Date: January 5, 2004 NAME, SURNAME:__________
Time: 9:00-11:30

MATH 202 NUMBER:__________

STUDENT ID:__________

MATH 202 FINAL EXAM SOLUTION KEY

IMPORTANT

1. Write your name, surname on top of each page.
2. The exam consists of 7 questions some of which have more than one part.
3. Please read the questions carefully and write your answers neatly under the corresponding questions.
4. Show all your work. Correct answers without sufficient explanation might not get full credit.
5. Calculators are not allowed.

1	2	3	4	5	6	7	TOTAL
20 pts	20 pts	20 pts	20 pts	20 pts	25 pts	25 pts	150 pts

1.)(a)[10] Use the change of variable $v = \dfrac{y}{t^2}$ to solve the initial value problem:

$$y' = \frac{2y}{t} + t\tan\frac{y}{t^2}\ , \quad y(1) = \frac{\pi}{6}\ .$$

Solution:

$y = vt^2$ which means $y' = v't^2 + 2tv$. Inserting into the equation: $v't = \tan t$ which is separable:

$$\frac{dv}{\tan v} = \frac{dt}{t} \Rightarrow \ln|\sin v| = \ln|t| + \ln|c| \Rightarrow \sin v = ct.$$

(Under mild domain restrictions) this is equivalent to: $v = \arcsin ct$, hence

$$y = t^2 \arcsin ct.$$

Imposing the IC it is found that $c = 1/2$. Thus:

$$\boxed{y = t^2 \arcsin\frac{t}{2}}$$

♦

(b)[10] Find a second solution of $t^2y'' + ty' - y = 0$, $t > 0$ if $y_1 = t$ is a given solution.

Solution:

Reduction of order: set $y_2 = tv(t)$ and substitute this together with y_2', y_2'' into the equation to get $tv'' + 3v' = 0$. Letting $u = v'$ it becomes a 1st order separable equation:

$$\frac{du}{u} = -3\frac{dt}{t} \Rightarrow u = t^{-3} = v'.$$

One integration yields $v = -t^{-2}/2$. Then $y_2 = -t^{-2}t/2 = -t^{-1}/2$. But the constant in front of t^{-1} is unimportant. Hence:

$$\boxed{y_2 = \frac{1}{t}}$$

Alternative way: You could explicitly state that this is an Euler differential equation and solutions are of the form t^r find r to be 1 or -1 and say that $r = -1$ corresponds to a second fundamental solution.

♦

2.)[20] Solve the initial value problem: $y'' - 2y' - 3y = 6 + e^{-t}$, $\quad y(0) = -2$, $y'(0) = 0$.

Solution:

Complementary solution: Characteristic equation is $r^2 - 2r - 3 = 0 \Rightarrow (r-3)(r+1) = 0 \Rightarrow r = 3, -1$. So $y_c = c_1e^{3t} + c_2e^{-t}$.

Particular solution: We can use the method of undetermined coefficients: setting $y_p = Ate^{-t} + B$ (since $e^{-t} \in y_c$), we find $y_p' = Ae^{-t} - Ate^{-t}$ and $y_p'' = Ate^{-t} - 2Ae^{-t}$. Insert these into the equation to get:

$$-4Ae^{-t} - 3B = 6 + e^{-t} \Rightarrow A = -1/4,\ B = -2.$$

So the general solution of the differential equation becomes:

$$y = c_1e^{3t} + c_2e^{-t} - \frac{1}{4}te^{-t} - 2.$$

Now we use the IC to find c_1 and c_2:

$$\begin{aligned}
&y(0) = c_1 + c_2 - 2 = -2 \Rightarrow c_2 = -c_1 \\
&y'(0) = 3c_1 - c_2 - 1/4 = 0 \Rightarrow 3c_1 + c_1 = 1/4 \Rightarrow \\
&c_1 = 1/16 \Rightarrow c_2 = -1/16.
\end{aligned}$$

The solution of the IVP is:

$$\boxed{y = \frac{1}{16}e^{3t} - \frac{1}{16}e^{-t} - \frac{1}{4}te^{-t} - 2}$$

One can also use the Laplace transform or variation of parameters to solve this problem.

♦

3.) Consider the differential equation: $xy'' + 2y' + xy = 0$.
(a)[04] Which values of x are ordinary, regular singular, or irregular singular points?

Solution:

Coefficient of y'' vanishes only when $x = 0$ which means all points but $x = 0$ are ordinary points. $x = 0$ is a singular point. To determine if regular or irregular we check the limits:

$$\left.\begin{array}{l} \lim_{x\to 0} x\dfrac{2}{x} = 2 \\ \\ \lim_{x\to 0} x^2\dfrac{x}{x} = 0 \end{array}\right\} \quad \text{both finite} \Rightarrow x = 0 \,\text{is a regular singular point.}$$

There is no irregular singular point.

♦

(b)[16] There is only one regular singular point, call it x_0. Find a fundamental solution near x_0 corresponding to the bigger root of the indicial equation. What is the sum of this series solution? How does it behave when x is close to 0?

Solution:

We first conclude that $x_0 = 0$. We start with $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ and compute

$$y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}.$$

Plugging into the equation we receive:

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-1} + 2\sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0.$$

We transform indices so that powers of x coincide. One way is in the last summand set $n+1 \to n-1$. It becomes $\sum_{n=2}^{\infty} a_{n-2}x^{n+r-1}$. Rewriting the equation:

$$a_0x^{r-1}(r(r-1)+2r) + a_1x^r(r(r+1)+2(r+1))$$

$$+\sum_{n=2}^{\infty}[a_n \underbrace{((n+r)(n+r-1)+2(n+r))}_{(n+r)(n+r+1)} + a_{n-2}]x^{n+r-1} = 0.$$

The minimum power gives the indicial equation, $a_0 \neq 0$, and it is: $r(r-1)+2r = 0 \Leftrightarrow r^2 + r = 0 \Leftrightarrow r_1 = 0,\ r_2 = -1$. These are exponents of singularity. Note that $r_1 - r_2 = 1$ is an integer. Still a non-logarithmic solution exists for the bigger root $r_1 = 0$. From this point on set $r = 0$. Recurrence relation becomes:

$$2a_1 = 0 \quad \text{and} \quad a_k = -\frac{a_{k-2}}{k(k+1)},\ k \geq 2.$$

We conclude that $a_{2k+1} = 0$ as $a_1 = 0$ and every coefficient is determined by the two-less indexed one. Let us understand the even numbered coefficients:

$$a_{2k} = -\frac{a_{2k_2}}{(2k+1)2k} = \frac{a_{2k-4}}{(2k+1)2k(2k-1)(2k-2)} = \cdots = (-1)^k\frac{a_0}{(2k+1)!}.$$

Then a fundamental solution (set $a_0 = 1$) for $r = 0$ is:

$$\begin{aligned} y &= x^0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \\ &= \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right) \\ &= \boxed{\frac{\sin x}{x}} \end{aligned}$$

Behaviour near $x = 0$: $\lim_{x\to 0} \frac{\sin x}{x} = 1$, meaning that it does not blow up.

Note that the other solution, for $r = -1$ is also a usual series without a logarithm. Namely $\cos x/x$.

♦

4.)[20] Find the solution of the initial value problem:

$$y'' + 4y' = \delta(t-\pi)\sin(t/6), \quad y(0) = 2,\ y'(0) = 1.$$

Solution:

Applying the Laplace transform:

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) &= \mathcal{L}\{\delta(t-\pi)\sin(t/6)\} \\ &= \int_0^\infty e^{st}\delta(t-\pi)\sin(t/6)dt \\ &= e^{-\pi s}\sin(\pi/6) = \frac{e^{-\pi s}}{2}. \end{aligned}$$

Using the given conditions and isolating $Y(s)$ we obtain:

$$\begin{aligned} Y(s) &= \frac{2}{s+4} + \frac{9}{s^2+4s} + \frac{e^{-\pi s}}{2(s^2+4s)} \\ &= \frac{2}{s+4} + \frac{9}{(s+2)^2-4} + \frac{e^{-\pi s}}{2[(s+2)^2-4]}. \end{aligned}$$

We are now ready for the inversion process:

$$\boxed{y(t) = 2e^{-4t} + \frac{9}{2}e^{-2t}\sinh 2t + \frac{1}{4}u_\pi(t)e^{-2(t-\pi)}\sinh 2(t-\pi)}$$

Or if you use partial fractions an equivalent answer is:

$$\boxed{y(t) = \frac{9}{4} - \frac{1}{4}e^{-4t} + \frac{1}{4}u_\pi(t)e^{-2(t-\pi)}\sinh 2(t-\pi)}$$

or

$$\boxed{y(t) = \frac{9}{4} - \frac{1}{4}e^{-4t} + \frac{1}{8}u_\pi(t)(1-e^{-4(t-\pi)})}$$

In the solution above two identities have been used:

1. $\mathcal{L}\{e^{ct}f(t)\} = F(s-c)$
2. $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$

♦

5.)[20] Find the fundamental matrix $\mathbf{\Psi}$ for the linear system:

$$\begin{aligned} x_1' &= -x_1 - x_2 \\ x_2' &= x_1 - x_2 \end{aligned} \quad \text{such that} \quad \boldsymbol{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Solution:

Let us first find the eigenvalue of the coefficient matrix $\boldsymbol{A} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$ by evaluating the determinant $|\boldsymbol{A} - \lambda \boldsymbol{I}|$:

$$\begin{vmatrix} -1-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = (1+\lambda)^2 + 1 = 0 \Rightarrow \lambda = -1 \pm i.$$

So, eigenvalues are complex conjugates. It suffices to find an eigenvector for one of these eigenvalues. Set $\lambda = -1 + i$ and determine the solution space of $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v} = 0$. Row reducing the coefficient matrix:

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -i \\ 1 & -i \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

Thus, components of solution satisfy $v_1 = iv_2$. We choose an eigenvector to be: $\boldsymbol{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$ and write the solution:

$$\boldsymbol{x} = \boldsymbol{v}e^{\lambda t} = \begin{bmatrix} i \\ 1 \end{bmatrix} e^{-t}(\cos t + i \sin t) = e^{-t} \begin{bmatrix} i\cos t - \sin t \\ \cos t + i \sin t \end{bmatrix}$$

Separating here real and imaginary parts:

$$\boldsymbol{x} = \underbrace{\begin{bmatrix} -e^{-t}\sin t \\ e^{-t}\cos t \end{bmatrix}}_{\boldsymbol{x}^{[1]}} + i \underbrace{\begin{bmatrix} e^{-t}\cos t \\ e^{-t}\sin t \end{bmatrix}}_{\boldsymbol{x}^{[2]}}$$

we obtain two fundamental solutions. The general solution is given by the superposition $\boldsymbol{x} = c_1\boldsymbol{x}^{[1]} + c_2\boldsymbol{x}^{[2]}$ where the constants are to be determined by the IC:

$$\boldsymbol{x}(0) = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow c_1 = 1,\ c_2 = 2.$$

Hence the fundamental matrix required is:

$$\boxed{\mathbf{\Psi} = \begin{bmatrix} -e^{-t}\sin t & 2e^{-t}\cos t \\ e^{-t}\cos t & 2e^{-t}\sin t \end{bmatrix}}$$

Note that columns of this matrix may be interchanged.

♦

6.) Consider $f(x) = \begin{cases} 1 & 0 < x \le \pi/2 \\ 2 & \pi/2 < x \le \pi \end{cases}, \quad f(x+\pi) = f(x).$

(a)[07] Sketch the graph of the function to which the Fourier series of $f(x)$ converges over $[-\pi, \pi]$ [Justify how you draw the picture].

Solution:

We use the convergence theorem for Fourier series:

Fourier series of $f(x) \to \dfrac{f(x+) + f(x-)}{2}$ at every x where f is discontinuous and it converges to $f(x)$ at every x where f is continuous.

(b)[06] Extend $f(x)$ as an *odd* function on $[-\pi, \pi]$ and sketch the graph of the extended function over this interval. What is the period of this extension?

Solution:

We extend f so that it becomes odd. To do that we need to enlarge the period, i.e. we define f on the symmetric interval in the following way:

$$f_{\text{odd}} = \begin{cases} 1 & 0 < x \le \pi/2 \\ 2 & \pi/2 < x < \pi \\ -1 & -\pi/2 \le x < 0 \\ -2 & -\pi < x < -\pi/2 \\ 0 & x = 0, \pi \end{cases}$$

The extended function has the fundamental period 2π.

(c)[12] Find the Fourier sine series of $f(x)$. Sketch the graph of the function to which the Fourier sine series converges over $[-\pi, \pi]$.

Solution:

The Fourier sine series of $f(x)$ is given by:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

where:

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, ...$$

In this problem $L = \pi$: half period for the odd extension. Using the definition of $f(x)$ as well, the coefficient integral becomes:

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx \\ &= \frac{2}{\pi} \int_0^{\pi/2} \sin nx dx + \frac{2}{\pi} \int_{\pi/2}^{\pi} 2 \sin nx dx \\ &= -\frac{2}{n\pi} \cos nx \Big|_0^{\pi/2} - \frac{4}{n\pi} \cos nx \Big|_{\pi/2}^{\pi} \\ &= \frac{2}{n\pi} \left[1 + \cos \frac{n\pi}{2} - 2 \cos n\pi\right] \end{aligned}$$

Hence the Fourier sine series is explicitly:

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \left[1 + \cos \frac{n\pi}{2} - 2 \cos n\pi\right] \sin nx.$$

♦

7.)[25] Using separation of variables solve the heat conduction problem described by:

$$\begin{aligned} &u_t = u_{xx}\ , \quad 0 < x < \pi \quad \text{subject to} \\ &u_x(0,t) = 0, \quad u_x(\pi,t) = 0, \\ &u(x,0) = \pi + 2\cos 3x - \pi^2 \cos 10x. \end{aligned}$$

Solution:

SoV means setting $u(x,t) = X(x)T(t)$ so that the heat equation becomes $XT' = X''T$. By separation we get:

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda \Rightarrow \begin{cases} X'' + \lambda X = 0 \\ T' + \lambda T = 0 \end{cases}$$

On the other hand as usual the boundary conditions, being homogeneous, can be imposed only on X:

$$\begin{aligned} u_x(0,t) = X'(0)T(t) = 0 \Rightarrow X'(0) = 0 \\ u_x(\pi,t) = X'(\pi)T(t) = 0 \Rightarrow X'(\pi) = 0 \end{aligned}$$

The initial condition remains to be considered later as it is not homogeneous. Let us now find the eigenvalues of the problem:

Case 1: $\lambda > 0$ Set $\lambda = a^2$. Solve $X'' + a^2X = 0$ together with the BC above. Clearly $X = c_1 \cos ax + c_2 \sin ax$ and $X' = -ac_1 \sin ax + ac_2 \cos ax$. Using the BC: $X'(0) = ac_2 = 0 \Leftrightarrow c_2 = 0$ $(a \neq 0)$. The other BC: $X'(\pi) = -ac_1 \sin a\pi = 0$, in order not to make the solution a trivial one we seek $a \neq 0$ such that $\sin a\pi = 0$. We find that $a = 1, 2, 3, ...$ work or $a = n$, $n = 1, 2, ...$ Conclusion:

$$\lambda = n^2 \quad \text{and} \quad X_n = \cos nx, \quad n = 1, 2, ...$$

are eigenvalues and eigenfunctions, respectively.

Case 2: $\lambda < 0$ Set $\lambda = -a^2$. Then $X = c_1e^{ax} + c_2e^{-ax}$ and $X' = ac_1e^{ax} - ac_2e^{-ax}$. Using the BC: $X'(0) = c_1 - c_2 = 0 \Rightarrow c_1 = c_2$, and using the other one $X'(\pi) = ac_1(e^{\pi a} - e^{-\pi a}) = 0$ which has no solution unless $c_1 = 0$ or $a = 0$ both of which are not allowed. No eigenvalue, no eigenfunction.

Case 3: $\lambda = 0$ We solve $X'' = 0$. Easily we find that $X = c_1 + c_2x$ and $X' = c_2$. Boundary conditions are on the derivative: $X'(0) = c_2 = 0$ and $X'(\pi) = c_2 = 0$ are both satisfied if $c_2 = 0$. Note that c_1 remains free. Hence $X = c_1$ solves the equation and satisfies the BC for any number c_1. Conclusion:

$$\lambda = 0 \quad \text{and} \quad X_0 = 1$$

are eigenvalue and eigenfunction respectively.

Case 1 and Case 3 can easily be combined to find the complete set of eigenvalues and eigenfunctions:

$$\lambda = n^2 \quad \text{and} \quad X_n = \cos nx, \quad n = 0, 1, 2, ...$$

Now solve the equation for T: but this is easy, $T = \exp(-\lambda t)$ gives fundamental solutions to be $T_n = e^{-n^2 t}$, for $n = 0, 1, 2, ...$

We now superpose all solutions and use the initial condition. Namely:

$$u(x,t) = \sum_{n=0}^{\infty} a_n X_n T_n = \sum_{n=0}^{\infty} a_n e^{-n^2 t} \cos nx.$$

We now impose the initial condition:

$$u(x,0) = \sum_{n=0}^{\infty} a_n \cos nx = \pi + 2\cos 3x - \pi^2 \cos 10x$$

which is satisfied only when

$$a_0 = \pi, \quad a_3 = 2, \quad a_{10} = -\pi^2$$

and all $a_j = 0$, for $j \neq 0, 3, 10$. Hence the solution of the initial-boundary value problem is:

$$\boxed{u(x,t) = \pi + 2e^{-9t}\cos 3x - \pi^2 e^{-100t}\cos 10x}$$

♦

Some basic Laplace transforms you might need

$\mathcal{L}\{1\} = \frac{1}{s}$, $s > 0$	$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, $s > a$	$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$, $s > 0$
$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$, $s > 0$	$\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$, $s > \|a\|$	$\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$, $s > \|a\|$

Solutions to Math 202: Final Exam. Jan. 06, 2005

1. (10 pnts) Given $y'' + \frac{1}{x}y' - y = 0$, show that although $x = 0$ is a regular singular point you can still **find (!)** a solution which is *analytic* (therefore *continuous*) at $x = 0$. **Prove** that its domain of convergence is the whole real line. **Solution:**$y(x) = \sum_{n=0}^{\infty} a_n x^n \Longrightarrow \frac{a_1}{x} + \sum_{n=2}^{\infty}[n^2 a_n - a_{n-2}]x^{n-2} = 0 \Longrightarrow a_1 = 0, a_n = \frac{a_{n-2}}{n^2}$,let $a_0 = 1 \Longrightarrow y = 1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2^2 4^2 6^6 \cdots (2n)^2} = 1 + \frac{1}{2^2}x^2 + \frac{1}{2^2 4^2}x^4 + \frac{1}{2^2 4^2 6^6}x^6 + ..$which is an analytic function at $x = 0$. Ratio Test gives $Lim_{n\to\infty}|\frac{a_{2(n+1)}}{a_{2n}}| = Lim_{n\to\infty}|\frac{x^2}{(2n+2)^2}| = 0 < 1$,i.e the radius of convergence is ∞, i.e. the domain of convergence is the whole real line

2. (5 pnts) Given the **integral equation** $f(t) = 2t - \int_0^t e^{(t-\alpha)} f(\alpha) d\alpha$ for the unknown function $f(t)$, solve it by Laplace transform.

   Solution: Observe that $f(t) = 2t - e^t * f(t)$ and **Convolution Theorem** gives:

   $F(s) = \frac{2}{s^2} - \frac{F(s)}{s-1} \Longrightarrow F(s) = \frac{2s-2}{s^3} = \frac{2}{s^2} - \frac{2}{s^3} \Longrightarrow f(t) = 2t - t^2$.

3. (5 pnts) Given the same **integral equation** $f(t) = 2t - \int_0^t e^{(t-\alpha)} f(\alpha) d\alpha$ by taking its derivative reduce it into an initial value problem of a **first order differential equation and then solve it**. You should use $\frac{d}{dx}\int_0^{h(x)} F(x,y)dy = \int_0^{h(x)} \frac{\partial}{\partial x}F(x,y)dy + F(x,h(x)) \cdot h'(x)$.

   Solution: First observe that $f(0) = 0$. Derivative gives: $f' = 2 + (-\int_0^t e^{(t-\alpha)} f(\alpha)d\alpha) - e^0 f(t) \Longrightarrow f' = 2 + (f - 2t) - f \Longrightarrow f' = 2 - 2t \Longrightarrow f = 2t - t^2 + c, f(0) = 0 \Longrightarrow f(t) = 2t - t^2$.

4. (10 pnts)

   (a) Prove that $\frac{d}{ds}Laplace\{f(t)\} = Laplace\{-t \ f(t)\}$.
   Solution:$\frac{d}{ds}Laplace\{f(t)\} = \frac{d}{ds}\int_0^{\infty} e^{-st} f(t)dt = -\int_0^{\infty} e^{-st} t \cdot f(t)dt = -Laplace\{t \cdot f(t)\}$.

   (b) Use the above fact to solve $ty'' + (t-1)y' + y = 0, y(0) = 0, y'(0) = \alpha = arbitrary$.
   Solution: $-\frac{d}{ds}Laplace\{y''\} - \frac{d}{ds}Laplace\{y'\} - Laplace\{y'\} + Laplace\{y\} = 0 \Longrightarrow -\frac{d}{ds}(s^2Y - s\cdot 0 - \alpha) - \frac{d}{ds}(sY - s\cdot 0) - (sY - s\cdot 0) + Y = 0 \Longrightarrow 2sY + s^2Y' + Y + sY' + sY - Y = 0 \Longrightarrow s(s+1)Y' + 3sY = 0 \Longrightarrow \frac{Y'}{Y} = -\frac{3}{s+1} \Longrightarrow Y(s) = \frac{c}{(s+1)^3} \Longrightarrow y(t) = ce^{-t}t^2$.

5. (10 pnts) Solve the following Boundary Value Probelem: $x^2y'' - 3xy' + (4 + \pi^2)y = 1, y(1) = 0, y(2) = 0$.

   Solution: This is a Cauchy-Euler type differential wquation: Putting $y = x^r \Longrightarrow r(r-1) - 3r + (4 + \pi^2) = 0 \Longrightarrow (r-2)^2 = -\pi^2 \Longrightarrow r = 2 \pm \pi i$

and $y_{HS} = x^2(c_1 \cos \pi \ln x + c_2 \sin \pi \ln x)$. Clearly $y_{PS} = \frac{1}{4+\pi^2}$, thus G.S. $y = x^2(c_1 \cos \pi \ln x + c_2 \sin \pi \ln x) + \frac{1}{4+\pi^2}$. Now the Boundary Conditions: $y(1) = c_1 + \frac{1}{4+\pi^2} = 0, y(2) = 4(c_1 \cos \pi \ln 2 + c_2 \sin \pi \ln 2) + \frac{1}{4+\pi^2} = 0 \Longrightarrow$ $c_1 = -\frac{1}{4+\pi^2}, c_2 =$sen bul!

6. (10 pnts) Solve the following Eigenvalue Probelem: $y'' + \lambda y = 0, y'(0) = 0, y'(\pi) = 0$. Assume that all eigenvalues are real.

   Solution: For $\lambda = 0$ we get $y = c_1 + c_2 x$,and $y' = c_2$. B.C. gives $c_2 = 0$ and we can take $c_1 = 1$. i.e.0. is an e.value with e. fn 1. For $\lambda < 0$ we can set $\lambda = -\mu^2$ with $\mu > 0$. D.E becomes $y'' - \mu^2 y = 0$, which gives $y = c_1 \cosh \mu x + c_2 \sinh \mu x$, with $y' = c_1 \mu \sinh x + c_2 \mu \cosh x$. B.C. gives $c_2 = 0$ and $c_1 \sinh \mu\pi = 0$ which gives $c_1 = 0$ because $\sinh \mu\pi \neq 0$,i.e. no negative eigenvalues. For $\lambda > 0$ we can set $\lambda = \mu^2$ with $\mu > 0$. D.E becomes $y'' + \mu^2 y = 0$ which gives $y = c_1 \cos \mu x + c_2 \sin \mu x$, with $y' = -c_1 \mu \sin \mu x + c_2 \mu \cos \mu x$. B.C. gives $c_2 = 0$ and $-c_1 \mu \sin \mu\pi = 0$ which forces us to set $\sin \mu\pi = 0$ for non-trivial solns, i.e. $\mu\pi = n\pi$ i.e. $\mu = n = 1, 2, 3, ...$ In short $\lambda = n^2$ are the positive eigenvalues with the corresponding eigenfunctions $\cos nx, n = 1, 2, 3, ...$ .To sum up what we found is that $\lambda_n = n^2$ are the (countably infinite) eigenvalues with their corresponding eigenfunctions $y_n(x) = \cos nx, n = 0, 1, 2, ...$

B U Department of Mathematics

Math 202 Differential Equations

Fall 2005 Final Exam

1. Suppose that e^{-t} and $1 - e^{-2t}$ are both solutions to ODE $y'' + by' + ky = q(t)$ where b and k are constants. What are b and k, and what is $q(t)$?

Solution:

Let $y_1 = 1 + e^{-t}$ & $y_2 = 1 - 2e^{-2t}$

$$\Rightarrow \begin{cases} y_1' = -e^{-t} & y_2' = 4e^{-2t} \\ y_1'' = e^{-t} & y_2'' = -8e^{-2t} \end{cases}$$

Since y_1 & y_2 both satisfy $y'' + by' + ky = q(t)$

$$\left.\begin{array}{r} (1-b+k)e^{-t} + k = q(t) \\ (-8+4b-2k)e^{-2t} + k = q(t) \end{array}\right\} \Rightarrow \quad (1-b+k)e^{-t} + k = (-8+4b-2k)e^{-2t} + k$$

$$\Rightarrow \quad 1 - b + k = 0 \quad \text{and} \quad -8 + 4b - 2k = 0$$

So, $b = 3$ and $k = 2$.
Then, $q(t) = 2$ follows.

OR: $y_1 - y_2 = e^{-t} + 2e^{-2t}$ is a solution of $y'' + by' + ky = 0$
$\Rightarrow \quad (D+1)(D+2)y = 0$
$\Rightarrow \quad (D^2 + 3D + 2)y = 0$
$\Rightarrow \quad b = 3 \quad \& \ k = 2.$

2. Given the function f(x) $= \begin{cases} 1 & if \;\; 0 \le x \le \pi/2 \\ 0 & if \;\; \pi/2 \le x \le \pi \end{cases}$

a) Find the Fourier cosine series of f;
b) Find the Fourier sine series of f;
c) Find the Fourier series of f;
d) Graph the three series found over the interval $[-3\pi, 3\pi]$;
e) Using one of the series found show that $\pi/4$ can be expressed as the sum of an alternating series.

Solution:

a) Fourier cosine series of f is the Fourier expansion of E_f, which is

$$E_f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi/2} 1dx = 1$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx = \frac{2}{\pi}\int_0^{\pi/2} \cos(nx)dx = \frac{2}{n\pi}\left[\sin(n\frac{\pi}{2})\right] = \begin{cases} 0 & ,n \;\; even \\ \frac{2}{n\pi}\sin(n\frac{\pi}{2}) & ,n \;\; odd. \end{cases}$$

So,

$$E_f \sim \frac{1}{2} + \frac{2}{\pi}\left[\cos x - \frac{1}{3}\cos(3x) + \frac{1}{5}\cos(5x) - \frac{1}{7}cos(7x) + ...\right]$$

b) Fourier cosine series of f is the Fourier expansion of O_f, which is

$$O_f \sim \sum_{n=1}^{\infty} b_n \sin(nx)$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx = \frac{2}{\pi}\int_0^{\pi/2} \sin(nx)dx = -\frac{2}{n\pi}\left[\cos(n\frac{\pi}{2}) - 1\right]$$

$$= \begin{cases} \frac{2}{n\pi} & ,n \;\; odd \\ 0 & n \;\; even\&n = 4k \\ \frac{4}{n\pi} & n \;\; even\&n \neq 4k \end{cases}$$

Hence,

$$O_f \sim \frac{2}{\pi}\left[\sin x + \sin(2x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + ...\right]$$

c) Now Fourier series; $p = \frac{\pi - 0}{2} = \frac{\pi}{2}$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(\frac{n\pi x}{p}) + b_n \sin(\frac{n\pi x}{p})\right]$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi/2} 1dx = 1$$

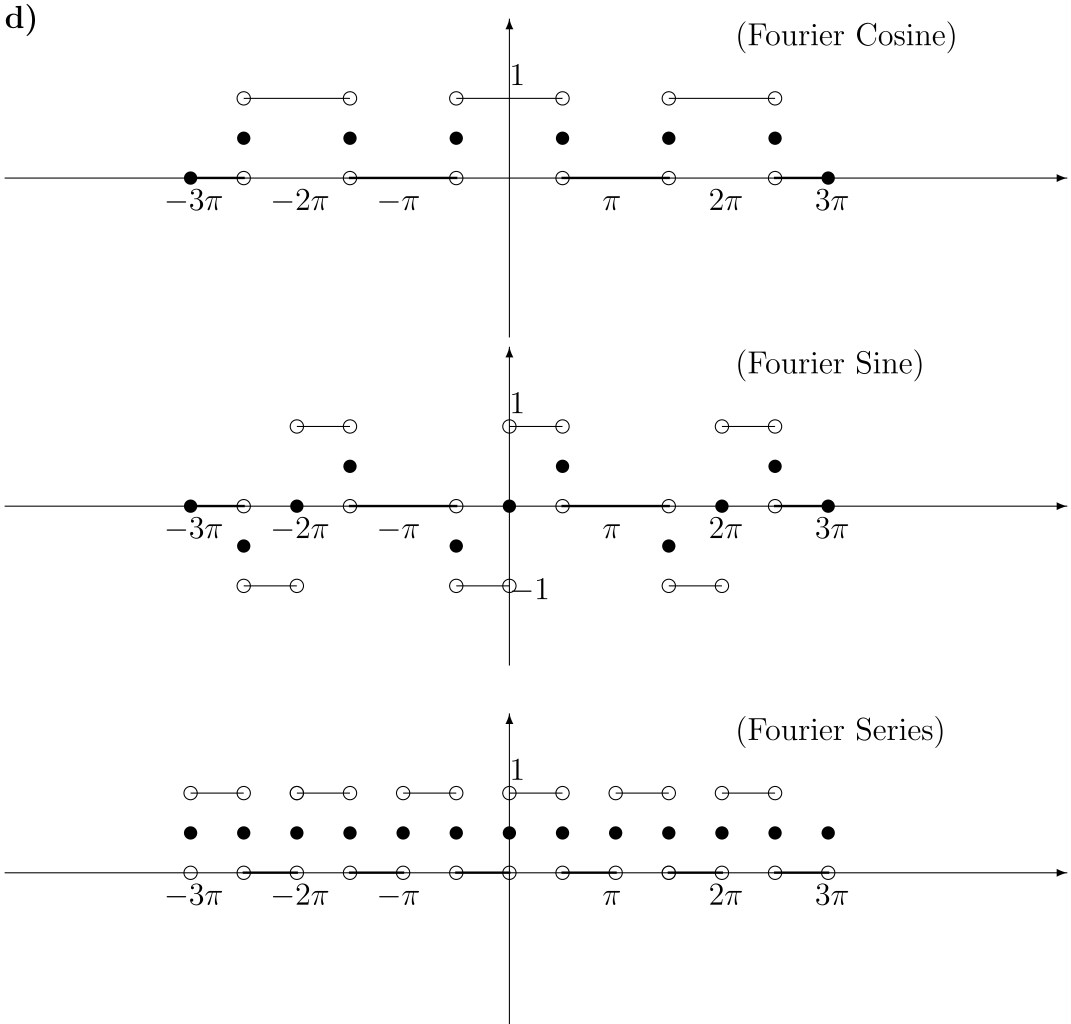
$$a_n = \frac{2}{\pi}\int_0^{\pi/2} 1.\cos(2nx)dx = \frac{1}{n\pi}\left[\sin(n\pi) - 0\right] = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi/2} 1.\sin(2nx)dx = -\frac{1}{n\pi}\left[\cos(n\pi) - 1\right] = \begin{cases} 0 & , n \;\; even \\ \dfrac{2}{n\pi} & , n \;\; odd. \end{cases} \quad \text{So,}$$

$$f(x) \sim \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left[2(2n-1)x\right]$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\sin(2x) + \frac{1}{3}\sin(6x) + \frac{1}{5}\sin(10x) + ...\right]$$

d)



e) Using part (c) for $x = \pi/4$

$$f(\frac{\pi}{4}) = 1 = \frac{1}{2} + \frac{2}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...\right]$$

$$\Rightarrow \frac{\pi}{4} = \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...\right]$$

3. An n^{th} order homogenous linear differential equation with constant coefficients has characteristic equation $f(r) = 0$. If all the roots of characteristic equation are negative, find limit as $x \to \infty$ of any solution of the differential equation, if this limit exists. What can you conclude about the behavior of all solutions on the interval $[0, \infty)$, if all the roots of the characteristic equation are non-positive ?

Solution:

Since differential equation is order n, $f(r) = 0$ has n roots. It's given that all roots are negative.

CASE1: If all n roots $r_1, r_2, ..., r_n$ of $f(r) = 0$ are distinct then any solution is of the form

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + ... + c_n e^{r_n x} \quad for\ some \quad c_1, c_2, ..., c_n \in \mathbb{R}.$$

For $r < 0$, $\lim_{x\to\infty} e^{rx} = 0$.
Hence $\lim_{x\to\infty} y = 0$ follows.

CASE2: If some root, say r, of $f(r) = 0$ is repeated then any solution involves a term $(c_1 + c_2 x + ... + c_k x^{k-1})e^{rx}$ for $2 \le k \le n$ & for some $c_1, c_2, ..., c_n \in \mathbb{R}$.

As $\lim_{x\to\infty} x^k e^{rx} = 0$ (for $r < 0$) again $\lim_{x\to\infty} y = 0$ for any solution y.

Now, if any root of $f(r) = 0$ satisfy that $r \le 0$ then

$$\lim_{x\to\infty} e^{rx} = 1 \quad if \quad r = 0$$
$$\lim_{x\to\infty} e^{rx} = 0 \quad if \quad r < 0$$

Hence any solution has a horizontal asymptote if root of $f(r) = 0$ are distinct.

In case that a root r of$f(r) = 0$ is repeated and $r = 0$, then any solution includes a term of the form $c_1 + c_2 x + ... + c_k x^{k-1}$ for $2 \le k \le n$ then limit of the solution as $x \to \infty$ becomes infinity.

4. a) What function f(t) has Laplace transform $\dfrac{1}{s(s^2+4s+8)}$?
b) Write down an Initial Value Problem whose solution is this function f(t).(Don't neglect the initial conditions!)

Solution:

a) $$\varphi(s) = \frac{1}{s(s^2+4s+8)} = \frac{(1/8)}{s} - \frac{(1/8)s+(1/2)}{s^2+4s+8}$$
$$= \frac{1}{8}\left(\frac{1}{s}\right) - \frac{1}{8}\frac{s+2}{(s+2)^2+2^2} - \frac{1}{8}\frac{2}{(s+2)^2+2^2}$$
$$\Rightarrow f(t) = \pounds^{-1}[\varphi(s)] = \frac{1}{8} - \frac{1}{8}e^{-2t}(\cos 2t + \sin 2t)$$

b) Consider a constant coefficient homogenous linear D.E. so that a solution is f(t).

The roots of its characteristic equation are,

$$0, \quad -2+2i, \quad -2-2i$$

As $(-2+2i)+(-2-2i) = -4$
$(-2+2i)\cdot(-2-2i) = 4+4 = 8$,

Characteristic Equation: $r(r^2+4r+8) = 0$

Hence D.E.: $y''' + 4y'' + 8y' = 0$

Now, $f(t) = \frac{1}{8} - \frac{1}{8}e^{-2t}(\cos 2t + \sin 2t)$ is a solution of this D.E.

For $f(0) = 0$

& $f'(t) = \frac{1}{4}e^{-2t}(\cos 2t + \sin 2t) - \frac{1}{8}e^{-2t}(-2\sin 2t + 2\cos 2t)$

& $f'(0) = \frac{1}{4} - \frac{2}{8} = 0$

& $f''(t) = e^{-2t}(-\sin 2t + \cos 2t)$

& $f''(0) = 1$

Hence IVP: $y''' + 4y'' + 8y' = 0, \qquad y(0) = y'(0) = 0 \ \& \ y''(0) = 1.$

5. a) Find a real 2×2 matrix A whose eigenvalues are 2 and -1, with corresponding eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ respectively;

b) Write down the general solution of the system of equations given by $\overrightarrow{X'(t)} = A\overrightarrow{X(t)}$;

c) Find the general solution of the non-homogenous system $\overrightarrow{X'(t)} = A\overrightarrow{X(t)} + \begin{bmatrix} e^t \\ 0 \end{bmatrix}$

Solution:

a) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

For $\lambda = 2$ & corresponding eigenvector $\overrightarrow{c} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$(A - 2I)\overrightarrow{c} = \overrightarrow{0} \quad \Rightarrow \quad a + 2b = 2 \quad and \quad c + 2d = 4$

For $\lambda = -1$ & corresponding eigenvector $\overrightarrow{c} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$(A + I)\overrightarrow{c} = \overrightarrow{0} \quad \Rightarrow \quad 2a + b = -2 \quad and \quad 2c + d = -1$

$\Rightarrow \quad A = \begin{bmatrix} -2 & 2 \\ -2 & 3 \end{bmatrix}$

b) General solution:

$$\overrightarrow{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} = \begin{pmatrix} e^{2t} & 2e^{-t} \\ 2e^{2t} & e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad c_1, c_2 \in \mathbb{R}$$

c) Let $\overrightarrow{X_p}(t) = \begin{pmatrix} e^{2t} & 2e^{-t} \\ 2e^{2t} & e^{-t} \end{pmatrix} \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}$ be a particular solution of the non-homogenous system.

Then $\begin{pmatrix} e^{2t} & 2e^{-t} \\ 2e^{2t} & e^{-t} \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} e^t \\ 0 \end{pmatrix}$ must hold.

$\Rightarrow c_1' = -\frac{1}{3}e^{-t}, \qquad c_2' = \frac{2}{3}e^{2t}$

$\Rightarrow c_1 = \frac{1}{3}e^{-t}, \qquad c_2 = \frac{1}{3}e^{2t} \qquad \Rightarrow \qquad \overrightarrow{X_p}(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$

Thus the general solution of the non-homogenous system

$$\overrightarrow{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{2t} & 2e^{-t} \\ 2e^{2t} & e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} e^t \\ e^t \end{pmatrix}, \quad c_1, c_2 \in \mathbb{R}$$

6. Find a linear differential equation with constant coefficients satisfied by all the given functions:

$$u_1(x) = \cosh x, \quad u_2(x) = \sinh x, \quad u_3(x) = x\cosh x, \quad u_4(x) = x\sinh x.$$

Solution:

$\cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$
$\sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$
So, u_1, u_2, u_3, u_4 satisfy $(D^2 - 1)^2 y = 0$
i.e. $y^{(4)} - 2y'' + y = 0$

7. Find the function $y(t)$ that satisfies the integral equation

$$y(t) = t^2 + \int_0^t y(u)\sin(t-u)du.$$

Solution:

$$\int_0^t y(u)\sin(t-u)du = \sin t * y(t)$$

Applying Laplace transform
$\mathcal{L}[y(t)] = \mathcal{L}[t^2] + \mathcal{L}[\sin t * y(t)]$

$$= \frac{2!}{s^3} + \underbrace{\mathcal{L}[\sin t]}_{\frac{1}{s^2+1}} \cdot \mathcal{L}[y(t)]$$

$$\Rightarrow \mathcal{L}[y(t)]\left(1 - \frac{1}{s^2+1}\right) = \frac{2}{s^3}$$

$$\Rightarrow \mathcal{L}[y(t)] = \frac{2(s^2+1)}{s^5} = 2\left[\frac{1}{s^3} + \frac{1}{s^5}\right]$$

$$\Rightarrow y(t) = 2\left[\frac{t^2}{2} + \frac{t^4}{4!}\right]$$

B U Department of Mathematics

Math 202 Differential Equations

Date: June 2, 2004	Full Name :
Time: 15:00-17:30	Math 202 Number :
	Student ID :

Spring 2004 Final Exam Solution Key

IMPORTANT

1. Write your name, surname on top of each page. **2.** The exam consists of 7 questions some of which have more than one part. **3.** Read the questions carefully and write your answers neatly under the corresponding questions. **4.** Show all your work. Correct answers without sufficient explanation might not get full credit. **5.** Calculators are not allowed.

Q1	Q2	Q3	Q4	Q5	Q6	Q7	TOTAL
20 pts	20 pts	20 pts	20 pts	20 pts	25 pts	25 pts	150 pts

1.)[20] Solve the following differential equation by finding an integrating factor $\mu = \mu(x)$:

$$(x+2)\sin y\; dx + x\cos y\; dy = 0.$$

Solution:

Realize that
$$\frac{\partial M}{\partial y} = (x+2)\cos y \neq \cos y = \frac{\partial N}{\partial x}$$
and the differential eqution is not exact. Then

$$p(x) = \frac{(x+2)\cos y - \cos y}{x\cos y} = \frac{x+1}{x} = 1 + \frac{1}{x}$$

and $\mu(x) = e^{\int p(x)dx} = e^{\int (1+\frac{1}{x})dx} = e^{x+\ln x} = xe^x$. So multiplying by this integrating factor, $\mu(x) = xe^x$, the following differential equation:

$$(xe^x(x+2)\sin y)dx + (x^2e^x\cos y)dy = 0$$

is exact.

Let $\Phi = \displaystyle\int (x^2e^x\cos y)dy = x^2e^x\sin y + \psi(x)$. Then

$$\frac{\partial \Phi}{\partial x} = \psi'(x) + 2xe^x\sin y + x^2e^x\sin y = xe^x(x+2)\sin y = x^2e^x\sin y + 2xe^x\sin y.$$

So $\psi'(x) = 0$ and $\psi(x) = c$ a constant.

The family of solutions is : $x^2e^x\sin y + c = 0$.

2.)[20] Find the general solution of the differential equation:

$$y''' - y' = 3t + \cos t.$$

Solution:

The characteristic equation is $r^3 - r = 0$. Then we get three distinct roots. $r = 0, +1, -1$. The homogeneous solution is

$$y_h(t) = c_1 + c_2 e^t + c_3 e^{-t}.$$

To find the particular solution we use the the method of undetermined coefficients. Let $y_p(t) = At + Bt^2 + C\cos t + D\sin t$. Then

$$y_p'(t) = A + 2Bt - C\sin t + D\cos t,$$

$$y_p''(t) = 2B - C\cos t - D\sin t,$$
$$y_p'''(t) = C\sin t - D\cos t.$$

Plugging into the differential equation, we get:

$$C\sin t - D\cos t - A - 2Bt + C\sin t - D\cos t = 3t + \cos t.$$

Solving for the coefficients: $A = 0, B = -3/2, C = 0, D = -1/2$. Then the particular solution is: $y_p(t) = -\frac{3}{2}t^2 - \frac{1}{2}\sin t$.

The general solution becomes: $y(t) = -\frac{3}{2}t^2 - \frac{1}{2}\sin t + c_1 + c_2 e^t + c_3 e^{-t}$.

3.) Consider the differential equation about the point $x = 0$:

$$y'' + 4x^2 y = 0.$$

(a)[10] Find the recurrence relation which defines two fundamental series solutions about $x = 0$.

Solution:

First note that $x = 0$ is an ordinary point. Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, and $y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$. Plugging into the DE,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 4\sum_{n=0}^{\infty} a_n x^{n+2} = 0.$$

By rearranging the coefficients and the indicies we get,

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=2}^{\infty} 4a_{n-2}x^n = 0.$$

From $n = 0$: $2a_2 = 0$ and from $n = 1$: $6a_3 = 0$. Both $a_2 = 0 = a_3$. The recurrence relation is:

$$(n+2)(n+1)a_{n+2} + 4a_{n-2} = 0.$$

(b)[10] Compute the first <u>three non-zero</u> terms of each fundamental solution about $x = 0$.

Solution:

As a_2 and a_3 is both zero, from the recurrence relation we see that $a_{4k+2} = 0$ and $a_{4k+3} = 0$ for all $k \in \mathbb{Z}^+$. Then

$$a_4 = -\frac{4(a_0)}{4(3)} = -\frac{a_0}{3}, \quad a_8 = -\frac{4(\frac{-a_0}{3})}{8(7)} = \frac{a_0}{42}.$$

$$a_5 = -\frac{4(a_1)}{5(4)} = -\frac{a_1}{5}, \quad a_9 = -\frac{4(\frac{-a_1}{5})}{9(8)} = \frac{a_1}{90}.$$

The two fundamental solutions y_1 and y_2 with three non-zero terms will be as follows:

$$y_1 = 1 - \frac{1}{3}x^4 + \frac{1}{42}x^8 + \cdots$$

$$y_2 = x - \frac{1}{5}x^5 + \frac{1}{90}x^9 + \cdots.$$

4.)[20] Solve the integro-differential equation:

$$y'(t) + 2y(t) + \int_0^t y(\tau)d\tau = \sin t \quad \text{where} \quad y(0) = 1.$$

Solution:

Start by taking the Laplace transform of both sides of the equation.

$$\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} + \mathcal{L}\{\int_0^t y(\tau)d\tau\} = \mathcal{L}\{\sin t\}$$

$$s\mathcal{L}\{y\} - y(0) + 2\mathcal{L}\{y\} + \mathcal{L}\{\int_0^t 1 \cdot y(\tau)d\tau\} = \frac{1}{s^2+1}$$

$$s\mathcal{L}\{y\} - 1 + 2\mathcal{L}\{y\} + \mathcal{L}\{1 \star y\} = \frac{1}{s^2+1}$$

$$(s+2)\mathcal{L}\{y\} + \mathcal{L}\{1\}\mathcal{L}\{y\} = \frac{1}{s^2+1} + 1$$

Simplifying

$$(s+2)\mathcal{L}\{y\} + \frac{1}{s}\mathcal{L}\{y\} = \frac{1}{s^2+1} + 1$$

$$\frac{(s+1)^2}{s}\mathcal{L}\{y\} = \frac{1}{s^2+1} + 1$$

$$\mathcal{L}\{y\} = \frac{1}{(s^2+1)}\frac{s}{(s+1)^2} + \frac{s}{(s+1)^2}$$

Split the right hand side into partial fractions:

$$\mathcal{L}\{y\} = \frac{As+B}{s^2+1} + \frac{Cs+D}{(s+1)^2} + \frac{s+1}{(s+1)^2} - \frac{1}{(s+1)^2}$$

$$\mathcal{L}\{y\} = \frac{1/2}{s^2+1} + \frac{-1/2}{(s+1)^2} + \frac{1}{s+1} - \frac{1}{(s+1)^2}.$$

After taking the inverse Laplace transform,

$$y(t) = \frac{1}{2}\sin t - \frac{1}{2}e^{-t}t + e^{-t} - e^{-t}t.$$

$$y(t) = \frac{1}{2}\sin t - \frac{3}{2}e^{-t}t + e^{-t}.$$

5.)[20] Let $f(t) = \llbracket t \rrbracket$ be the function of greatest integer value less than or equal to t for $t \geq 0$. That is, if $n \in \mathbb{Z}^+ \cup \{0\}$ and $n \leq t < n+1$, then $f(t) = n$. Find the Laplace transform of $f(t)$.
[Hint: $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ if $|x| < 1$.]

Solution:

Now, realize that the greatest integer function $f(t)$ can be represented in terms of step functions as follows:

$$f(t) = (u_1 - u_2) + 2(u_2 - u_3) + 3(u_3 - u_2) + 4(u_4 - u_3) + \cdots$$

$$f(t) = u_1 + u_2 + u_3 + u_4 + \cdots$$

So if we take the Laplace transform of $f(t)$ we get:

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u_1(t)\} + \mathcal{L}\{u_2(t)\} + \mathcal{L}\{u_3(t)\} + \cdots$$

$$\mathcal{L}\{f(t)\} = \sum_{n=1}^{\infty} \mathcal{L}\{u_n(t)\}$$

$$\mathcal{L}\{f(t)\} = \sum_{n=1}^{\infty} \frac{e^{-ns}}{s}$$

By using the hint and the fact that $|e^{-s}| < 1$ when $s > 0$,

$$\mathcal{L}\{f(t)\} = \frac{1}{s} \sum_{n=1}^{\infty} (e^{-s})^n = e^{-s} \left(\frac{1}{s} \sum_{n=0}^{\infty} (e^{-s})^n \right) = e^{-s} \left(\frac{1}{s} \frac{1}{(1 - e^{-s})} \right).$$

After simplifying

$$\mathcal{L}\{f(t)\} = \frac{1}{s} \frac{1}{(e^s - 1)}.$$

Some basic Laplace transforms you might need for Q4 and Q5

$\mathcal{L}\{1\} = \frac{1}{s},\ s > 0$	$\mathcal{L}\{e^{at}\} = \frac{1}{s-a},\ s > a$	$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2},\ s > 0$
$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2},\ s > 0$	$\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2},\ s > \|a\|$	$\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2},\ s > \|a\|$

6.)[25] Solve the following system of differential equations:

$$\begin{aligned} x_1' - 2x_1 &= 0 \\ x_2' + x_1 - 4x_2 &= 0 \\ x_3' + 3x_1 - 6x_2 - 2x_3 &= 0, \end{aligned}$$

where $x_i = x_i(t)$ for each $i = 1, 2, 3$. Show that the solutions you have found are linearly independent.

Solution:

We can write the system as a matrix equation in the following manner.

$$\mathbf{X}' = \mathbf{A}\mathbf{X} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & 0 \\ -3 & 6 & 2 \end{bmatrix} \mathbf{X}$$

where $\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{X}' = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$.

We find the eigenvalues for $\mathbf{A}$. $0 = \det(\mathbf{A} - \lambda\mathbf{I}) = (2-\lambda)(4-\lambda)(2-\lambda)$. We get two eigenvalues $\lambda_1 = 2$ with multiplicity two and $\lambda_2 = 4$ with multiplicity one.

For $\lambda_1 = 2$:

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & 0 \\ -3 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We get $x_1 = -2x_2$, and x_2 and x_3 are free. Hence, we get two linearly independent eigenvectors: $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda_2 = 4$:

$$\begin{bmatrix} -2 & 0 & 0 \\ -1 & 0 & 0 \\ -3 & 6 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -6 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

We get $x_1 = 0$, $x_2 = \frac{1}{3}x_3$, and x_3 is free. Hence, we get an eigenvector: $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$.

The solution to the system becomes:

$$\mathbf{X} = c_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} e^{4t}.$$

To show that the solutions are linearly independent, realize that $\mathbf{v}_1 e^{2t}, \mathbf{v}_3 e^{4t}$ and $\mathbf{v}_2 e^{2t}$ form the columns of the 3×3 lower triangular matrix:

$$\begin{bmatrix} 2e^{2t} & 0 & 0 \\ e^{2t} & e^{4t} & 0 \\ 0 & 3e^{4t} & e^{2t} \end{bmatrix}$$

which is also invertible. (Its determinant $2e^{8t}$ is non-zero.)

7.)[25] Find the eigenvalues $\lambda \in \mathbb{R}$ and eigenfunctions for the two-point boundary value problem:

$$y'' + 2y' + \lambda y = 0 \quad \text{with} \quad y(0) = y(1) = 0.$$

[Hint: You are supposed to find an infinite number of eigenvalues.]

Solution:

The characteristic equation of the DE is $r^2 + 2r + \lambda = 0$. We obtain two roots $r_{1,2} = -1 \pm \sqrt{1-\lambda}$. We will analyze the three possible cases:

CASE 1: $1 - \lambda > 0$ Let $1 - \lambda = \alpha^2$, then

$$y = c_1 e^{(\alpha-1)t} + c_2 e^{(-\alpha-1)t}.$$

Plug in the boundary conditions, $y(0) = 0 = c_1 + c_2$, so $c_2 = -c_1$. Also $y(1) = 0 = c_1(e^{(\alpha-1)} - e^{(-\alpha-1)}$ implies $e^{(\alpha-1)} = e^{(-\alpha-1)}$. Then $\alpha - 1 = -\alpha - 1$, and $\alpha = 0$. This is contradictory to α^2 being positive. Hence $c_1 = 0 = c_2$. The only solution is the trivial solution, $y = 0$.

CASE 2: $1 - \lambda = 0$ In this case, there are repeated roots, $r_1 = -1 = r_2$. The solution becomes:

$$y = c_1 e^{-t} + c_2 t e^{-t}.$$

Plug in the boundary conditions, $y(0) = 0 = c_1$ and $y(1) = 0 = c_2/e$. So $c_2 = 0$ as well. We again get the trivial solution.

CASE 3: $1 - \lambda < 0$ Let $1 - \lambda = -\alpha^2$, then we get the solution as

$$y = e^{-t}(c_1 \cos \alpha t + c_2 \sin \alpha t).$$

Plug in the boundary conditions, $y(0) = 0 = 1c_1(1)$ and $y(1) = 0 = c_2 \sin \alpha$. If $\sin \alpha = 0$, then $\alpha = k\pi$ for any $k \in \mathbb{Z}^+$. Then $1 - \lambda = -\alpha^2 = -(k\pi)^2$. The eigenvalues are

$$\lambda = 1 + k^2\pi^2 \text{ and the eigenfunctions are } y = e^{-t} \sin k\pi t \text{ for all } k \in \mathbb{Z}^+.$$

B U Department of Mathematics

Math 202 Differential Equations

Spring 2005 Final Exam

1. Let $\mathcal{L}\big(f(t)\big) = F(s)$. Show that

$$\mathcal{L}\big(f(at)\big) = \frac{1}{a}F(\frac{s}{a}), \quad a > 0.$$

Solution:

$$\begin{aligned}\mathcal{L}\big(f(at)\big) &= \int_0^\infty e^{-st} f(at)dt \\ &= \int_0^\infty \frac{1}{a}e^{-\frac{su}{a}} f(u)du \quad (u = at;\ du = adt) \\ &= \frac{1}{a}F(\frac{s}{a}).\end{aligned}$$

2. Solve the following initial value problem and discuss the interval of existence

$$(1+t)x' + x = \cos t; \quad x(-\frac{\pi}{2}) = 0.$$

Solution:

Observe that the left hand side equals $\frac{d}{dt}((1+t)x)$. Then

$$(1+t)x = \int \cos t\ dt = \sin t + C, \qquad (C \in \mathbb{R}).$$

Inserting the initial condition we get:

$$0 = \sin(-\frac{\pi}{2}) + C.$$

Hence $C = 1$ and $x(t) = \dfrac{\sin t + 1}{1+t}$.

3. Given that $y(t) = e^{-t}\sin t$ is a solution of the constant-coefficient differential equation

$$9y''' + 11y'' + 4y' - 14y = 0,$$

find the general solution of this equation.

Solution:

It is hard to solve r in $p(r) = 9r^3 + 11r^2 + 4r - 14 = 0$. But since $y(t) = e^{-1\cdot t}\sin(1\cdot t)$ is a solution of the DE, we deduce that $(-1+i)$ and hence its conjugate $(-1-i)$ are roots of $p(r)$. Therefore $(r+1-i)(r+1+i) = r^2+2r+2$ divides $p(r)$. We divide and obtain $p(r) = (r^2+2r+2)(9r-7)$. So the general solution for the DE is:

$$y(t) = e^{-t}(C_1\cos t + C_2\sin t) + C_3 e^{\frac{7}{9}t}; \qquad (C_1, C_2, C_3 \in \mathbb{R}).$$

4. Find a fundamental set of solutions for the differential equation:

$$y'' + xy' + 2y = 0$$

by means of power series about $x = 0$.
Find the recurrence relation, the general term in each solution found and also estimate the radius of convergence of the solutions. Verify that the solutions form a fundamental set.

Solution:

First we observe that $x = 0$ is a regular point of the DE.

Now assume that the solution is of the form $y = \sum_{k=0}^{\infty} a_k x^k$. Then

$$y' = \sum_{k=1}^{\infty} k a_k x^{k-1}; \quad y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k.$$

Insert these in the DE to get:

$$\begin{aligned} 0 &= \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=1}^{\infty} k a_k x^k + 2 \sum_{k=0}^{\infty} a_k x^k \\ &= 2(a_0 + a_2) + \sum_{k=1}^{\infty} \big((k+2)(k+1) a_{k+2} + (k+2) a_k \big) x^k. \end{aligned}$$

It follows that the recurrence relation is $(k+1)a_{k+2} = -a_k, \quad k \geq 0$.

Now first let $a_1 = 0$. Then the terms with odd index vanish and the general term becomes:

$$a_{2k} = \frac{(-1)^k}{(2k-1)(2k-3)\cdots 3 \cdot 1} a_0 = \frac{(-1)^k 2^k k!}{(2k)!} a_0.$$

Similarly, let $a_0 = 0$. Then the general term becomes:

$$a_{2k+1} = \frac{(-1)^k}{(2k)(2k-2)\cdots 4 \cdot 2} a_1 = \frac{(-1)^k}{2^k k!} a_1.$$

Hence the general solution is given as

$$y = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k 2^k k!}{(2k)!} x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} x^{2k+1}.$$

The radius of convergence is $+\infty$ because the coefficient functions are analytic.

These two solutions form a fundamental set because they are linearly independent; one contains the even powers of x while the other contains the odd powers.

5. Solve the following equation by using the Laplace transform

$$f(t) = 4t - 3 \int_0^t f(x) \sin(t - x) dx.$$

Solution:

The term containing the integral is the convolution of $f(t)$ and $\sin t$. Letting $\mathcal{L}\big(f(t)\big) = F(s)$, the Laplace transform of the identity reads:

$$F(s) = \frac{4}{s^2} - 3F(s)\frac{1}{s^2+1}$$

and we have

$$F(s) = \frac{4}{s^2}\frac{s^2+1}{s^2+4} = 4\frac{s^2+1}{s^2(s^2+4)} = 4\left(\frac{1}{s^2+4} + \frac{1}{s^2(s^2+4)}\right) = 2\frac{2}{s^2+4} + \frac{1}{s^2} + \frac{3}{s^2+4}.$$

Therefore,

$$f(t) = 2\sin 2t + t + \frac{3}{2}\sin 2t = t + \frac{7}{2}\sin 2t.$$

6. Whatever real number α we choose, show that at least one nontrivial solution of the following system tends to $\pm\infty$ as t goes to $+\infty$:

$$\mathbf{x}' = \begin{bmatrix} 1 & \alpha \\ -1 & 1 \end{bmatrix}\mathbf{x}.$$

Solution:

Assuming a solution of the form $\begin{bmatrix} a \\ b \end{bmatrix} e^{\lambda t}$ takes us to finding the eigenvalues of the above matrix. We require

$$0 = \begin{vmatrix} 1-\lambda & \alpha \\ -1 & 1-\lambda \end{vmatrix} = (\lambda-1)^2 + \alpha$$

to get $\lambda = 1 \pm \sqrt{-\alpha}$. We should investigate the behaviour of the solutions for all values of α.

If $\alpha = 0$ then $\lambda = 1$ is a double root and the general solution in this case has terms e^t and te^t both of which go to $+\infty$ as t goes to $+\infty$.

If $\alpha < 0$, let $\alpha = -c^2$ with $c \in \mathbb{R}^+$. Then $\lambda = 1 \pm c$. The general solution contains the terms $e^{(1+c)t}$ and $e^{(1-c)t}$; the former goes to $+\infty$ as $t \to +\infty$ because $1 + c > 0$.

If $\alpha > 0$, let $\alpha = c^2$ with $c \in \mathbb{R}^+$. Then $\lambda = 1 \pm ic$. The general solution contains the terms $e^{(1\pm ic)t} = e^t(\cos ct \pm i \sin ct)$ which go to $+\infty$ as $t \to +\infty$ because e^t does so and the second factor is bounded in magnitude.

7. (a) Find the Fourier series expansion on the interval $[-1, 1]$ of the function

$$f(x) = \begin{cases} 1, & -1 \le x < 0 \\ 2, & 0 \le x < 1 \end{cases}$$

(b) Draw the graph of the series found in part (a).
(c) Using the Fourier series found in part (a), find an infinite series expansion for π.

Solution:

B U Department of Mathematics

Math 202 Differential Equations

Summer 2000 Final Exam

1. Solve the initial value problem

$$\begin{aligned} x_1' &= 12x_1 - 15x_2, \\ x_2' &= 4x_1 - 4x_2, \\ x_1(0) &= 18, \\ x_2(0) &= 8. \end{aligned}$$

Solution:

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $A = \begin{bmatrix} 12 & -15 \\ 4 & -4 \end{bmatrix}$; $\vec{x}' = A\vec{x}$. Then $\vec{x}$ is of the form $\vec{x} = \vec{\xi} e^{rt}$, where r is a eigenvalue and $\vec{\xi}$ is a corresponding eigenvector. Now

$$p(r) = \det(A - rI) = r^2 - 8r + 12 = 0$$

gives $r_1 = 2$, $r_2 = 6$. Hence

$$\begin{aligned} (A - 2I)\vec{\xi}^{(1)} = \vec{0} \Rightarrow \vec{\xi}^{(1)} &= \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \\ (A - 6I)\vec{\xi}^{(2)} = \vec{0} \Rightarrow \vec{\xi}^{(2)} &= \begin{bmatrix} 5 \\ 2 \end{bmatrix}. \end{aligned}$$

So, the general solution is

$$\vec{x} = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^{6t}.$$

But $\vec{x}(0) = \begin{bmatrix} 18 \\ 8 \end{bmatrix} = \begin{bmatrix} 3c_1 + 5c_2 \\ 2c_1 + 2c_2 \end{bmatrix}$ gives $c_1 = 1$ and $c_2 = 3$. Therefore,

$$\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{2t} + \begin{bmatrix} 15 \\ 6 \end{bmatrix} e^{6t}$$

is the solution.

2. Solve the following equations.

(a) $(xy + x + 2y + 1)dx + (x + 1)dy = 0$.

(b) $(\sec^2 y)y' - 3\tan y + 1 = 0$.

Solution:

(a) $\mu(x) = e^x$ is an integrating factor. $\psi_x = e^x(xy + x + 2y + 1)$, $\psi_y = e^x(x + 1)$ gives

$$\psi = e^x(x + 1)y + h(x),$$

with $h'(x) = e^x(x + 1)$ implies $h(x) = xe^x$; $\psi = e^x(x + 1)y + xe^x$.

A one-parameter family of solutions is, therefore,

$$e^x(xy + y + x) = c, \quad c \in \mathbb{R}.$$

Since the equation is linear this is the general solution.

(b) Let $u = \tan y$, $u' = y' \sec^2 y$. Differential equation now becomes

$$u' - 3u = -1,$$

a linear equation. Then

$$\begin{aligned}(e^{-3x}u)' &= -e^{-3x} \\ e^{-3x}u &= \frac{1}{3}e^{-3x} + c \\ u &= ce^{3x} + \frac{1}{3} = \tan y\end{aligned}$$

Thus,

$$y(x) = \arctan{(ce^{3x} + \frac{1}{3})}.$$

3. (a) Suppose that $p(x)$ and $q(x)$ are continuous on (a, b) and $\{y_1, y_2\}$ is a set of solutions of

$$y'' + p(x)y' + q(x)y = 0,$$

on (a, b) such that either $y_1(x_0) = y_2(x_0) = 0$ or $y_1'(x_0) = y_2'(x_0) = 0$ for some x_0 in (a, b). Show that $\{y_1, y_2\}$ is not a fundamental set.

(b) Find a particular solution of $y'' - 2y' + y = 14x^{\frac{3}{2}}e^x$, $x > 0$.

Solution:

(a) Wronskian $W[y_1(x_0), y_2(x_0)] = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = 0$ since either the first or the second row is zero.

This implies $W[y_1(x), y_2(x)] = 0$ for all $x \in (a, b)$.

Let $\phi(x) = c_1y_1(x) + c_2y_2(x)$. Then $\phi(x_0) = \phi'(x_0) = 0$. Since the solution must be unique, $\phi(x) = 0$, for all $x \in (a, b)$. Thus $\{y_1, y_2\}$ is linearly dependent.

(b) $y = e^x$, $y = xe^x$ are homogeneous solutions. Let $y_p = e^x v(x)$. Then $v'' = 14x^{\frac{3}{2}}$, hence, $v(x) = \frac{8}{5}x^{\frac{7}{2}}$ and so

$$y_p = \frac{8}{5}x^{\frac{7}{2}}e^x.$$

4. (a) Using Laplace transform solve the integral equation

$$y(t) = 1 + 2\int_0^t y(\tau)\cos{(t - \tau)}d\tau.$$

(b) Find the inverse Laplace transform $f(t)$ of the function

$$F(s) = \frac{e^{-s}}{s^3}(1 + se^{-s}).$$

Solution:

(a) By convolution theorem $\mathfrak{L}[y] = \dfrac{1}{s} + 2\mathfrak{L}[y]\mathfrak{L}[\cos t]$. Hence $\left(1 - \dfrac{2s}{s^2+1}\right)\mathfrak{L}[y] = \dfrac{1}{s}$ implies

$$\mathfrak{L}[y] = \frac{s^2+1}{s(s-1)^2},$$

i.e., $\mathfrak{L}[y] = \dfrac{1}{s} + \dfrac{2}{(s-1)^2}$. Thus $y(t) = \mathfrak{L}^{-1}\left[\dfrac{1}{s} + \dfrac{2}{(s-1)^2}\right]$ gives

$$y(t) = 1 + 2te^t.$$

(b) $F(s) = \dfrac{1}{s^3}e^{-s} + \dfrac{1}{s^2}e^{-2s}$ implies

$$f(t) = \mathfrak{L}^{-1}[F(s)] = u_1(t)\frac{1}{2}(t-1)^2 + u_2(t)(t-2),$$

where $u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geqslant c \end{cases}$ is the unit step function.

5. Using the method of power series construct the general solution of

$$(1+x^2)y'' - 8xy' + 20y = 0,$$

about the point $x = 0$.

Solution:

$x = 0$ is an ordinary point, so $y = \displaystyle\sum_{n=0}^{\infty} c_n x^n$. Plugging y into the differential equation yields

$$2c_2 + 20c_0 + (6c_3 + 12c_1)x + \sum_{k=2}^{\infty}[(k^2 - 9k + 20)c_k + (k+1)(k+2)c_{k+2}]x^k = 0.$$

Hence $c_2 = -10c_0$, $c_3 = -2c_1$ and $c_{k+2} = -\dfrac{(k-4)(k-5)}{(k+1)(k+2)}c_k$, for $k = 2, 3, 4, \ldots$. Thus $c_n = 0$ for $n \geqslant 6$. But $c_4 = -\dfrac{1}{2}c_2 = 5c_0$, $c_5 = -\dfrac{1}{10}c_3 = \dfrac{1}{5}c_1$. Therefore both solutions of the differential equation are polynomials

$$y(x) = c_0(1 - 10x^2 + 5x^4) + c_1(x - 2x^3 + \frac{1}{5}x^5).$$

6. **(a)** Using the method of separation of variables determine the function $u(x,t)$ which obeys

$$u_{xx} + 2u = tu_t,$$
$$u(0,t) = u(\pi,t) = u(x,0) = 0.$$

(b) Expand the function $f(x) = \sin x$, $0 < x < \pi$ as a Fourier cosine series of period 2π. Sketch the graph of the sum of this series.

Solution:

(a) Let $u(x,t) = X(x)T(t)$. Then plugging u into the equation gives $\frac{X''}{X} + 2 = t\frac{T'}{T}$, hence,

$$\frac{X''}{X} = \lambda, \quad \frac{tT'}{T} = 2 + \lambda.$$

Consider $X'' = \lambda X$ with boundary conditions $X(0) = X(\pi) = 0$. Then $\lambda = -n^2$ with $n = 1, 2, 3, \ldots$ implies

$$X_n = B_n \sin(nx).$$

$tT' = (2 - n^2)T$ gives $T_n(t) = t^{2-n^2}$. Thus, $u_n(x,t) = B_n t^{2-n^2} \sin(nx)$. Initial condition $T(0) = 0$ implies $n = 1$ and

$$u(x,t) = B_1 t \sin x,$$

where B_1 is arbitrary.

(b) Even extension of f is $E_f(x) = \begin{cases} \sin x, & 0 < x < \pi, \\ -\sin x, & -\pi < x < 0 \end{cases}$. For $E_f(x)$, $b_n = 0$ for $n = 1, 2, 3, \ldots$. As $E_f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$ with $a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx dx$ one finds that $a_0 = \frac{4}{\pi}$, $a_1 = 0$ and $a_n = -\frac{2[1 + (-1)^n]}{\pi(n^2 - 1)}$ for $n = 2, 3, \ldots$. Thus

$$\begin{aligned} f(x) &= \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{[1 + (-1)^n]}{n^2 - 1} \cos nx \\ &= \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos 2kx}{4k^2 - 1}. \end{aligned}$$

B U Department of Mathematics

Math 202 Differential Equations

Summer 2001 Final Exam

1) Without using Laplace transforms solve the initial value problem :

$$\begin{aligned} x_1' &= -3x_1 - 4x_2 \\ x_2' &= x_1 - 7x_2 \end{aligned}$$

$$x_1(0) = 2\ ,\ x_2(0) = 3.$$

Solution:

$\overrightarrow{x'} = A\overrightarrow{x}$ where $A = \begin{bmatrix} -3 & -4 \\ 1 & -7 \end{bmatrix}$, $\det(A - rI) = (r+5)^2 = 0$

then we have $r_1 = r_2 = -5$a repeated real eigenvalue. $(A + 5I)\overrightarrow{\xi} = \overrightarrow{0} \Rightarrow \overrightarrow{\xi} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\overrightarrow{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-5t}$. Let $(A + 5I)\overrightarrow{\eta} = \overrightarrow{\xi}$ up to a multiple of $\overrightarrow{\xi}$, $\overrightarrow{\eta} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$\overrightarrow{x}^{(2)} = \overrightarrow{\xi} te^{-5t} + \overrightarrow{\eta} e^{-5t}$ Then $\overrightarrow{x}^{(2)} = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{-5t}$.

$\{\overrightarrow{x}^{(1)}, \overrightarrow{x}^{(2)}\}$is a fundamental set gen. soln. $\overrightarrow{x} = c_1\overrightarrow{x}^{(1)} + c_2\overrightarrow{x}^{(2)}$

$= \left(c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{-5t}$.

$\overrightarrow{x}(0) = \begin{bmatrix} 2c_1 + c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow c_1 = 3\ ,\ c_2 = -4$.Unique solution to this initial value problem is therefore,

$$\overrightarrow{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-5t} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} te^{-5t}.$$

2) Given the differential equation:

$$x^2(1 - 2x)y'' + x(4x - 5)y' + (9 - 4x)y = 0,$$

a) Locate and classify all of its singular points in the finite plane.

b) Find a series solution about the point $x = 0$. Determine the general term and the radius of convergence of this series. Describe the nature of the second, linearly independent series solution about the same point but do not compute its coefficients.

Solution:

a) $p(x) = \dfrac{(4x-5)}{x(1-2x)}$, $q(x) = \dfrac{(9-4x)}{x^2(1-2x)} \Rightarrow x = 0$ and $x = \dfrac{1}{2}$ are regular singular points
All other finite points are ordinary points.

b) Let $y=\sum_{n=0}^{\infty} c_n x^{n+\alpha}$ Substituting the series for y in differential equation we get,

$$(\alpha-3)^2x^\alpha \quad +\sum_{k=1}^{\infty}[(k+\alpha)(k+\alpha-6)+9]c_k x^{k+\alpha}$$
$$+\sum_{k=1}^{\infty}[2(k+\alpha-1)(4-k-\alpha)-4]c_{k-1}x^{k+\alpha}=0$$

$$\Rightarrow \alpha_1=\alpha_2=3\ ,\ c_k=\frac{2(1+k)}{k}c_{k-1}\ ,\ k=0,1,2,\ldots \Rightarrow c_1=4c_0\ ,$$

$c_2=2^2(3)c_0\,,\ldots\ c_k=2^k(k+1)c_0\ for\ k=0,1,2,\ldots$ therefore

$$y_1=x^3\sum_{n=0}^{\infty}2^n(n+1)x^n.\ R=\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|=\frac{1}{2}.$$

Since the two exponents are the same , the second solution must have the form

$$y_2=y_1\ln x+x^3\sum_{n=1}^{\infty}b_n x^n.\text{(it turns out that } b_n=-2^n n\ for\ n=1,2,3,\ldots)$$

3)

a) A function $y(t)$ is known to satisfy the initial value problem :

$$y''+y=(t-3)u_3(t),\quad y(0)=y'(0)=0.$$

Compute $y\left(\frac{\pi}{2}\right)$ and $y\left(\frac{\pi}{2}+3\right)$.

Solution:

Let $Y(s)=\mathfrak{L}[y(t)]$ then we have the differential equation as $(s^2+1)Y=\frac{e^{-3s}}{s^2}$

$$Y(s)=\frac{e^{-3s}}{s^2(s^2+1)}=e^{-3s}\left[\frac{1}{s^2}-\frac{1}{(s^2+1)}\right]\Rightarrow y(t)=\mathfrak{L}^{-1}[Y(s)]$$

$=u_3(t)[(t-3)-\sin(t-3)]$ since $u_3(t)=0$ for $t<3$, $y\left(\frac{\pi}{2}\right)=0.$

$$y\left(\frac{\pi}{2}+3\right)=\frac{\pi}{2}-\sin\left(\frac{\pi}{2}\right)=\frac{\pi}{2}-1$$

b) Solve the integrodifferential equation :

$$\frac{dy}{dt}=\cos(t)+\int_0^t y(\tau)\cos(t-\tau)d\tau\ ,\ y(0)=1.$$

Solution:

Take Laplace transform and use convolution theorem. Let $Y(s)=\mathfrak{L}[y(t)]$

$$\begin{aligned}
sY-1 &= \frac{s}{(s^2+1)}+Y\frac{s}{(s^2+1)}\\
\frac{s^3}{(s^2+1)}Y &= 1+\frac{s}{(s^2+1)}\\
Y(s) &= \frac{s^2+1}{s^3}+\frac{1}{s^2}\\
Y(s) &= \frac{1}{s}+\frac{1}{s^2}+\frac{1}{s^3}\\
y(t) &= \mathfrak{L}^{-1}[Y(s)]=1+t+t^2\frac{1}{2}
\end{aligned}$$

4) Find the general solution of the following equations:

a) (t+y)dy+dt=0

Solution:

This is a linear equation for t(y):
$\frac{dt}{dy}+t=-y$, $(te^y)'=-ye^y$ Then $te^y=e^y-ye^y+c$, $t=1-y+ce^{-y}$ where $c\in\mathbb{R}$

b) $y''-2y'+y=12\frac{e^t}{t^5}, \quad t>0$

Solution:

Use either reduction of order or variation of parameters. Since e^t is a homogeneous solution let $y=e^t v(t)$. Then we have $v''=12t^{-5}, \quad v=t^{-3}+c_1+c_2t$
$y=c_1e^t+c_2te^t+t^{-3}e^t$

5)

a) Determine and sketch the graph of the Fourier series of the function:

$$f(x)=x, \quad -4\le x\le 4$$

b) Using the method of separation of variables find a solution of the heat equation

$$u_t=\alpha^2 u_{xx}, \quad 0<x<L, \quad 0<t, \quad L\in\mathbb{R}$$

that satisfies the boundary conditions: $u_x(0,t)=u_x(L,t)=0$, and matches the initial condition: $u(x,0)=7+3\cos\left(\frac{2\pi}{L}x\right)$

Solution:

a) $f(x)=x$, is odd, $a_0=a_k=0, \quad k=0,1,2,\ldots$
$b_k=\frac{2}{4}\int_0^4 x\sin\left(\frac{k\pi}{4}x\right)=\frac{8(-1)^{k+1}}{\pi k}$
$x=\frac{8}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin\left(\frac{k\pi}{4}x\right)$

b) Let $u(x,t)=T(t)X(x)$. Heat equation $\Rightarrow\frac{T'}{T}=\frac{\alpha^2X''}{X}=k,\ T'=kT$
$X''=\frac{k}{\alpha^2}X$. Boundary conditions $X'(0)=X'(L)=0$ $k\le 0$ to satisfy the boundary conditions.Letting $k=-\alpha^2\beta^2$ one gets $X_n=\cos\left(\frac{n\pi}{L}x\right)$. $\beta=\frac{n\pi}{L}$
$\Rightarrow T_n=B_ne^{-(\frac{n\pi}{L}\alpha)^2t}$, $n=0,1,2,\ldots u_n=T_nX_n,\ u=\sum_{n=0}^{\infty}u_n$.
Let$B_0=\frac{1}{2}a_0,\ B_k=a_k\ k=1,2,\ldots$ Hence we get a Fourier series at t=0:
$u(x,0)=\frac{1}{2}a_0+\sum_{k=1}^{\infty}a_k\cos\left(\frac{k\pi}{L}x\right)=7+3\cos\left(\frac{2\pi}{L}x\right)\Rightarrow a_0=14,\ a_2=3$ all other
$a_n=0$ Therefore, $u(x,t)=7+3e^{-(\frac{2\pi}{L}\alpha)^2t}\cos\left(\frac{2\pi}{L}x\right)$

6)

a) Show that if z satisfies the second order linear equation:
$z'' + p(x)z' + q(x)z = 0$ and $z \neq 0$, then $y = \dfrac{z'}{z}$ must satisfy the Ricatti equation.

Solution:

$$y' = \frac{z''}{z} - \frac{(z')^2}{z^2} = -p(x)\frac{z'}{z} - q(x) - \frac{(z')^2}{z^2} = -p(x)y - q(x) - y^2$$

$$y' + y^2 + p(x)y + q(x) = 0 \quad \text{a Ricatti equation}$$

b) Let $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ where λ is an arbitrary real number. Determine exp(At).

Solution:

$$\text{A=D+N} \text{ , } D = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad N^2 = 0, \ \text{DN=ND}$$

$$e^{At} = e^{(D+N)t} = e^{Dt}e^{Nt}, e^{Dt} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix}, \quad e^{Nt} = I + Nt = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

B U Department of Mathematics

Math 202 Differential Equations

Summer 2002 Final Exam

1. Using Laplace transforms solve the initial value problem:

$$\begin{aligned} x_1' &= -x_1 - 3\,x_2\,, \\ x_2' &= 3\,x_1 + 5\,x_2\,, \quad x_1(0) = 2,\ x_2(0) = -3. \end{aligned}$$

Solution:

Let u(s) = L[x_1] , v(s) = L[x_2].
The Laplace transform of the system gives

$$\begin{aligned} (s+1)u + 3v &= 2 \\ -3u + (s-5)v &= -3 \end{aligned}$$

$$\Rightarrow \quad u = \frac{2s-1}{(s-2)^2} = \frac{2}{s-2} + \frac{3}{(s-2)^2}\,,$$

$$v = \frac{3-3s}{(s-2)^2} = \frac{-3}{s-2} - \frac{3}{(s-2)^2}\,.$$

$$\Rightarrow \quad \begin{aligned} x_1(t) = L^{-1}[u] &= (2+3t)e^{2t}\,, \\ x_2(t) = L^{-1}[v] &= -3(1+t)e^{2t}\,, \end{aligned}$$

2. **a)** Determine and sketch the graph of the Fourier series of the function:

$$f(x) = \begin{cases} 0 & -1 < x < 0 \\ x & 0 \le x < 1 \end{cases},$$

b) Using the results of (a) evaluate the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

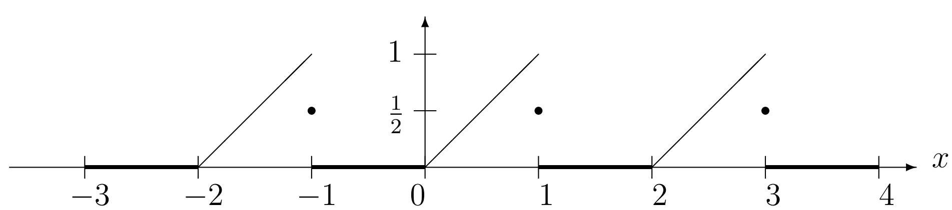
Solution:

(a)

$$\begin{aligned}
f(x) &= \frac{1}{2}\,a_0 + \sum_{k=1}^{\infty}(a_k \cos k\pi x + b_k \sin k\pi x) \\
a_0 &= \int_{-1}^{1} f(x)dx = \int_{0}^{1} x\; dx = \frac{1}{2} \\
a_k &= \int_{-1}^{1} f(x) \cos k\pi x\; dx = \int_{0}^{1} x \cos k\pi x\; dx = \frac{(-1)^k - 1}{\pi^2 k^2} \\
b_k &= \int_{-1}^{1} f(x) \sin k\pi x\; dx = \int_{0}^{1} x \sin k\pi x\; dx = \frac{(-1)^{k+1}}{\pi k}
\end{aligned}$$

Since $a_k = 0$ for $k =$ even, we can write

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-2)}{\pi^2(2n-1)^2} \cos(2n-1)\pi x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n} \sin n\pi x.$$



(b)

$f(x)$ is piecewise smooth.

At $x = 0$ series converges pointwise to $f(0) = 0 \Rightarrow 0 = \dfrac{1}{4} - \dfrac{2}{\pi^2} \displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

3. **a)** Find the general solution of

$$(x-1)\,y'' + 2\,y' = 0,$$

by constructing the two linearly independent power series solutions about the point $x = 0$. Determine the radii of the convergence and identify the functions which are represented by these series.

b) Find the general solution of the same equation by another method and show that the two answers are the same.

Solution:

(a)

$x = 0$ is an ordinary point. Let $y = \sum_{n=0}^{\infty} c_n x^n$.

$$DE \;\Rightarrow\; \sum_{n=2}^{\infty} n\,(n-1)\,c_n\,(x^{n-1} - x^{n-2}) + \sum_{n=1}^{\infty} 2n\,c_n\,x^{n-1} = 0$$

$$\Rightarrow\; 2\,(c_1 - c_2) + \sum_{k=1}^{\infty} (k+1)(k+2)(c_{k+1} - c_{k+2})x^k = 0$$

$$\Rightarrow\; c_2 = c_1\,,\quad c_{n+1} = c_n\,,\quad n = 2, 3, \ldots$$
$$\Rightarrow\; c_0,\; c_1 \text{ arbitrary},\quad c_n = c_1 \text{ for } n = 2, 3, \ldots$$

Therefore, $y = c_0 + c_1 \sum_{n=1}^{\infty} x^n,\ |x| < 1$

Noting that $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$ for $|x| < 1$

$$y = (c_0 - c_1) + c_1(1-x)^{-1}$$

$$R_1 = \infty\,,$$
$$R_2 = 1$$

(b)

Let $y' = v$. DE : $\dfrac{v'}{v} = \dfrac{-2}{x-1}$

$\Rightarrow v = y' = \dfrac{a_2}{(1-x)^2}\,,\quad y = a_1 + a_2(1-x)^{-1}\quad a_1, a_2 \in \mathbb{R}$

Identify: $a_1 = c_0 - c_1\,,\quad a_2 = c_1$

4. **a)** Find the Laplace transform of the function $f(t)$ defined by

$$f(t) = \int_0^t (e^x - 3e^{2x})(t-x)^3 dx$$

b) Let $g(t)$ be the inverse Laplace transform of $F(s) = \dfrac{(1-e^{-3s})(1+4e^{-3s})}{s^2}$.
Evaluate $g(4)$ and $g(7)$.

Solution:

(a)

Let $g(t) = e^t - 3e^{2t}$, $h(t) = t^3$.
Clearly, $f(t) = (g * h)(t)$.
Convolution theorem gives

$$\begin{aligned} L[f(t)] &= L[g]L[h] \\ &= \frac{6}{s^4}\left[\frac{1}{s-1} - \frac{3}{s-2}\right]. \end{aligned}$$

(b)

$$\begin{aligned} F(s) &= \frac{1}{s^2}\left[1 + 3e^{-3s} - 4e^{-6s}\right] \\ g(t) &= L^{-1}[F(s)] = t + 3u_3(t)(t-3) - 4u_6(t)(t-6) \\ \Rightarrow \quad g(4) &= 4 + 3 = 7 \ , \\ g(7) &= 7 + 3(4) - 4 = 15. \end{aligned}$$

5. **a)** Find the general solution of $y'' - 16y = 8\sin^2 x$.

b) Prove that the Wronskian $W[y_1, y_2]$ of the two solutions y_1, y_2 of $L[y] = y'' + p(x)y' + q(x)y = 0$, where $p(x)$ and $q(x)$ are continuous on an interval I, satisfies the Abel's formula :

$$W = c \ \exp(-\int p(x)dx), \quad c \in \mathbb{R}.$$

Solution:

(a)

$y_H = c_1e^{4x} + c_2e^{-4x}$, $c_1, c_2 \in \mathbb{R}$
$8\sin^2 x = 4(1 - \cos 2x)$.
Let $y_p = A + B\cos 2x$.

$$\begin{aligned} DE \quad &\Rightarrow \quad -16A - 20B\cos 2x = 4 - 4\cos 2x. \\ &\Rightarrow \quad A = -\frac{1}{4} \ , \quad B = \frac{1}{5} \end{aligned}$$

General solution : $y = y_H + y_p = c_1e^{4x} + c_2e^{-4x} - \dfrac{1}{4} + \dfrac{1}{5}\cos 2x$

(b)

$$\begin{aligned} W &= y_1y_2' - y_2y_1' \ , \quad W' = y_1y_2'' - y_2y_1'' \\ W' &= y_1(-py_2' - qy_2) + y_2(py_1' + qy_1) \\ &= -p(y_1y_2' - y_2y_1') = -p\ W \\ &\Rightarrow \ln|W| = -\int p(x)dx + k\ , \quad k \in \mathbb{R} \\ &\Rightarrow W = c\exp(-\int p(x)dx). \ \ (c = \pm e^k) \end{aligned}$$

6. **a)** Using the method of separation of variables solve the partial differential equation : $u_{xy} - u = 0$.

   b) Find the solution of the initial value problem: $y' = 2(3x + y)^2 - 1\ , \ \ y(0) = 1$.

Solution:

(a)

Let $u(x, y) = \mathbb{X}(x)\mathbb{Y}(y)$.

DE : $\mathbb{X}'\mathbb{Y}' = \mathbb{X}\mathbb{Y}$ $\qquad \dfrac{\mathbb{X}'}{\mathbb{X}} = \dfrac{\mathbb{Y}}{\mathbb{Y}'} = k$

$\mathbb{X}' - k\mathbb{X} = 0 \quad \mathbb{Y}' - \dfrac{1}{k}\mathbb{Y} = 0$

$\Rightarrow \mathbb{X}(x) = (const.)e^{kx}\ , \ \ \mathbb{Y}(y) = (const.)e^{y/k}$

$u(x, y) = Ce^{kx+y/k}\ , \ \ (k, c \in \mathbb{R})$

(b)

Let $u = 3x + y$

$\Rightarrow y' = u' - 3 = 2u^2 - 1\ ;\ \ u' = 2(u^2 + 1)$ separable

$\Rightarrow \tan^{-1} u = 2x + c.\ \ y(0) = 1 \Rightarrow u(0) = 1$

$\Rightarrow c = \tan^{-1} 1 = \dfrac{\pi}{4},$

$\tan^{-1}(3x + y) = 2x + \dfrac{\pi}{4},$

$y = \tan(2x + \dfrac{\pi}{4}) - 3x.$

B U Department of Mathematics

Math 202 Differential Equations

Summer 2005 Final Exam

1. Let $A = \begin{pmatrix} -5 & 1 \\ -2 & -2 \end{pmatrix}$. Find the general solution of the system $\overrightarrow{x}' = A\overrightarrow{x}$ without using Laplace transforms and show that as $t \to \infty$, $\overrightarrow{x} \to \overrightarrow{0}$.

Solution:

Let $\overrightarrow{x} = \overrightarrow{\xi} e^{rt}$ so that $A\overrightarrow{\xi} = r\overrightarrow{\xi}$.

$p(r) = det(A - rI) = \begin{vmatrix} -5-r & 1 \\ -2 & -2-r \end{vmatrix} = r^2 + 7r + 12 = 0.$

$p(r) = (r+3)(r+4) = 0 \Rightarrow r_1 = -3, r_2 = -4$

Since we have two distinct eigenvalues we shall have two linearly independent eigenvectors.

$(A - r_1 I)\overrightarrow{\xi}^{(1)} = (A + 3I)\overrightarrow{\xi}^{(1)} = 0$, $(A - r_2 I)\overrightarrow{\xi}^{(2)} = (A + 4I)\overrightarrow{\xi}^{(2)} = \overrightarrow{0}$.

$\overrightarrow{\xi}^{(1)} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $\overrightarrow{\xi}^{(2)} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$,

$\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow a_2 = 2a_1$, $b_2 = b_1$. Choose $a_1 = 1$, $b_1 = 1$. Then

$\overrightarrow{x}^{(1)} = \overrightarrow{\xi}^{(1)} e^{r_1 t} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-3t}$, $\overrightarrow{x}^{(2)} = \overrightarrow{\xi}^{(2)} e^{r_2 t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-4t}$ is a fundamental set of solutions.

General solution $\overrightarrow{x}(t) = c_1\overrightarrow{x}^{(1)} + c_2\overrightarrow{x}^{(2)} = c_1 e^{-3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Note that $\lim_{t\to\infty} \overrightarrow{x}^{(1)}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\lim_{t\to\infty} \overrightarrow{x}^{(2)}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Therefore, $\lim_{t\to\infty} \overrightarrow{x}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

2. (a) Solve the initial value problem: $(2x+3)y' = y + (2x+3)^{\frac{1}{2}}$, $y(-1) = 0$.

Solution:

D.E. is linear and $\mu(x) = (2x+3)^{-\frac{1}{2}}$ is an integrating factor:
$\left[(2x+3)^{-\frac{1}{2}}y\right]' = \dfrac{1}{2x+3}, \quad (2x+3 \geq 0)$.
$(2x+3)^{-\frac{1}{2}}y = \frac{1}{2}\ln(2x+3) + c$
$y(-1) = 0 \Rightarrow c = 0$. Therefore

$$y = \frac{1}{2}(2x+3^{\frac{1}{2}})\ln(2x+3)$$

(b) Find the general solution of $2x^2y'' + 10xy' + 8y = x^3$ for $x > 0$.

Solution:

This is a nonhomogeneous Euler equation. Let $x = e^u$.
$xy' = \dfrac{dy}{du}$,
$x^2y'' = \dfrac{d^2y}{du^2} - \dfrac{dy}{du}$ and DE becomes
$2\dfrac{d^2y}{du^2} + 8\dfrac{dy}{du} + 8y = e^{3u}$,
$\dfrac{d^2y}{du^2} + 4\dfrac{dy}{du} + 4y = \dfrac{1}{2}e^{3u}$
$y(u) = y_H(u) + y_P(u)$.
Characteristic equation: $r^2 + 4r + 4 = (r+2)^2 = 0$
$\Rightarrow \{e^{-2u}, ue^{-2u}\}$ is a fundamental set of homogeneous solutions.
$y_H = c_1e^{-2u} + c_2ue^{-2u}$. Let $y_P(u) = Ae^{3u}$
DE $\Rightarrow (9A + 12A + 4A)e^{3u} = \frac{1}{2}e^{3u} \Rightarrow A = \dfrac{1}{50}$.
$y = c_1e^{-2u} + c_2ue^{-2u} + \frac{1}{50}e^{3u}$

$$y(x) = c_1x^{-2} + c_2x^{-2}\ln x + \frac{1}{50}x^3.$$

3. Given the equation $xy''+3y'-y=0$ determine the two exponents at the point $x=0$. Prove that the two Frobenius series constructed about $x=0$ with these exponents are linearly dependent. Describe the nature of the second linearly independent solution about the same point but do not determine it completely.

Solution:

$x=0$ is a regular singular point. Let $y=\sum_{n=0}^{\infty} c_n x^{n+\alpha}$.

DE $\Rightarrow \displaystyle\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)c_n x^{n+\alpha-1}+\sum_{n=0}^{\infty}(n+\alpha)3c_n x^{n+\alpha-1}-\sum_{n=0}^{\infty}c_n x^{n+\alpha}=0.$

$\displaystyle\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha+2)c_n x^{n+\alpha-1}-\sum_{n=0}^{\infty}c_n x^{n+\alpha}\Rightarrow$

$\displaystyle\alpha(\alpha+2)c_0x^{\alpha-1}+\sum_{k=0}^{\infty}\left[(k+\alpha+1)(k+\alpha+3)c_{k+1}-c_k\right]x^{k+\alpha}=0$

Indicial eqn.: $\alpha(\alpha+2)=0$. Exponents: $\alpha_1=0$, $\alpha_2=-2$.

$(k+\alpha+1)(k+\alpha+3)c_{k+1}-c_k=0,\ k=0,1,2,...$

For $\alpha=\alpha_1$, the recurrence relation gives $c_n=\dfrac{2c_0}{n!(n+2)!},\quad n=1,2,...$

$c_0\neq 0$, arbitrary.

$\Rightarrow y_1=c_0\displaystyle\sum_{n=0}^{\infty}\frac{2}{n!(n+2)!}x^n$, converges for $|x|<\infty$.

For $\alpha=\alpha_2=-2$, recurrence relation is $(k-1)(k+1)c_{k+1}-c_k=0$.

$c_1=0\Rightarrow c_0=0$ and for $n\geq 2$, $c_n=\dfrac{2c_2}{n!(n-2)!}$.

c_2 is arbitrary. Thus $y_2=c_2\displaystyle\sum_{n=2}^{\infty}\frac{2}{n!(n-2)!}x^{n-2}=c_2\sum_{k=0}^{\infty}\frac{2}{k!(k+2)!}x^k$

Thus $c_0y_2=c_2y_1$ and y_1,y_2 are linearly dependent.

Note that $\alpha_1-\alpha_2=2$. Since the two exponents differ by an integer,

$y_2=Cy_1\ln x+\displaystyle\sum_{n=0}^{\infty}b_nx^{n+\alpha_2}$, $b_0\neq 0$ is the second solution of the fundamental set and although C can, in general be zero, in this problem is not zero.

4. (a) Solve the initial value problem: $y' + 2y = f(t)$, $y(0) = 3$, where $f(t) = 0$ is $t < 1$ and $f(t) = 5$ if $t \geq 1$.

Solution:

$f(t) = 5u(t)$.

Taking the Laplace transform of the DE gives:

$$(s+2)Y(s) = 3 + 5e^{-s},\ Y(s) = \mathfrak{L}\{y(t)\}.$$

$$Y(s) = \frac{3}{s+2} + \frac{5e^{-s}}{s(s+2)} = \frac{3}{s+2} + \frac{5}{2}\left[\frac{e^{-s}}{s} - \frac{e^{-s}}{s+2}\right]$$

$$y(t) = \mathfrak{L}^{-1}\{Y(s)\} = 3e^{-2t} + \tfrac{5}{2}\left[1 - e^{-2(t-1)}\right]u(t)$$

(b) Using the method of seperation of variables reduce the partial differential equation

$$\frac{\partial}{\partial x}(ax\frac{\partial u}{\partial x}) = \frac{\partial^2 u}{\partial t^2},\quad (a \in \mathbb{R}),$$

to two ordinary differential equations. Classify all finite points of these differential equations but do not construct the solutions.

Solution:

Let $U(x,t) = X(x)T(t)$. DE $\Rightarrow T\dfrac{d}{dx}(ax\dfrac{dX}{dx}) = X\dfrac{d^2T}{dt^2}$

$\Rightarrow \dfrac{T''}{T} = \dfrac{(axX')'}{X} = k$ (a constant)

$\Rightarrow T'' - kT = 0$

$(axX')' - kX = 0 \Rightarrow axX'' + aX' - kX = 0$

$T'' - kT = 0 \Rightarrow$ all finite points are ordinary points.

$X'' + \dfrac{1}{x}X' - \dfrac{k}{a}X = 0,\ a \neq 0.$

For $a \neq 0$, $x = 0$ is a regular singular point.

If $a = 0$, $k = 0$ but the PDE does not fit the x-dependence:

$$U_{tt} = 0,\ U(x,t) = f(x)t + g(x).$$

(f,g are arbitrary functions.)

5. (a) Determine Fourier series expansion of the function $f(x) = x^2 + x + 3$, $x \in [-\pi, \pi]$.

(b) Using the result of part (a) prove that $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}$. Justify your proof.

Solution:

$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi} \left[\frac{x^3}{3} + \frac{x^2}{2} + 3x \right]_{-\pi}^{\pi} = \frac{2\pi^2}{3} + 6$

$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos kx dx,$

since $x \cos kx$ is odd and 3 is orthogonal to $\cos kx$ for $k \geq 1$.

Integrating by parts twice gives $a_k = \frac{4}{k^2}(-1)^k, \quad k = 1, 2, 3, ...$

$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin kx dx,$

since $x^2 \sin kx$ is odd and 3 is orthogonal to $\sin kx$ for $k \geq 1$.

Integrating by parts gives $b_k = \frac{2}{k}(-1)^{k+1}$.

$$\begin{aligned} f(x) &= \frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \\ &= 3 + \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \left(\frac{4(-1)^k}{k^2} \cos kx + \frac{2(-1)^{k+1}}{k} \sin kx \right) \end{aligned}$$

Obviously, f and f' are continuous in $[-\pi, \pi]$.

Therefore Fourier series converges pointwise:

$f(0) = 3 + \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \left(\frac{4(-1)^k}{k^2} \cos(0) + \frac{2(-1)^{k+1}}{k} \sin(0) \right)$

Since $f(0) = 3$ and $\sin(0) = 0$, $\cos(0) = 1$, we get

$\frac{\pi^2}{3} + 4 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = 0$, and therefore

$$\sum_{k=1}^{\infty} \frac{(-1)^{(k+1)}}{k^2} = \frac{\pi^2}{12}$$

6. (a) Let $f(t) = [t]$, where $[t]$ denotes the greatest integer $\leq t$. Find the Laplace transform of $f(t)$.

Solution:

$$
\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty [t]e^{-st}dt = 0\int_0^1 e^{-st}dt + 1\int_1^2 e^{-st}dt + 2\int_2^3 e^{-st}dt + ... \\
&= -\frac{1}{s}\left[\ e^{-2s} - e^{-s} + 2(e^{-3s} - e^{-2s}) + 3(e^{-4s} - e^{-3s}) + ...\ \right] \\
&= -\frac{1}{s}\left[\ -e^{-s} - e^{-2s} - e^{-3s} - e^{-4s} - ...\ \right] \\
&= \frac{1}{s}\left[\ e^{-s} + e^{-2s} + e^{-3s} + e^{-4s} + ...\ \right] \\
&= \frac{e^{-s}}{s}\sum_{n=0}^{\infty} e^{-ns} \\
&= \frac{e^{-s}}{s}\frac{1}{(1-e^{-s})} \\
\text{Hence } \mathcal{L}\{f(t)\} &= \frac{1}{s(e^s - 1)}
\end{aligned}
$$

(b) Without using partial fractions find the inverse Laplace transform of the function:

$$F(s) = \frac{s}{(s-3)(s^2+9)}$$

Solution:

$F(s) = \dfrac{s}{(s-3)(s^2+9)} = \mathcal{L}\{e^{3t}\}\mathcal{L}\{\cos 3t\}.$

By convolution theorem $\mathcal{L}^{-1}\{F(s)\} = e^{3t} * \cos 3t$

$$
\begin{aligned}
&= \int_0^t e^{3(t-\tau)}\cos 3\tau d\tau = e^{3t}\int_0^t e^{-3\tau}\cos 3\tau d\tau \\
&= e^{3t}\left[\frac{1}{6}e^{-3\tau}(\sin 3\tau - \cos 3\tau)\right]_0^t \text{ integrate by parts twice} \\
&= e^{3t}\left[\frac{1}{6}e^{-3t}(\sin 3t - \cos 3t) + \frac{1}{6}\right] \\
&= \frac{1}{6}\left[e^{3t} + \sin 3t - \cos 3t\right]
\end{aligned}
$$