x^2}\, y = 0$ for $0 < x < \infty$.
(a)[10] Find the solutions of the form $y = x^r$ to this equation. Have you found *all* solutions of the equation? Explain.

Solution:

Letting $y = x^r$ we get $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting into the equation:

$$(r^2 - r - 2)x^{r-2} = 0 \Longrightarrow r^2 - r - 2 = 0 \Longrightarrow r = 2, \quad r = -1.$$

Hence two fundamental solutions are $\boxed{y_1 = x^2}$ and $\boxed{y_2 = x^{-1}}$

Yes, every solution is of the form $c_1y_1+c_2y_2$ as a second order homogeneous *linear* equation has exactly two fundamental solutions.

■

(b)[15] Find a particular solution of the nonhomogeneous equation $y'' - \frac{2}{x^2}\, y = x$.

Solution:

We must use variation of parameters by setting $y_p = v_1(x)x^2 + v_2(x)x^{-1}$ as a particular solution for some v_1 and v_2. This is true if:

$$\begin{aligned} v_1'x^2 + v_2x^{-1} &= 0 \\ 2v_1'x - v_2'x^{-2} &= x. \end{aligned}$$

Solving this algebraic system by $x \cdot$ (2nd eqn.) + (1st eqn.) we receive:

$$v_1' = 1/3 \Longrightarrow v_1 = x/3 \quad \text{then} \quad v_2' = -x^3/3 \Longrightarrow v_2 = -x^4/12.$$

Collecting these in y_p a particular solution is established to be:

$$\boxed{y_p = \frac{x^3}{4}}$$

■

4.)[25] Let $y'' + p(x)y' + q(x)y = x^{-1}$ be a second order linear differential equation defined for $x > 0$ with continuous $p(x)$ and $q(x)$ on $(0, \infty)$. Suppose that $y_1(x)$ and $y_2(x)$ are the solutions of the corresponding homogeneous equation with Wronskian $W(y_1, y_2)(x) = 1/x$. Suppose furthermore that $y_p(x) = x$ is a particular solution. Find the solution of the initial value problem described by the given *nonhomogeneous* differential equation and the initial conditions $y(1) = 0$ and $y'(1) = 2$.

Solution:

$W(y_1, y_2)(x) = \dfrac{1}{x} = c \exp\left[-\int p(x)dx\right]$ by Abel's formula. But this entails immediately:

$$-\int p(x)dx = -\ln x - \ln c,$$

which, after one differentiation, is equivalent to $p(x) = 1/x$.

On the other hand, $y_p = x$ is given to be a solution of the nonhomogeneous equation. Using this fact (substitute into the eqn.) easily obtained that $p(x) + xq(x) = 1/x$. But $p(x) = 1/x$ with the immediate consequence $q(x) = 0$.

Since a particular solution is given we need to find the complementary solution of the equation, namely solve:

$$y'' + \frac{y'}{x} = 0.$$

We observe that this DE does not contain y, so order is reducible via $y' = u$. We now solve:

$$u' + \frac{u}{x} = 0$$

which is separable:

$$\frac{du}{u} = -\frac{dx}{x} \Longrightarrow u = \frac{c_1}{x} = y'.$$

Integrating this once more to find y:

$$y_c = c_1 \ln x + c_2.$$

Thus the general solution is:

$$y = c_1 \ln x + c_2 + x.$$

Now using the initial conditions given:

$$y(1) = c_2 + 1 = 0 \Longrightarrow c_2 = -1$$
$$y'(1) = c_1 + 1 = 2 \Longrightarrow c_1 = 1.$$

Solution of the IVP is $\boxed{y = x + \ln x - 1}$

■

B U Department of Mathematics
Math 202 Fall 2004 First Midterm

1. **a**) Prove that $y = x$ can NOT possibly be a solution to a **first order linear homogeneous** differential equation **with constant coefficients**.

Solution:

For such a problem Wronskian can never be zero. $\forall x \in \mathbb{R}$ but $W(x) = x$ which is zero at $x = 0$ or $y = x$ can never be a solution to $ay' + by = 0$; $a, b \in \mathbb{R}$ since $a.1 + b.x = 0 \Rightarrow a = b = 0$

b)Show that $y = x$ can be a solution to a (higher order) linear homogeneous differential equation **with constant coefficients**.

Solution:

$y'' = 0$

c)Show that $y = x$ can be a solution to a **linear non-homogeneous** differential equation **with constant coefficients**..

Solution:

$y = x$ is an example (0^{th} order d.e.)

$y' = 1$ is an another example(1^{st} order d.e.).

2. Find the general solution of the differential equation $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = \sin e^x$

Solution:

Fundamental solutions $\{e^{-x}, e^{-2x}\}$

Variation of parameters: $y = c_1(x)e^{-x} + c_2(x)e^{-2x}$ Since constant coefficients

$c_1 = e^x \sin e^x \Rightarrow c_1 = -\cos e^x$

$c_2' = -e^{2x} \sin e^x = -e^x.e^x \sin e^x \Rightarrow c_2 = e^x \cos x - \sin e^x$

$\Rightarrow Y_p = (-\cos e^x)e^{-x} + (e^x \cos x - \sin e^x)e^{-2x} = -e^{-2x} \sin e^x$

$y = c_1 e^{-x} + c_2 e^{-2x} - e^{-2x} \sin e^x$

3. Find the general solution of the differential equation $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 4e^x + x$

Solution:

Fundamental Solutions:$\{e^x, \cos x, \sin x\}$

Since e^x is repeated on the R.H.S. we must try $Y_p = Axe^x + Bx + C$

$Y_p' = Ae^x + Axe^x + B \Rightarrow Y_p'' = 2Ae^x + Axe^x \Rightarrow Y_p''' = 3Ae^x + Axe^x$

$\Rightarrow 3Ae^x + Axe^x - 2Ae^x - Axe^x + Ae^x + Axe^x + B - Axe^x - Bx - C = 4e^x + x$

$\Rightarrow 2A = 4 \Rightarrow A = 2, B = -1, C = -1$

$\Rightarrow Y_p = 2xe^x - x - 1$

4.Consider the non-homogeneous linear differential equation $(x-1)\dfrac{d^2y}{d^2x} - x\dfrac{dy}{dx} + y = 4(x-1)^2 e^{-x}$, for $x > 1$,

a)Check that $y_1 = e^x$ is a solution to the associated **homogeneous equation**. Using this solution find a second solution y_2 so that y_1 and y_2 are linearly independent.

Solution:

$(x-1)e^x - xe^x + e^x - xe^x - e^x - xe^x + e^x = 0$

$y_2 = c(x)e^x$ $y_2' = c'e^x + ce^x$ $y_2'' = 2c'e^x + ce^x + c''e^x$ put these into the homogenous equation and get;

$(x-1)c'' + (x-2)c' = 0 \Rightarrow \dfrac{c''}{c'} = \dfrac{1}{x-1} - 1 \Rightarrow \ln c' = \ln|x-1| - x \Rightarrow c' = (x-1)e^{-x}$

$\Rightarrow c(x) = (1-x)e^{-x} + e^x = -xe^{-x} \Rightarrow y_2 = -xe^{-x}e^x = -x$

i.e. $y_2 = x$ is an another fundamental solution $W(x, e^x) = (x-1)e^x$ nonzero for $x > 1$.

b) Now solve the **non-homogeneous** equation.

Solution:

Use Variation of Parameters. $y = c_1(x)e^x + c_2(x)x$

$y' = c_1'e^x + c_1e^x + c_2'x + c_2$ and we set $c_1'e^x + c_2'x + c_2 = 0$.....(*)

$y'' = c_1'e^x + c_1e^x + c_2' \Rightarrow (x-1)(c_1'e^x + c_1e^x + c_2') - x(c_1e^x + c_2) + c_1e^x + c_2x = 4(x-1)e^{-x}$

$\Rightarrow (x-1)e^x c_1' + (x-1)c_2' = 4(x-1)e^{-x} \Rightarrow c_1'e^x + c_2' = 4(x-1)e^{-x}$.....(**)

Solve (*) and (**): $c_1' = 4xe^{-2x} \Rightarrow c_1 = -(2x+1)e^{-2x}$

$c_2' = -\frac{e^x}{x}c_1' = -4e^{-x} \Rightarrow c_2 = 4e^{-x}$

$Y_p = c_1e^x + c_2x = -(2x+1)e^{-x} + 4xe^{-x} = (2x-1)e^{-x}$

G.S.:$y = c_1e^x + c_2x + (2x-1)e^{-x}$

B U Department of Mathematics

Math 202 Differential Equations

Fall 2005 First Midterm

1. Let $2xyy^{'} + (1+x)y^2 = e^x, \ x > 0$

i) By suitable change of variable transform this equation to a linear first order equation.

ii) Solve the equation

Solution:

i) $\underbrace{(2yy^{'})}\, x + (1+x)\, \underbrace{y^2} = e^x$

Let $y^2 = u \Rightarrow 2yy^{'} = u^{'}$

So $u^{'}x + (1+x)u = e^x$ is a linear 1^{st} order equation.

ii)

$$u^{'} + (\frac{1+x}{x}) = \frac{1}{x}e^x$$

$$\mu = IF = \exp[\int (\frac{1}{x+1})\ dx] = \exp[\ln(x) + x] = x\ e^x$$

$$x\ e^x\ u^{'} + \ (\frac{1}{x} + 1)\ x\ e^x\ u = x\ e^x\ (\frac{1}{x}\ e^x)$$

$$\Rightarrow\ x\ e^x\ u^{'} + (1+x)\ e^x\ u = e^{2x}$$

$$\Rightarrow\ \frac{d}{dx}\ (x\ e^x u) = e^{2x}$$

$$(x\ e^x u) = \frac{1}{2}\ e^{2x} + \ c, \quad c \in \mathbb{R}$$

$$u = \frac{1}{2x}\ e^x + \frac{c}{x}\ e^{-x}, \quad c \in \mathbb{R}$$

$$\Rightarrow u = \frac{e^x + c^{'} e^{-x}}{2x}, \quad c^{'} \in \mathbb{R}$$

Hence from $y^2 = u$

$$\Rightarrow y^2 = \frac{e^x + c^{'} e^{-x}}{2x}, \quad c^{'} \in \mathbb{R}$$

is the implicit solution of the given differential equation.

2. Consider the Initial Value Problem: $y' = \frac{x}{y+1}$, $y(0) = 0$

i) Discuss the existence of the solution of the Initial Value Problem given above; if there exists a solution how is the existence guaranteed?

(ii) Solve, if possible, the given Initial Value Problem <u>explicitly.</u>

Solution:

i) $y' = \dfrac{x}{y+1} = f(x, y)$, $y(0) = 0$.

$f(x, y) = \dfrac{x}{y+1}$ and $\dfrac{\partial(f)}{\partial(y)} = \dfrac{-x}{(y+1)^2}$ are continuous in an open rectangular region R if R does not contain $y = -1$ line. So, for such region R involving the point $(0, 0)$ we can find an open interval I with $0 \in I$. So, the IVP has a unique solution in I.

ii) Given DE, $(y + 1)dy = xdx$ and hence $\dfrac{y^2}{2} + y = \dfrac{x^2}{2} + c$.

$y(0) = 0 \Rightarrow c = 0$,

therefore $y^2 + 2y = x^2$ is the implicit solution of the IVP.

For the explicit solution,

$$y^2 + 2y - x^2 = 0$$

$$y = \frac{-2 \pm \sqrt{4 + 4x^2}}{2} = -1 \pm \sqrt{1 + x^2}$$

For $y(0) = 0$ to hold, take

$$y = -1 + \sqrt{1 + x^2}.$$

3. Given that the differential equation $y'' + (4x)y' + q(x)y = 0$ has two solutions of the form $y_1 = u(x)$, $y_2 = xu(x)$ where $u(0) = 1$, determine both $u(x)$ and $q(x)$ explicitly.

Solution:

y_1, y_2 are linearly independent $\Rightarrow W(y_1, y_2) = \begin{vmatrix} u & xu \\ u' & u + xu' \end{vmatrix} = u^2 \neq 0$

By Abel's theorem

$$\begin{aligned} u^2 &= Cexp[-\int 4x dx] \\ &= Cexp(-2x^2). \end{aligned}$$

For $u(0) = 1 \Rightarrow C = 1$.
So, $u^2(x) = e^{-2x^2}$

$$\underline{u(x) = e^{-x^2}}$$

Hence $u'(x) = (-2x)e^{-x^2}$ and $u''(x) = (-2 + 4x^2)e^{-x^2}$.
Therefore, from DE
$(-2 + 4x^2)e^{-x^2} + 4x(-2x)e^{-x^2} + q(x)e^{-x^2} = 0$
$-2 + 4x^2 - 8x^2 + q(x) = 0$

$$\underline{q(x) = 4x^2 + 2}$$

4. Given that $y_1(t = t$ and $y_1(t = t^{-3}$ are solutions to homogeneous equation $t^2y''(t) + 3ty'(t) - 3y(t) = 0$ find the general solution to $t^2y''(t)+3ty'(t)-3y(t) = \frac{1}{t}$ by using the <u>Variation of Parameters</u> method.

Solution:

DE: $\quad y'' + (3t^{-1})y' - (3t^{-2})y = t^{-3}.$

So $g(t) = t^{-3}$.

Now $y_h = c_1t + c_2t^{-3}, \qquad c_1, c_2 \in \mathbb{R}$.

Let $y_p = c_1(t)t + c_2\mathrm{t}^{-3}t^{-3}$ be the particular solution where

$$c_1'(t) = \frac{\begin{vmatrix} 0 & t^{-3} \\ t^{-3} & -3t^{-4} \end{vmatrix}}{W(y_1, y_2)(t)} \qquad c_2'(t) = \frac{\begin{vmatrix} t & 0 \\ 1 & t^{-3} \end{vmatrix}}{W(y_1, y_2)(t)}$$

where

$$W(y_1, y_2)(t) = \begin{vmatrix} t & t^{-3} \\ 1 & -3t^{-4} \end{vmatrix} = -4t^{-3}$$

$$\Rightarrow c_1'(t) = \frac{-t^{-6}}{-4t^{-3}} = \frac{1}{4}t^{-3} \qquad \Rightarrow \qquad c_1(t) = \frac{1}{8}t^{-2}$$

$$\& \;\; c_2'(t) = \frac{t^{-2}}{-4t^{-3}} = -\frac{1}{4}t \qquad \Rightarrow \qquad c_2(t) = -\frac{t^2}{8}$$

Hence

$$y_p = -\frac{t^{-2}}{8}(t)t - \frac{t^2}{8}\mathrm{t}^{-3}t^{-3} = -\frac{1}{4}t^{-1}$$

Thus the general solution

$$y(t) = c_1t + c_2t^{-3} + -\frac{1}{4}t^{-1}, \qquad c_1, c_2 \in \mathbb{R}$$

5. Solve implicitly; $(x^2 \ln x)\, dy + (xy - 1)\, dx = 0, \qquad x > 0$

Solution:

$(xy - 1)\, dx + (x^2 \ln x)\, dy = 0$
$M = xy - 1 \quad \Rightarrow \quad M_y = x \quad \& \quad N = x^2 \ln x \quad \Rightarrow \quad N_x = 2x \ln x + x$
$M_y \neq N_x \Rightarrow$ Equation is not exact.

$$\frac{M_y - N_x}{N} = -\frac{2}{x} \qquad \Longrightarrow \qquad IF = \exp\left[-\int \frac{2}{x} dx\right] = x^{-2}$$

Then we get

$$\left(\frac{y}{x} - x^{-2}\right) dx + (\ln x)\, dy = 0$$

$$P = \left(\frac{y}{x} - x^{-2}\right) \;\; \& \;\; Q = (\ln x) \;\; \Rightarrow \;\; P_y = \frac{1}{x} = Q_x \Rightarrow \text{New equation is exact}$$

There is

$$u = u\ (x,\ y) \qquad \texttt{s. t.} \qquad \frac{\partial u}{\partial x} = P, \quad \frac{\partial u}{\partial y} = Q$$

$$\frac{\partial u}{\partial y} = \ln x \qquad \Rightarrow \qquad u\ (x,\ y) = \int (\ln x)\, dy = y \ln x + h(x)$$

So

$$\frac{\partial u}{\partial x} = \frac{y}{x} + h'(x) = \frac{y}{x} - x^{-2}$$

$$h'(x) = -x^{-2}$$

$$h(x) = \frac{1}{x} + k$$

$$\Rightarrow \quad u\ (x,\ y) = y \ln x + \frac{1}{x} + k$$

As $u\ (x,\ y) = constant$ gives the implicit solution

$$y \ln x + \frac{1}{x} = c, \qquad c \in \mathbb{R}$$

is the implicit solution.

B U Department of Mathematics

Math 202 Differential Equations

Date: April 9, 2004	Full Name :
Time: 18:10-19:25	Math 202 Number :
	Student ID :

Spring 2004 First Midterm - Solution Key

IMPORTANT

1. Write your name, surname on top of each page. **2.** The exam consists of 4 questions some of which have more than one part. **3.** Read the questions carefully and write your answers neatly under the corresponding questions. **4.** Show all your work. Correct answers without sufficient explanation might not get full credit. **5.** Calculators are not allowed.

Q1	Q2	Q3	Q4	TOTAL
20 pts	25 pts	30 pts	25 pts	100 pts

1.)[20] Solve $(\sec^2 y)y' + \dfrac{\tan y}{1+x} = \dfrac{1}{\sqrt{1+x}}$ by using the substitution $u = \tan y$.

Solution:

Using the given substitution we get:

$$u = \tan y \Rightarrow u' = (\sec^2 y)y'.$$

The transformed DE for u reads:

$$u' + \frac{u}{1+x} = \frac{1}{\sqrt{1+x}},$$

which obviously linear (but not separable). Hence we have to find an integrating factor $\mu(x)$ by:

$$\mu(x) = \exp\left[\int \frac{dx}{1+x}\right] = 1 + x.$$

Multiplying the equation by this $\mu(x)$ factor we get $(1+x)u' + u = \sqrt{1+x}$, so that the left hand side becomes a total derivative:

$$[(1+x)u]' = \sqrt{1+x} \Rightarrow (1+x)u = \frac{2}{3}(1+x)^{3/2} + c.$$

Leaving u alone on the left we get the u function to be:

$$u = \frac{2}{3}\sqrt{1+x} + \frac{c}{1+x},$$

which then implies after going back to y:

$$y = \arctan\left(\frac{2}{3}\sqrt{1+x} + \frac{c}{1+x}\right).$$

2.) Consider the differential equation $y^{(4)} + 6y''' + 9y'' = f(t)$.
(a)[15] Find the general solution of this differential equation when $f(t) = 50e^{2t} + 18$.

Solution:

We first need to find the complementary solution. This is a constant coefficient linear DE, so we can use the characteristic equation: $r^4 + 6r^3 + 9r^2 = 0$ which is equivalent to $r^2(r+3)^2 = 0$. We have the roots to be $r = 0, -3$ both double. Hence the complementary solution is:

$$y_c = c_1 + c_2 t + c_3 e^{-3t} + c_4 t e^{-3t}.$$

(Alternatively: you can set $y'' = u$ and reduce the order by 2, i.e. solve $u'' + 6u' + 9u = 0$ then find y after two integrations.)

Now we can look for a particular solution and the most appropriate way to do this is the method of undetermined coefficients. So set $y_p = Ae^{2t} + B$. But $B \in y_c$ so multiply by t. Still $Bt \in y_c$. Multiply once more by t. Now Bt^2 is not a complementary solution. Then the correct form is $y_p = Ae^{2t} + Bt^2$. Compute derivatives to substitute in the DE: $y_p' = 2Ae^{2t} + 2Bt$, $y_p'' = 4Ae^{2t} + 2B$, $y_p''' = 8Ae^{2t}$ and $y^{(4)} = 16e^{2t}$. After plugging into the DE we get:

$$16Ae^{2t} + 48Ae^{2t} + 9(4Ae^{2t} + 2B) = 50e^{2t} + 18.$$

Comparing the coefficients yields: $A = 1/2$ and $B = 1$. Hence the general solution is:

$$y = y_c + \frac{e^{2t}}{2} + t^2.$$

(b)[10] Suppose that the motion of a particle is described by the differential equation above with $f(t) = 0$. Suppose furthermore that both the initial ($t = 0$) position and initial velocity of the particle are zero, and the initial acceleration is 6 m/sec^2. If the velocity becomes constantly 1 m/sec in large time, find the position of this particle at time $t = 1$.

Solution:

Since $f(t) = 0$ we need only the complementary solution found above. From the problem we understand that the following conditions are attached to this solution: $y(0) = 0$, $y'(0) = 0$, $y''(0) = 6$ and the large time behaviour $\lim_{t\to\infty} y'(t) = 1$. Now apply these conditions to $y = c_1 + c_2 t + c_3 e^{-3t} + c_4 t e^{-3t}$. First compute the necessary derivatives: $y' = c_2 - 3c_3 e^{-3t} + c_4 e^{-3t} - 3tc_4 e^{-3t}$ and $y'' = 9c_3 e^{-3t} - 6c_4 e^{-3t} + 9tc_4 e^{-3t}$. Then the conditions imply:

$$\begin{aligned} y(0) = 0 &\Rightarrow c_1 + c_3 = 0 \\ y'(0) = 0 &\Rightarrow c_2 - 3c_3 + c_4 = 0 \\ y''(0) = 6 &\Rightarrow 9c_3 - 6c_4 = 6 \\ \lim_{t\to\infty} y'(t) = 1 &\Rightarrow c_2 = 1. \end{aligned}$$

With this found c_2 there remain three equations for three unknowns which are easily solved to find $c_1 = 0$, $c_3 = 0$ and $c_4 = -1$. Then the unique solution describing the dynamics of the particle is:

$$y = t - te^{-3t}.$$

Hence the position at $t = 1$ is simply: $y(1) = 1 - e^{-3}$ m.

3.) Given $x^2y'' - x(x+2)y' + (x+2)y = 2x^3$ for $x > 0$ and $y_1 = x$ is a solution of the corresponding homogeneous differential equation.

(a)[12] Find a second linearly independent solution y_2 of the homogeneous differential equation.

Solution:

The corresponding homogeneous equation is: $x^2y'' - x(x+2)y' + (x+2)y = 0$ with a given solution $y_1 = x$. We will apply the reduction of order scheme: let $y_2 = y_1 v(x) = xv(x)$ and find $v(x)$ from the homogeneous DE. Computing the derivatives: $y_2' = v + xv'$ and $y_2'' = 2v' + xv''$ and inserting into the DE we receive:

$$x^2(2v' + xv'') - x(x+2)(v + xv') + (x+2)xv = 0.$$

After the necessary simplification (v terms cancel out obviously) the remaining DE for v reads: $x^3v'' - x^3v' = 0$ in which x^3 can be cancelled out as $x > 0$. Hence the final DE is:

$$v'' - v' = 0.$$

Letting $v' = u$ we reduce the order by 1 as aimed. This is, we need to solve the easy DE: $u' = u$ which is apparently separable. Solving yields $u = c_1e^x$ which then entails $v = c_1e^x + c_2$. We can but set $c_2 = 0$ as it gives the previous solution after multiplied by $y_1 = x$. Hence $y_2 = xv = c_1xe^x$. Since we are seeking a linearly independent solution it suffices to choose one nonzero c_1, so let $c_1 = 1$ and we find:

$$y_2 = xe^x.$$

(b)[6] Show that $\{y_1, y_2\}$ forms a fundamental set of solutions.

Solution:

We have now $y_1 = x$ and $y_2 = xe^x$. Checking whether $\{y_1, y_2\}$ is a fundamental set means showing their Wroskian $W(y_1, y_2)$ never vanishes whenever the solutions exist. Thus:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & xe^x \\ 1 & xe^x + e^x \end{vmatrix} = x^2e^x \neq 0 \quad \text{as } x > 0.$$

Consequently $\{y_1, y_2\}$ forms a fundamental set of solutions.

(c)[12] Find a particular solution y_p for the nonhomogeneous differential equation.

Solution:

Having obtained two linearly independent solutions of the homogeneous equation we are now able to find a particular solution y_p. The suitable technique is variation of parameters (note that undetermined coefficients method is not acceptable because given equation is not constant coefficient). Writing the DE leaving the highest derivative with coefficient one:

$$y'' - \frac{(x+2)}{x}y' + \frac{(x+2)}{x^2}y = 2x.$$

We are now ready to implement the method: let $y_p = v_1 y_1 + v_2 y_2 = xy_1 + xe^x v_2$ where v_1 and v_2 are functions of x. We require the solutions v_1' and v_2' of the following algebraic equations:

$$\left.\begin{array}{rcl} y_1 v_1' + y_2 v_2' & = & 0 \\ y_1' v_1' + y_2' v_2' & = & 2x \end{array}\right\} \Leftrightarrow \left\{\begin{array}{rcl} xv_1' + xe^x v_2' & = & 0 \\ v_1' + (xe^x + e^x)v_2' & = & 2x \end{array}\right.$$

After this point you can use any method (Cramer's rule or ordinary elimination). Using the first equation in the second we get:

$$v_2' = 2e^{-x} \Rightarrow v_2 = -2e^{-x}.$$

Note that the integration constant may be taken as zero. Using v_2 in the first equation again:

$$v_1' = -2 \Rightarrow v_1 = -2x.$$

Creating a particular solution y_p by the above recipe:

$$y_p = x(-2x) + xe^x(-2e^{-x}) = -2x^2 - 2x.$$

It is good but not necessary to observe that $-2x$ part in y_p is in fact a complementary solution. Since we can always add complementary solutions to a particular solution the above answer is correct. Nonetheless a finer answer would be obtained after removing this part, namely:

$$y_p = -2x^2.$$

4.) Let f and g be any two solutions of the differential equation:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = 0,$$

in the interval $[a, b]$, where $p(x)$ is differentiable function that does not vanish in this interval.
(a)[15] Show that $p(x)[f(x)g'(x) - g(x)f'(x)] = k$, where k is any constant.

Solution:

Recognizing the quantity $f(x)g'(x) - g(x)f'(x)$ as the Wroskian of f and g, i.e.:

$$W(f, g)(x) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}(x) = f(x)g'(x) - g(x)f'(x).$$

The DE, when differentiation is explicitly used, is:

$$p(x)y'' + p'(x)y' + q(x)y = 0.$$

We now use Abel's formula to find the Wronskian:

$$W(f, g)(x) = k \exp\left[-\int \frac{p'(x)}{p(x)}dx\right] = k(p(x))^{-1}.$$

This ends the proof: $p(x)W(f, g)(x) = k$ where k is a constant.

(b)[10] Suppose that $f(x) > 0$ and $g(x) > 0$ in $[a, b]$. Show that if $\ln(f/g)$ has a local maximum at x_0 in (a, b), then f and g are linearly dependent.

Solution:

Since $f, g > 0$ in $[a, b]$ their logarithmic quotient is well defined. $\ln(f/g)$ has a local maximum in the open interval (a, b) means it has vanishing first derivative at $x_0 \in (a, b)$. Writing this:

$$\left.\frac{d}{dx}\ln(f/g)\right|_{x=x_0} = -\frac{f(x_0)g'(x_0) - g(x_0)f'(x_0)}{f(x_0)g(x_0)} = 0.$$

But this could only happen if the numerator is zero (note that the denominator is not zero due to the assumption $f, g > 0$). We again recognize the quantity in the numerator as $W(f, g)(x_0)$. Thus we have shown that $W(f, g)(x_0) = 0$. Independence requires nonzero Wronskian at all points. So we conclude that f and g are linearly dependent.

Indeed, from part (a) we know that $p(x)W(f, g)(x) = k$ for all x, being the same constant for every x. Using this at x_0 we get:

$$p(x_0)W(f, g)(x_0) = k \Rightarrow k = 0,$$

which implies $p(x)W(f, g)(x) = 0$ for every $x \in [a, b]$. As $p(x)$ never vanishes throughout this interval, the only possibility is:

$$W(f, g)(x) = 0, \quad \text{for all } x \in [a, b],$$

meaning that $W(f, g)$ is identically zero.

B U Department of Mathematics

Math 202 Differential Equations

Spring 2005 First Midterm

1. Consider the differential equation $(xy-1)dx+(x^2-xy)dy=0$.
(a) Show that this is not an exact equation.
(b) Find an integrating factor μ to make it exact.
(c) Multiply the equation by μ on both sides and solve the differential equation.
(d) State conditions on x and y required to make your solution valid.

Solution:

Let $M(x,y)=xy-1$ and $N(x,y)=x^2-xy$. Then $\frac{\partial}{\partial y}M=x$; $\frac{\partial}{\partial x}N=2x-y$. Since $M_y \neq N_x$, the DE is not exact.

We find $\mu(x,y)$ such that $\frac{\partial}{\partial y}(\mu M)=\frac{\partial}{\partial x}(\mu N)$. We require:

$$\mu_y(xy-1)+x\mu=\mu_x(x^2-xy)+(2x-y)\mu.$$

Assuming that $\mu=\mu(x)$ i.e. $\mu_y=0$ we get:

$$-x\mu_x=\mu$$

which is in accordance with our assumption. Hence we get as an integrating factor, for example, $\mu=\frac{1}{x}$, $x\neq 0$.

Multiplying both sides by μ, the DE becomes:

$$(y-\frac{1}{x})dx+(x-y)dy=0.$$

Assume the left hand side equals $d\psi$. Finding ψ gives an implicit expression between x and y. Since $\psi_x=y-\frac{1}{x}$, we have $\psi=xy-ln|x|+\phi(y)$. Meanwhile, $\psi_y=x-y=x+\phi_y$. So, $\phi(y)=-\frac{y^2}{2}+C$ so that

$$\psi=xy-\ln|x|-\frac{y^2}{2}+C$$

and $\psi=K$ gives the desired expression. Here $C,K\in\mathbb{R}$.

The solution is valid for arbitrary y and for $x\neq 0$.

2. Find **the general solution** of the following homogenous differential equation over $(0,\infty)$ given that $y_1(t)=\dfrac{1}{t}$ is a solution:

$$t^2y''+4ty'+2y=0.$$

Solution:

Let the second solution be given as $y_2(t) = u(t)y_1(t)$. Insert in the DE to get:

$$\begin{aligned}
& t^2(u''y_1 + 2u'y_1' + uy_1'') + 4t(u'y_1 + uy_1') + uy_1 = 0 \\
\to\ & t^2u''y_1 + 2t^2u'y_1' + 4tu'y_1 = 0 \\
\to\ & tu'' + 3u' = 0 \\
\to\ & \frac{u''}{u'} = -\frac{3}{t} \\
\to\ & u' = Ct^{-3}, \quad (C \in \mathbb{R}) \\
\to\ & u(t) = Ct^{-2} + C_1, \quad (C_1 \in \mathbb{R})
\end{aligned}$$

Hence we get $y_2(t) = t^{-3}$ so that the general solution for the DE becomes

$$y(t) = C_1 t^{-1} + C_2 t^{-3}; \quad C_1, C_2 \in \mathbb{R}.$$

3. Using **the method of undetermined coefficients**, solve the following initial value problem:

$$y'' + 2y' + 2y = 2 + \cos 2t, \quad y(0) = 1, \ \ y'(0) = 0.$$

Solution:

Consider the corresponding homogenous equation $y'' + 2y' + 2y = 0$. The natural frequencies satisfy: $r^2 + 2r + 2 = 0$. Therefore $r_{1,2} = -1 \pm \sqrt{2}$ and the homogenous solution is

$$y_h = e^t(C_1 \cos \sqrt{2}t + C_2 \sin \sqrt{2}t).$$

As for the particular solution, let $y_p = D_1 + D_2 \cos 2t + D_3 \sin 2t$. Inserting in the DE we get:

$$\begin{aligned}
& -4D_2 \cos 2t - 4D_3 \sin 2t + 4(-D_2 \sin 2t + D_3 \cos 2t) + 2(D_1 + D_2 \cos 2t + D_3 \sin 2t) = 2 + \cos 2t \\
\to\ & D_1 = 1; \quad -2D_2 + 4D_3 = 1; \quad -4D_2 - 2D_3 = 0 \\
\to\ & D_1 = 1; \quad D_2 = -\frac{1}{10}; \quad D_3 = \frac{1}{5}.
\end{aligned}$$

The general solution is

$$y = e^t(C_1 \cos \sqrt{2}t + C_2 \sin \sqrt{2}t) + 1 - \frac{1}{10}\cos 2t + \frac{1}{5}\sin 2t.$$

Now, since $y(0) = 1 = C_1 + \frac{9}{10}$, $C_1 = -\frac{9}{10}$. Similarly, since $y'(0) = 0 = C_1 + \sqrt{2}C_2 + \frac{2}{5}$, $C_2 = \frac{1}{2\sqrt{2}}$.

4. Determine the interval(s) in which the solutions exist for the following nonhomogenous differential equation:

$$\frac{d^2y}{dx^2} + 4y = \sec^2 2x.$$

Now find **the general solution** of the differential equation in one of the intervals determined above.

Solution:

Solution exists whenever $\sec^2 2x$ is defined, i.e. whenever $x \neq \frac{\pi}{4} + k\pi$, $k \in \mathbb{Z}$.

Let us restrict ourselves to the interval $(-\frac{\pi}{4}, \frac{\pi}{4})$. The homogenous solution is $C_1 \cos 2x + C_2 \sin 2x$. Vary the parameters: assume that the particular solution is of the form:

$y_p = u(x)\cos 2x + v(x)\sin 2x$. Then $y_p' = u'\cos 2x + v'\sin 2x - 2u\sin 2x + 2v\cos 2x$. As usual we let:

$$u'\cos 2x + v'\sin 2x = 0. \tag{1}$$

So we have $y_p'' = -4u\cos 2x - 4v\sin 2x - 2u'\sin 2x + 2v'\cos 2x$. Inserting in the DE, we obtain:

$$-2u'\sin 2x + 2v'\cos 2x = \sec^2 2x. \tag{2}$$

Solving (1) and (2) simultaneously by Cramer's rule, we get:

$$u' = -\frac{1}{2}\sec^2 2x \sin 2x = -\frac{1}{2}\sec 2x \tan 2x; \quad v' = \frac{1}{2}\sec^2 2x \cos 2x = \frac{1}{2}\sec 2x.$$

Hence,

$$u = -\frac{1}{4}\sec 2x + C_1; \quad v = \frac{1}{4}\ln|\sec 2x + \tan 2x| + C_2.$$

5. (a) Find a second order differential equation with constant coefficients whose general solution is in the form

$$y(t) = c_1 e^{-t} + c_2 e^{2t} + 2te^{-t} + 5$$

where c_1, c_2 are arbitrary real numbers.

Solution:

The homogenous equation $y'' - y' - 2y = 0$ has general solution as the first two terms of the given solution above. We require that $2te^{-t} + 5$ is a particular solution. To get the first term here, the nonhomogenous equation must have $C_1 e^{-t}$ on the right hand side and for the second term, there must be C_2 on the right hand side. So the equation is

$$y'' - y' - 2y = C_1 e^{-t} + C_2.$$

Then inserting the particular solution we obtain $C_1 = -3$ and $C_2 = -10$.

(b) Consider the differential equation $y''' + e^t y'' + (\sin t)^2 y' + 4y = 0$. Let $\psi(t)$ be a solution of the equation and let $\psi(1) = 0$, $\psi'(1) = 0$, $\psi''(1) = 0$. Show that $\psi(2) \neq 1$.

Solution:

First observe that $\psi(t)$ is the solution of the initial value problem

$$y''' + e^t y'' + (\sin t)^2 y' + 4y = 0; \;\; y(1) = 0, \; y'(1) = 0, \; y''(1) = 0.$$

The zero function $y(t) = 0$ is also a solution of this initial value problem. Since e^t, $(\sin t)^2$ and 4 are continuous functions, this initial value problem must have a unique solution by the Theorem. Hence, $\psi(t)$ is the zero function and, of course, $\psi(2) \neq 1$ as required.

B U Department of Mathematics

Math 202 Differential Equations

Summer 2000 First Midterm

1. **(a)** Solve the initial value problem: $y' = 2(2t - y), \quad y(0) = 1$

Solution:

Equation is linear: $y' + 2y = 4t; \quad (e^{2t}y)' = 4te^{2t}$

$y(t) = 2t - 1 + ce^{-2t}$ is the general solution.

$y(0) = -1 + c = 1 \Rightarrow c = 2$

$y(t) = 2t - 1 + 2e^{-2t}$ is the unique solution of the initial value problem.

(b) Solve: $2tyy' = 9t^2 + 3y^2$

Solution:

Equation is homogeneous: Let $y = tv(t)$

$\Rightarrow 2tvv' = 9 + v^2$ is a separable equation.

$$\frac{2vdv}{9+v^2} = \frac{1}{t}dt$$

$\ln(9 + v^2) = \ln|t| + c_0 \quad \Rightarrow \quad 9 + v^2 = c|t| \quad \Rightarrow \quad v = \sqrt{c|t| - 9} \quad (c|t| \geq 0)$

Therefore, $y = t\sqrt{c|t| - 9}$

2. Given the differential equation: $y \ln y dx + (x - \ln y) dy = 0$,
   (a) Show that it is not exact.
   (b) Find an integrating factor and a one-parameter family of solutions.

Solution:

(a) $M = y \ln y, \qquad N = x - \ln y$

$M_y = \ln y + 1 \neq N_x = 1 \quad \Rightarrow$ Equation is not exact.

(b) Either by inspection or by solving $(\mu M)_y = (\mu N)_x$ for $\mu = \mu(y)$ one finds the integrating factor

$$\mu(y) = \frac{1}{y}$$

Therefore,

$$\ln y dx + \left(\frac{x}{y} - \frac{\ln y}{y}\right) dy = 0$$

$\Rightarrow \psi_x = \ln y, \qquad \psi_y = \dfrac{x}{y} - \dfrac{\ln y}{y}$

$\Rightarrow \psi_x = \ln y, \quad \Rightarrow \quad \psi = x \ln y + f(y)$

and $\psi_y = \dfrac{x}{y} - \dfrac{\ln y}{y} \quad \Rightarrow \quad f(' y) = -\dfrac{\ln y}{y}$

Integration by parts gives

$$\int \frac{\ln y}{y} dy = \frac{1}{2}(\ln y)^2$$

$\Rightarrow \psi = x \ln y - \dfrac{1}{2}(\ln y)^2$

Solutions: $\psi(x, y) = c$

$\Rightarrow x \ln y - \dfrac{1}{2}(\ln y)^2 = c$

3. **(a)** Solve the initial value problem: $y'' + 4y' + 5y = 10, \quad y(0) = 0, \quad y^{'}(0) = 0.$

Solution:

The characteristic equation: $r^2 + 4r + 5 = 0$ gives the roots $r_{1,2} = -2 \pm i$

$\Rightarrow y_h = (c_1 \cos t + c_2 \sin t)e^{-2t}$

$y_p = 2$

$y = y_h + y_p \quad \Rightarrow \quad y = (c_1 \cos t + c_2 \sin t)e^{-2t} + 2$ is the general solution.

$y(0) = 0 \quad \Rightarrow \quad c_1 = -2$

$y^{'}(0) = 0 \quad \Rightarrow \quad c_2 = 2c_1 = -4$

Therefore,

$$y(t) = 2(1 - e^{-2t} \cos t - 2e^{-2t} \sin t)$$

(b) Find a homogeneous, second order linear equation which admits

$$\left\{ \frac{e^{2t}}{t}, \frac{e^{-2t}}{t} \right\},$$

as a fundamental set of solutions for $t > 0$.

Solution:

Let $z_1 = ty_1 = e^{2t}, \qquad z_2 = ty_2 = e^{-2t}$

$\{z_1, z_2\}$ is a fundamental set for

$$z^{''} - 4z = 0 \quad and \quad z = ty.$$

Hence it follows that

$$ty^{''} + 2y^{'} - 4ty = 0$$

4. **(a)** Find the general solution of:

$$ty'' + (t-1)y' - y = 0,$$

given that y $= t - 1$ is a solution of this equation.

Solution:
$y = e^{-t}$ is also a solution!

$\{t-1, e^{-t}\}$ is a linearly independent set.

Therefore general solution is

$$y = c_1 e^{-t} + c_2(t-1), \qquad c_1, c_2 \in \mathbb{R}$$

(b) Find the general solution of the forth order equation

$$y^{(iv)} - 8y'' + 16y = 0.$$

Solution:

Characteristic equation: $r^4 - 8r^2 + 16 = 0$

$\Rightarrow (r^2 - 4)^2 = 0 \quad \Rightarrow \quad r_1 = r_2 = 2 \quad and \quad r_3 = r_4 = -2$

$\Rightarrow y(t) = (c_1 + c_2 t)e^{2t} + (c_3 + c_4 t)e^{-2t} \qquad c_1, c_2, c_3, c_4 \in \mathbb{R}$

B U Department of Mathematics

Math 202 Differential Equations

Summer 2001 First Midterm

1. Find one-parameter family of solutions of the differential equation:
   $y' = 1 + x - (1 + 2x)y + xy^2$
   Solution:

   This is a Ricatti eqn., $y = 1$ is a solution.
   Let $y = 1 + \dfrac{1}{z}$
   DE $\Rightarrow \quad z' - z = -x$ (a linear DE)
   Integrating factor $\mu = e^{-x}$
   $(e^{-x}z)' = -xe^{-x}, \;\; z = 1 + x + ce^{x}$
   $y = 1 + \dfrac{1}{1 + x + ce^{x}} \quad (c \in \mathbb{R})$

2. **a)** Show that the equation: $(x^2+y^2)\,dx+2xy\,dy = 0$ is exact and find a one-parameter family of solutions.
   Solution:

   $M = x^2 + y^2, \quad N = 2xy$
   $\Rightarrow \quad M_y = 2y = N_x$
   $\Rightarrow \quad$ DE is exact

   $$\text{Let } \psi_x = x^2 + y^2 \quad \Rightarrow \quad \psi = \frac{1}{3}x^3 + xy^2 + h(y)$$
   $$\Rightarrow \quad \psi_y = 2xy + h'(y) = 2xy$$
   $$\Rightarrow \quad h'(y) = 0$$
   Thus $\psi(x, y) = \dfrac{1}{3}x^3 + xy^2 + c_0, \quad (c_0 \in \mathbb{R})$
   and a one-parameter family of solutions is
   $\psi(x, y) = \dfrac{1}{3}x^3 + xy^2 = c$

b) Find at least three distinct, real solutions to the initial value problem:
$y' = 3x(y-2)^{1/3}, \quad y(0) = 2$, on $(-\infty, \infty)$ and discuss why the solution is not unique.
Solution:

Obviously, $y = 2$ is a solution and it obeys the initial condition.
DE is separable. Integration gives
$(y-2)^{2/3} = x^2 + c, \quad y(0) = 2$
$\Rightarrow \quad c = 0 \quad \Rightarrow \quad y - 2 = \pm x^3$

$$\left.\begin{array}{rcl} y &=& 2 + x^3 \\ y &=& 2 - x^3 \\ y &=& 2 \end{array}\right\} \text{all solve the above initial value problem}$$

Let $y' = f(x,y) = 3x(y-2)^{1/3}$
$\Rightarrow \quad \dfrac{\partial f}{\partial y} = x(y-2)^{-2/3}$ We see that $\dfrac{\partial f}{\partial y}$ is not continuous at $(0,2)$
So uniqueness theorem does not apply.

3. **a)** Find the general solution of: $y'' - 4y' + 13y = 0$
Solution:

Characteristic equation: $r^2 - 4r + 13 = 0$
$\Rightarrow \quad r_{1,2} = 2 \pm 3i$ (complex roots)
$\{e^{2x}\cos(3x), e^{2x}\sin(3x)\}$ is a fundamental set of solutions
General solution:
$y = e^{2x}(c_1\cos(3x) + c_2\sin(3x)) \quad c_1, c_2 \in \mathbb{R}$

b) A particular solution of $(1-x)y'' + xy' - y = 5$ is $y = -5$
Find the general solution.
Solution:

General solution: $y = y_h + y_p$ with $y_p = -5$
For y_h we need a fundamental set of solutions of $(1-x)y'' + xy' - y = 0$
By inspection one can see $y_1 = x$, $y_2 = e^x$
are two lin. indep. homog. solns.
Thus $y_h = c_1x + c_2e^x, \quad c_1, c_2 \in \mathbb{R}$
$\Rightarrow \quad y = c_1x + c_2e^x - 5$ is the general soln.
(Notify only $y_1 = x$ and letting $y = xv, \quad w = v'$
gives $(x - x^2)w' + (x^2 - 2x + 2)w = 0$
$\Rightarrow \quad w = \dfrac{x-1}{x^2}e^x \quad \Rightarrow \quad v = \dfrac{1}{x}e^x + const$
$\Rightarrow \quad y_2 = e^x$ The same answer could be obtained by using Abel's formula)

4. **a)** Suppose that on an open interval I the functions $p_1(x)$, $p_2(x)$ and $q_1(x)$, $q_2(x)$ are continuous and the equations: $y'' + p_1y' + q_1y = 0$, $y'' + p_2y' + q_2y = 0$ have the same solutions. Show that on the interval I $p_1(x) = p_2(x)$ and $q_1(x) = q_2(x)$
Solution:

Take a fundamental set of solns. $\{y_1, y_2\}$ since this solves both eqns., The Wronskien must be the same:
$W(x) = ce^{-\int p_1(x)\,dx} = ce^{-\int p_2(x)\,dx}$
$\Rightarrow \quad p_1(x) = p_2(x)$ on I using some $y(x)$ and this information, The difference of the equations then give $[q_1(x) - q_2(x)]y = 0$

Since $y_1(x)$ and $y_2(x)$ cannot vanish at the same point in I, it follows that $q_1(x) = q_2(x)$ on I

b) Solve the initial value problem:
$y'' + 4y' + 4y = 0, \quad y(0) = 1,\ y'(0) = 0$ Solution:

Characteristic eqn.: $r^2 + 4r + 4 = (r+2)^2 = 0$
$r_1 = r_2 = -2 \quad \Rightarrow \quad$ General soln. is $y = c_1 e^{-2x} + c_2 x e^{-2x}$
$y(0) = c_1 = 1$
$y'(0) = c_2 - 2c_1 = 0 \quad \Rightarrow \quad c_2 = 2$
Hence the unique solution to the initial value problem is $y = e^{-2x}(1 + 2x)$.

B U Department of Mathematics

Math 202 Differential Equations

Summer 2002 First Midterm

1. Show that the differential equation

$$2xy^3dx + (3x^2y^2 + x^2y^3 + 1)dy = 0,$$

is not exact. Find an integrating factor and a one-parameter family of solutions for this equation.

Solution:

$M = 2xy^3$, $N = 3x^2y^2 + x^2y^3 + 1$

$M_y = 6xy^2$, $N_x = 6xy^2 + 2xy^3 \Rightarrow M_y \neq N_x$;

DE is not exact.

$N_x - M_y = 2xy^3 = M \Rightarrow$ An integrating factor $\mu = \mu(y)$ exists:

$\dfrac{\mu'}{\mu} = \dfrac{N_x - M_y}{M} = 1 \Rightarrow \mu(y) = e^y$

Let $\psi = \psi(x, y)$ be such that $\psi_x = 2xy^3e^y, \psi_y = (3x^2y^2 + x^2y^3 + 1)e^y$

$\psi_x = 2xy^3e^y \Rightarrow \psi = x^2y^3e^y + f(y)$

$\psi_y = e^y(x^2y^3 + 3x^2y^2) + f'(y) \Rightarrow f'(y) = e^y, f(y) = e^y + const$

$$\psi(x, y) = (x^2y^3 + 1)e^y;\ \psi(x, y) = c,\ c \in \mathbb{R},$$

gives us a one-parameter family of solutions:

$$(1 + x^2y^3)e^y = c$$

2. (a) Solve the initial value problem: $2ty' - y = t^4 y^{-3}$, $(t > 0)$, $y(2) = 2$.
Is the solution unique? Justify your answer.

Solution:

Obviously, $y = t$ is the solution of the IVP. This is a Bernoulli eqn.

Let $f(t, y) = \dfrac{1}{2t} y + \dfrac{1}{2} t^3 y^{-3}$.

$y' = f(t, y)$, $f_y = \frac{1}{2t} - \frac{3}{2} t^3 y^4$ and around the point $(2, 2)$ in the ty-plane there is

a region $(t > 0,\ y > 0)$ in which f and f_y are continuous.

Therefore, this must be the unique solution of the IVP.

(Letting $u = y^4$ gives $u' - \frac{2}{t} u = 2t^3$, $u(2) = 16$, $u(t) = t^4 + ct^2$, $c = 0$,

$u(t) = t^4$, unique.)

(b) Let y_1, y_2, y_3 be three solutions of a normal, first-order linear differential equation on an interval I. Let $t_0 \in I$ such that $y_1(t_0) \neq y_3(t_0)$. Show that in I

$$\frac{y_1(t) - y_2(t)}{y_1(t) - y_3(t)} = \frac{y_1(t_0) - y_2(t_0)}{y_1(t_0) - y_3(t_0)}$$

Solution:

Consider $y' + p(t)y = g(t)$. The difference of any two solutions is

a homogeneous solution:

$(y_i - y_j)' = -p(t)(y_i - y_j)$,

$y_i - y_j = [y_i(t_0) - y_j(t_0)] exp(-\int_{t_0}^{t} p(s)ds)$, $t_0 \in I$

$$\Rightarrow \frac{y_1(t) - y_2(t)}{y_1(t) - y_3(t)} = \frac{y_1(t_0) - y_2(t_0)}{y_1(t_0) - y_3(t_0)}$$

3. (a) Find the general solution of $4y'' + 20y' + 61y = 0$.

Solution:

Characteristic equation: $4r^2 + 20r + 61 = 0$ and the roots are complex:

$r_1 = -\frac{5}{2} + 3i$, $r_2 = \overline{r}_1$

$\Rightarrow \{e^{-5t/2}cos3t, e^{-5t/2}sin3t\}$ is a fundamental set of solutions.

$$\Rightarrow y = e^{-5t/2}[(c_1cos3t + c_2sin3t)]$$

(b) Solve the initial value problem: $y'' - 18y' + 81y = 0$, $y(0) = 2$, $y'(0) = 25$.

Solution:

Characteristic equation is now: $r^2 - 18r + 81 = (r - 9)^2 = 0$

and we have a repeated real root: $r_1 = r_2 = 9$.

$\Rightarrow \{e^{9t}, te^{9t}\}$ is a fundamental set.

General solution is $y = e^{9t}[c_1 + tc_2]$, $c_1, c_2 \in \mathbb{R}$.

$y(0) = c_1 = 2$,

$y' = e^{9t}[9c_1 + (1 + 9t)c_2]$, $y'(0) = 9c_1 + c_2 = 18 + c_2 = 25 \Rightarrow c_2 = 7$.

Therefore, $y = e^{9t(2+7t)}$ is the unique solution of the IVP.

4. (a) Find the general solution of $y'' + 3y' - 4y = 18e^{2t}$

Solution:

Characteristic eqn. : $r^2 + 3r - 4 = (r-1)(r+4) = 0$

$\Rightarrow r_1 = 1,\ r_2 = -4$

$\Rightarrow \{e^t, e^{-4t}\}$ is a fundamental set of (homogeneous) solutions.

$$y_H = c_1 e^t + c_2 e^{-4t},\ \ (c_1, c_2 \in \mathbb{R})$$

Let $y_P = Ae^{2t}$. DE $\Rightarrow 6A = 18,\ A = 3.$

Hence the general solution is:

$$y = y_H + y_P = c_1 e^t + c_2 e^{-4t} + 3e^{2t}.$$

(b) Let $u(t)$, $v(t)$ be differentiable functions on an interval I and suppose that $u(t)$ never vanishes in I. Let $W[u(t), v(t)]$ be the Wronskian. Prove that if $W = 0$ in I, then $u(t)$ and $v(t)$ are linearly dependent on I.

Solution:

$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - vu'.$

Consider $\dfrac{W}{u^2} = \dfrac{v'}{u} - \dfrac{vu'}{u^2} = \dfrac{d}{dt}\left(\dfrac{v}{u}\right)$

Since $u(t) \neq 0$, $t \in I$, $W = 0$

$\Rightarrow \dfrac{d}{dt}\left(\dfrac{v}{u}\right) = 0 \Rightarrow \dfrac{v}{u} = c, \ \ c \in \mathbb{R}$

$v = cu$ and therefore, u and v are linearly dependent on I.

B U Department of Mathematics

Math 202 Differential Equations

Summer 2003 First Midterm

1. Determine the constants a and b so that $\mu(x,y) = x^a y^b$ is an integrating factor for the equation

$$y dx - (x + x^6) dy = 0$$

Find a one-parameter family of solutions of this equation.(All integrals that may arise should be evaluated.)

Solution:

$$(\mu M)_y = (\mu N)_x : (x^a y^{b+1})_y = -(x^{a+1} y^b + x^{a+6} y^b)_x$$

$$\Rightarrow (b+1) x^a y^b = -y^b[(a+1)x^a + (a+b)x^{a+5}]$$

$$\Rightarrow a + 6 = 0,\ a + 1 = -(b+1) \Rightarrow a = -6,\ b = 4$$

$\mu(x,y) = x^{-6} y^4$ Multiplying the DE by $\mu(x,y)$:

$$x^{-6} y^5 dx - (x^{-5} y^4 + y^4) dy = 0$$

$$\Rightarrow \psi_x = x^{-6} y^5, \psi_y = -(x^{-5} + 1) y^4$$

$$\Rightarrow \psi(x,y) = -\frac{1}{5}\ x^{-5} y^5 + f(y), f'(y) = -y^4$$

$$\psi(x,y) = -\frac{1}{5}\ (x^{-5} y^5 + y^5)$$

A one-parameter family of solutions is given by $\psi(x,y) = \text{const}$, i.e. by

$$y^5 = \frac{cx^5}{1 + x^5},\ c \in \mathbb{R}.$$

2. **a)** Solve: $t^2y' + 2y = 2e^{1/t}y^{1/2}$ for $t > 0$.

Solution:

$y = 0$ is a solution. Let $y \neq 0,\ u(t) = y^{1/2},\ u' = \dfrac{1}{2}y^{-1/2}y'$

DE $\Rightarrow\ 2t^2uu' + 2u^2 = 2e^{1/t}u \Rightarrow t^2u' + u = e^{1/t}$, linear.

$u' + \dfrac{1}{t^2}u = \dfrac{1}{t^2}e^{1/t}$ Integrating factor: $\mu = e^{-1/t}$

$(e^{-1/t}u)' = \dfrac{1}{t^2},\ e^{-1/t}u = c - \dfrac{1}{t},\ c \in \mathbb{R}$

$u = e^{1/t}(c - \dfrac{1}{t}),\ y(t) = u^2 = e^{2/t}(c - \dfrac{1}{t})^2$

b) Solve the initial value problem: $y' - ty^2 = 2y^2,\ y(0) = 1$ and prove that the solution attains its minimum value at $t = -2$.

Solution:

DE separable: $\dfrac{1}{y^2}y' = 2 + t,\ (y \neq 0)$

$\Rightarrow\ -\dfrac{1}{y} = \dfrac{1}{2}t^2 + 2t + c,\ y(0) = 1\ \Rightarrow\ c = -1$

$\dfrac{1}{y} = 1 - \dfrac{1}{2}t^2 - 2t,\ y(t) = (1 - \dfrac{1}{2}t^2 - 2t)^{-1}$

For a minimum at $t = t_0,\ y'(t_0) = 0,\ y''(t_0) > 0$

DE $\Rightarrow\ y' = 0$ iff $t + 2 = 0,\ t = -2$

$y'' = (2 + t)2yy' + y^2,\ y''(-2) = y^2(-2) > 0$

Therefore, the solution attains its minimum value at $t = -2$. $y(-2) = \dfrac{1}{3}$.

3. **a)** Find the general solution of $y'' - 8y' + 16y = 2t$.

Solution:

$y = y_h + y_p$, y_h :Characteristic equation: $r^2 - 8r + 16 = (r-4)^2 = 0$, $r_1 = r_2 = 4$

$\{e^{4t}, te^{4t}\}$ is a fundamental set. $y_h = e^{4t}(c_1 + c_2)$, c_1, $c_2 \in \mathbb{R}$

For y_p let $y_p = A + Bt \quad \Rightarrow \; y_p{}' = B,\; y_p{}'' = 0$

DE $\Rightarrow -8B + 16A + 16Bt = 2t \;\Rightarrow\; 2A = B,\; B = \dfrac{1}{8}$

$y_p = \dfrac{1}{16} + \dfrac{1}{8}t \;\; y_p = e^{4t}(c_1 + tc_2) + \dfrac{1}{16}(1 + 2t)$

b) Solve the initial value problem: $y'' + 6y' + 10y = 0$, $y(0) = 2$, $y'(0) = 1$

Solution:

Characteristic equation: $r^2 + 6r + 10 = 0$, $r_{1,2} = \dfrac{-6 \pm \sqrt{36 - 40}}{2}$

$r_1 = -3 + i$, $r_2 = -3 - i = \overline{r_1} \;\Rightarrow\; \{e^{-3t}sint,\, e^{-3t}cost\}$ is a fundamental set

of solutions.

$y(t) = e^{-3t}(c_1 cost + c_2 sint)$, $(c_1, c_2 \in \mathbb{R})$, (general solution)

$y(0) = c_1 = 2$, $\quad y'(t) = -3y(t) + e^{-3t}(-c_1 sint + c_2 cost)$

$y'(0) = -6 + c_2 = 1 \;\Rightarrow\; c_2 = 7$

Hence the unique solution of the IVP is $y(t) = e^{-3t}(2cost + 7sint)$

4. **a)** Find a second order, homogeneous linear differential equation which admits $\{t, e^{2t}\}$ as a fundamental set of solutions.

Solution:

DE: $y'' + p(t)y' + q(t)y = 0$

$y = t \Rightarrow p + tq = 0$ and $y = e^{2t} \Rightarrow 4 + 2p + q = 0$

$$\Rightarrow \begin{bmatrix} 1 & t \\ 2 & 1 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

$$p(t) = \frac{\begin{vmatrix} 0 & t \\ -4 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & t \\ 2 & 1 \end{vmatrix}} = \frac{4t}{1-2t}, \qquad q(t) = \frac{\begin{vmatrix} 1 & 0 \\ 2 & -4 \end{vmatrix}}{\begin{vmatrix} 1 & t \\ 2 & 1 \end{vmatrix}} = \frac{-4}{1-2t} \qquad (t \neq \frac{1}{2})$$

Therefore,

$$y'' + \frac{4t}{1-2t}y' - \frac{4}{1-2t}y = 0; \quad (2t-1)y'' - 4ty' + 4y = 0$$

b) Consider the differential equation $(t^2 - 4)y'' + 4ty' + 2y = 0$ for $-2 < t < 2$.

Given that $y_1 = \dfrac{1}{t-2}$ is a solution, find the general solution of this equation.

Solution:

Can use Abel's formula: $y_1 y_2' - y_2 y_1' = ce^{-\int p(t)dt}$

$$p(t) = \frac{4t}{t^2-4}; \quad \frac{1}{t-2}y_2' + \frac{1}{(t-2)^2}y_2 = \frac{c}{(t+2)^2}$$

$$\Rightarrow \quad (t-2)y_2' + y_2 = \frac{c}{(t+2)^2}; \quad [(t-2)y_2]' = \frac{c}{(t+2)^2}$$

$$(t-2)y_2 = \frac{-c}{t+2} + constant. \quad (\text{Let } constant = 0)$$

$$y_2 = \frac{-c}{t^2-4} = \frac{c}{4}(\frac{1}{t+2} - \frac{1}{t-2})$$

Thus identify $y_2 = \dfrac{1}{t+2}$. (Equivalently, $y_2 = \dfrac{1}{t^2-4}$)

General solution $\quad y(t) = \dfrac{c_1}{t-2} + \dfrac{c_2}{t+2}$.

B U Department of Mathematics

Math 202 Differential Equations

Summer 2005 First Midterm

1. Using Abel's Theorem and the fact that $y_1 = \dfrac{1}{t+1}$ is a particular solution of the differential equation, solve the initial value problem: $(t^2-1)y'' + 4ty' + 2y = 0$, $y(0) = -5$, $y'(0) = 1$. What is the interval of validity?

Solution:

$$y_1y_2' - y_2y_1' = ce^{-\int p(t)\,dt},\ p(t) = \frac{4t}{t^2-1} \text{ choose } c = 1$$

$$\frac{1}{t+1}y_2' + \frac{1}{(t+1)^2}y_2 = \frac{1}{(t^2-1)^2} \quad \Rightarrow \quad (t+1)y_2' + y_2 = \frac{(t+1)^2}{(t^2-1)^2}$$

$$\Rightarrow \quad ((t+1)y_2)' = \frac{1}{(t-1)^2}$$

$$\Rightarrow \quad (t+1)y_2 = -\frac{1}{t-1} + k, \quad k \in \mathbb{R}$$

$$\Rightarrow \quad y_2 = \frac{1}{t-1} \qquad [\text{or, equivalently, } y_2 = \frac{1}{t^2-1}]$$

Hence the general solution is

$$y(t) = \frac{c_1}{t+1} + \frac{c_2}{t-1} \qquad (t \neq \pm 1)$$

$$\Rightarrow \quad y'(t) = \frac{-c_1}{(t+1)^2} - \frac{c_2}{(t-1)^2}$$

$$\Rightarrow \quad y(0) = c_1 - c_2 = -5, \quad \Rightarrow \quad y'(0) = -c_1 - c_2 = 1$$

$$\Rightarrow \quad c_1 = -3, \quad c_2 = 2$$

$$\Rightarrow \quad y(t) = \frac{2}{t-1} - \frac{3}{t+1}$$

The interval of validity is $-1 < t < 1$

2. **a)** Show that $(3x^2y^2 + 6xy^3)dx + (2x^3y + 9x^2y^2)dy = 0$ is exact

Solution:

Let $M(x,y) = 3x^2y^2 + 6xy^3$, $N(x,y) = 2x^3y + 9x^2y^2$
DE is exact iff $M_y = N_x$
$M_y = 6x^2y + 18xy^2$
$N_x = 6x^2y + 18xy^2$
$\Rightarrow \quad M_y = N_x$

b) Find a one-parameter family of solutions of $y' = -\dfrac{3xy + 6y^2}{2x^2 + 9xy}$.

Solution:

If we multiply the numerator and the denominator of the RHS by xy
We get the DE of part (a).
Hence $\mu(x, y) = xy$ is an integrating factor of the DE.
Let $\psi(x, y)$ be such that
$\psi_x = 3x^2y^2 + 6xy^3$, $\psi_y = 2x^3y + 9x^2y^2$
Integrate ψ_x
$\psi(x, y) = x^3y^2 + 3x^2y^3 + f(y)$.
Then $\psi_y = 2x^3y + 9x^2y^2 + f'(y) = 2x^3y + 9x^2y^2$
$\Rightarrow \quad f'(y) = 0$
A one-parameter family of solutions is therefore

$$\psi(x, y) = c, \quad c \in \mathbb{R}$$

i.e.

$$x^3y^2 + 3x^2y^3 = c$$

3. Determine the solution of the following differential equations:

a) $y'' - 4y' + 13y = 0$
Solution:

$y = e^{rt}$
$\Rightarrow \quad r^2 - 4r + 13 = 0$ (characteristic eqn.)
$\Rightarrow \quad r_{1,2} = \dfrac{4 \pm \sqrt{16 - 52}}{2} = 2 \pm 3i$
$\Rightarrow \quad \{e^{2t}\cos(3t), e^{2t}\sin(3t)\}$ is a fundamental set of solns.
$\Rightarrow \quad$ Gen. soln.:$y(t) = e^{2t}(c_1\cos(3t) + c_2\sin(3t)) \quad (c_1, c_2 \in \mathbb{R})$

b) $y'' - y' - 6y = 8e^{2t} - 5e^{3t}$
Solution:

Gen. Soln. is of the form $y = y_h + y_p$
$y_h: \quad r^2 - r - 6 = 0 = (r+2)(r-3) = 0$
$\Rightarrow \quad \{e^{-2t}, e^{3t}\}$ is a fundamental set of homog. solns.
$\Rightarrow \quad y_h = c_1e^{-2t} + c_2e^{3t}$
Let $y_p = Ae^{2t} + Bte^{3t}$
$\Rightarrow \quad y_p' = 2Ae^{2t} + Be^{3t} + 3Bte^{3t}$
$\Rightarrow \quad y_p'' = 4Ae^{2t} + 6Be^{3t} + 9Bte^{3t}$
So $y_p'' - y_p' - 6y_p = -4Ae^{2t} + 5Be^{3t} = 8e^{2t} - 5e^{3t}$
$\Rightarrow \quad A = -2, \quad B = -1, \quad y_p = -2e^{2t} - te^{3t}$
$\Rightarrow \quad \underline{y = y_h + y_p = c_1e^{-2t} + c_2e^{3t} - 2e^{2t} - te^{3t}}$

4. **a)** Prove that the change of variable $v = y'/y$ reduces the second-order homogeneous linear differential equation $y''+p(t)y'+q(t)y = 0$ to the Riccati equation $v'+v^2+p(t)v+q(t) = 0$. Find the general solution of $y'' - y = 0$ by solving the associated Riccati equation. (No credit will be given for other approaches)

Solution:

$$v' = \frac{y''}{y} - \frac{(y')^2}{y^2} = \frac{y''}{y} - v^2$$

$\Rightarrow \quad y'' = y(v' + v^2)$

$\Rightarrow \quad y'' + p(t)y' + q(t)y = y(v' + v^2) + p(t)vy + q(t)y = 0$

Thus if $y \neq 0$, $\quad v' + v^2 + p(t)v + q(t) = 0$ (a Riccati eqn.)

$y'' - y = 0 \quad \Rightarrow \quad p = 0, \; q = -1$ and

the associated Riccati eqn. is $v' + v^2 - 1 = 0$

Two particular solutions are given by $v_{1,2} = \pm 1$

$v_1 = 1 \quad \Rightarrow \quad y_1' = y_1 \quad \Rightarrow \quad y_1 = e^t$

$v_2 = -1 \quad \Rightarrow \quad y_2' = -y_2 \quad \Rightarrow \quad y_2 = e^{-t}$

$\Rightarrow \quad \{e^t, e^{-t}\}$ is a fund. set

$\Rightarrow \quad$ Gen Soln: $y = c_1 e^t + c_2 e^{-t}$

Equivalently, $v' + v^2 - 1$ is a separable eqn.

$$\frac{dv}{1-v^2} = dt, \qquad \frac{1}{2}\left(\frac{1}{1+v} + \frac{1}{1-v}\right) dv = dt$$

$$\Rightarrow \quad v = \frac{ke^{2t} - 1}{ke^{2t} + 1}, \qquad k \in \mathbb{R}$$

$$v = \frac{y'}{y} = 1 - \frac{2}{ke^{2t} + 1}.$$

Letting $u = \dfrac{2}{ke^{2t} + 1}$ and integrating gives the same answer

b) Let $r > 0$ and $k > 0$ be real numbers. Find the general solution of $y' = (r - ky)y$. Determine the limit of $y(t)$ as $t \to \infty$

Solution:

This eqn. is important in population dynamics.

$y' - ry = -ky^2 \quad$ a Bernoulli equation

Let $u = y^{-1} \quad \Rightarrow \quad u' + ru = k \quad$ is a linear eqn.

$$\Rightarrow \quad (e^{rt}u)' = ke^{rt} \quad \Rightarrow \quad e^{rt}u = \frac{k}{r}e^{rt} + c, \qquad c \in \mathbb{R}$$

$$\Rightarrow \quad u = k/r + ce^{-rt}, \quad y = u^{-1} = \frac{1}{k/r + ce^{-rt}}$$

$$\Rightarrow \quad y = \frac{r}{k + cre^{-rt}}$$

$$\Rightarrow \quad \lim_{t\to\infty} y(t) = \frac{r}{k}$$