MAT261E Linear Algebra FINAL EXAM January 16, 2012

QUESTION 1 The blanks below will be filled by students. (Except the score)

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For the solution of this question please use only the front face and if necessary the back face of this page.

[12pt] a) Let $\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 & 4 \\ -2 & 1 & 3 & -1 \\ 0 & 7 & 7 & 7 \end{bmatrix}$. i) Find the rank of **A**.

ii) Find a basis for the column space of **A**.

iii) Find the dimension of the column space of **A**.

[13pt] b) Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_2 \\ x_1 - x_3 \end{bmatrix}$.

i) Find the kernel of the transformation L, $(\ker(L))$.

ii) If $S = \left\{ \mathbf{x} = \begin{bmatrix} 0 \\ a \\ b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$, find $L(S)$.

a. $\underset{\sim}{A} = \begin{bmatrix} 1 & 3 & 2 & 4 \\ -2 & 1 & 3 & -1 \\ 0 & 7 & 7 & 7 \end{bmatrix} \xrightarrow{S_2 \leftarrow S_2 + 2S_1} \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 7 & 7 & 7 \\ 0 & 7 & 7 & 7 \end{bmatrix} \xrightarrow[S_3 \leftarrow S_3 - 7S_2]{S_2 \leftarrow S_2/7} \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

i) $\operatorname{rank}(\underset{\sim}{A}) = 2$

ii) $\underset{\sim}{b} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix} \right\}$ a basis for column space

iii) $\dim(\underset{\sim}{b}) = 2$

b. i) $\ker(L) = \{ \underset{\sim}{v} \in \mathbb{R}^3 \mid L(\underset{\sim}{v}) = 0_{\mathbb{R}^2} \}$

$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_2 \\ x_1 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_2 = 0 \\ x_1 = x_3 \end{matrix}$

$\ker(L) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \mid \alpha \in \mathbb{R} \right\}$

a basis.

ii) $L\left(\begin{bmatrix} 0 \\ a \\ b \end{bmatrix}\right) = \begin{bmatrix} -a \\ 0 - b \end{bmatrix} = \begin{bmatrix} -a \\ -b \end{bmatrix} = a \begin{bmatrix} -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$L(S) = \left\{ \underset{\sim}{x} = \alpha \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ -1 \end{bmatrix} \mid \alpha, \beta \in \mathbb{R} \right\}$

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QUESTION 2 The blanks below will be filled by students. (Except the score)

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[12pt] **a)** Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{S} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$. Find vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ so that S will be the transition matrix from the basis $[\mathbf{v}_1, \mathbf{v}_2]$ to the basis $[\mathbf{u}_1, \mathbf{u}_2]$.

[13pt] **b)** Let $L : P_3 \to P_2$ be the linear transformation defined by $L(p(x)) = p'(x) + p(0)$.

i) Find the matrix **S** representing L with respect to the ordered bases $E = [1, x, x^2]$ and $F = [2, 1+x]$.

ii) Use the matrix **S** to determine $L(2x^2 + x - 1)$.

a. $[v_1, v_2] \xrightarrow{V} [e_i] \xrightarrow{U^{-1}} [u_1, u_2] \Rightarrow S = U^{-1}V$

$\Rightarrow U^{-1} = SV^{-1} \Rightarrow U = (SV^{-1})^{-1} \Rightarrow U = VS^{-1}$

$$S^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 1 & 0 \end{bmatrix}$$

$$[u_1, u_2] = \left[\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \end{bmatrix}\right]$$

b. $L(p(x)) = p'(x) + p(0)$

i. $L(1) = 1' + 1 = 0 + 1 = 1$

$L(x) = x' + x\big|_{x=0} = 1 + 0 = 1$

$L(x^2) = (x^2)' + x^2\big|_{x=0} = 2x + 0 = 2x$

$(1)_F = \frac{1}{2}.(2) + 0.(1+x) \Rightarrow (1)_F = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$

$(2x)_F = -1.(2) + 2(1+x) \Rightarrow (2x)_F = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$S = \begin{bmatrix} 1/2 & 1/2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

ii. $v_F = S v_E \Rightarrow p(x) = 2x^2 + x - 1 \Rightarrow v_E = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

$$v_F = \begin{bmatrix} 1/2 & 1/2 & -1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \Rightarrow L(2x^2 + x - 1) = -2.(2) + 4.(1+x)$$

$= -4 + 4 + 4x$

$= 4x$

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QUESTION 3 The blanks below will be filled by students. (Except the score)

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Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

[17pt] **a) Find the eigenvalues and the corresponding eigenvectors of the matrix A.**

[8pt] **b) Find a matrix X and a diagonal matrix D such that $\mathbf{A} = \mathbf{X}\mathbf{D}\mathbf{X}^{-1}$.**

a. $|A - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = (-\lambda).(2-\lambda).(-\lambda) + 2.(-1).2.(2-\lambda) = 0$$

$$\Rightarrow \lambda^2(2-\lambda) - 4(2-\lambda) = 0 \Rightarrow (2-\lambda)(\lambda^2-4) = 0$$

$$\lambda_1 = -2, \quad \lambda_{2,3} = 2$$

for $\lambda_1 = -2$

$$(A - \lambda_1 I)\xi^{(1)} = 0 \Rightarrow \begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \\ \xi_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} \xi_1^{(1)} = -\xi_3^{(1)} \\ \xi_2^{(1)} = 0 \end{array}$$

$$\Rightarrow \xi^{(1)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

for $\lambda_{2,3} = 2$

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{bmatrix} \begin{bmatrix} \xi_1^{(2)} \\ \xi_2^{(2)} \\ \xi_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \xi_1^{(2)} = \xi_3^{(2)}$$

$$\Rightarrow \begin{bmatrix} \alpha \\ \beta \\ \alpha \end{bmatrix} = \alpha \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\Downarrow\ \xi^{(2)}} + \beta \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\Downarrow\ \xi^{(3)}}$$

b. $X = \left[\xi^{(1)}, \xi^{(2)}, \xi^{(3)} \right] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

QUESTION 4 The blanks below will be filled by students. (Except the score)

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[12pt] **a)** Let $\mathbf{x} = [x_1, x_2]^T$ and $\mathbf{y} = [y_1, y_2]^T$ be vectors in $\mathbb{R}^2$.

Is $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$ an inner product on $\mathbb{R}^2$? Why?

[13pt] **b)** Given the nonorthogonal basis

$$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

for R^3, obtain an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

a. i. $\langle \underset{\sim}{x}, \underset{\sim}{x} \rangle = x_1^2 - x_1x_2 - x_2x_1 + 3x_2^2 = x_1^2 - 2x_1x_2 + x_2^2 + 2x_2^2 = (x_1 - x_2)^2 + 2x_2^2 \geqslant 0$

$\Rightarrow x = 0 \iff \langle x, x \rangle = 0$

ii. $\langle \underset{\sim}{x}, \underset{\sim}{y} \rangle = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$

$= y_1x_1 - y_2x_1 - y_1x_2 + 3y_2x_2$

$= y_1x_1 - y_1x_2 - y_2x_1 + 3y_2x_2 = \langle \underset{\sim}{y}, \underset{\sim}{x} \rangle$

iii. $\langle \alpha \underset{\sim}{x} + \beta \underset{\sim}{y}, \underset{\sim}{z} \rangle = (\alpha x_1 + \beta y_1) z_1 - (\alpha x_1 + \beta y_1) z_2 - (\alpha x_2 + \beta y_2) z_1$

$+ 3(\alpha x_2 + \beta y_2) z_2$

$= \alpha x_1 z_1 - \alpha x_1 z_2 - \alpha x_2 z_1 + \alpha 3 x_2 z_2$

$+ \beta y_1 z_1 - \beta y_1 z_2 - \beta y_2 z_1 + \beta 3 y_2 z_2$

$= \alpha \langle \underset{\sim}{x}, \underset{\sim}{z} \rangle + \beta \langle \underset{\sim}{y}, \underset{\sim}{z} \rangle$

Then, $\langle \underset{\sim}{x}, \underset{\sim}{y} \rangle$ is an inner product on $\mathbb{R}^2$

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[12pt] **a)** Let $\mathbf{x} = [x_1, x_2]^T$ and $\mathbf{y} = [y_1, y_2]^T$ be vectors in $\mathbb{R}^2$.

Is $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2$ an inner product on $\mathbb{R}^2$? Why?

[13pt] **b)** Given the nonorthogonal basis

$$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

for R^3, obtain an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

b. $x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \|x_1\| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$, $x_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$u_1 = \dfrac{x_1}{\|x_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$, $\langle x_2, u_1 \rangle = x_2^T u_1 = 1 \cdot \frac{1}{\sqrt{2}} + 0 \cdot \frac{1}{\sqrt{2}} + 1 \cdot 0 = \frac{1}{\sqrt{2}}$

$p_1 = \langle x_2, u_1 \rangle u_1 = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$

$x_2 - p_1 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$, $\|x_2 - p_1\| = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \dfrac{\sqrt{6}}{2}$

$u_2 = \dfrac{x_2 - p_1}{\|x_2 - p_1\|} = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$ $\qquad x_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\langle x_3, u_1 \rangle = 0 \cdot \frac{1}{\sqrt{2}} + 1 \cdot \frac{1}{\sqrt{2}} + 1 \cdot 0 = \frac{1}{\sqrt{2}}$

$p_2 = \langle x_3, u_1 \rangle u_1 + \langle x_3, u_2 \rangle u_2$

$\langle x_3, u_2 \rangle = 0 \cdot \frac{1}{\sqrt{6}} + 1 \cdot \left(-\frac{1}{\sqrt{6}}\right) + 1 \cdot \frac{2}{\sqrt{6}} = \frac{1}{\sqrt{6}}$

$$p_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} + \frac{1}{\sqrt{6}} \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1/6 \\ -1/6 \\ 2/6 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$x_3 - p_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 2/3 \\ 2/3 \end{bmatrix}$, $\|x_3 - p_2\| = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{4}{9}} = \dfrac{2\sqrt{3}}{3}$

$$u_3 = \frac{x_3 - p_2}{\|x_3 - p_2\|} = \begin{bmatrix} -\sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix}$$