B U Department of Mathematics

Math 201 Matrix Theory

Fall 2002 Second Midterm

1. (a) Find the equation **(do not solve)** for the coefficients C, D, E in

$$b = C + Dt + Et^2,$$

the parabola which best fits the four points:$(t, b) = (0, 0), (1, 1), (1, 3)$ and $(2, 2)$.

Solution:

We have

$$C = 0, \quad C + D + E = 1, \quad C + D + E = 3, \quad C + 2D + 4E = 2.$$

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$, $x = \begin{pmatrix} C \\ D \\ E \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 2 \end{pmatrix}$.

We have the system $Ax = b$ which is inconsistent. We need to look for its least square solution and solve the system:

$$A^T A\bar{x} = A^T b,$$

i.e.

$$\begin{pmatrix} 4 & 4 & 6 \\ 4 & 6 & 10 \\ 6 & 10 & 18 \end{pmatrix} \begin{pmatrix} \bar{C} \\ \bar{D} \\ \bar{E} \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 12 \end{pmatrix}$$

(b) **(Fill in the blanks)** In solving this problem you are projecting the vector (____b____) onto the subspace spanned by (____the columns of A____)

2. Let $A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{pmatrix}$

(a) Find orthonormal vectors e_1, e_2 and e_3 so that e_1 and e_2 form a basis for the column space of A.

(b) Find the projection matrix P which projects onto the left nullspace of A.

Solution:

(a) Let

$$a = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}, \quad e_1 = \frac{a}{\|a\|} = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

$$b' = b - (e_1^T b)e_1, \quad e_2 = \frac{b'}{\|b'\|} = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

Then e_3 must be in the left nullspace $\mathcal{N}(A^T)$:

$$A^T y = \begin{pmatrix} 1 & 2 & -2 \\ 1 & -1 & 4 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow y_1 = -2y_3, \quad y_2 = 2y_3,$ and y_3 is free.
$\Rightarrow dim \mathcal{N}(A^T) = 1$

A basis for $\mathcal{N}(A^T)$ is : $c = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$ .Then

$$e_3 = \frac{c}{\|c\|} = \frac{1}{3}\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}.$$

Now $\{e_1, e_2, e_3\}$ is now an ON basis for $\mathbb{R}^3$.

(b) We have

$$P = e_3(e_3^T e_3)^{-1} e_3^T = e_3 e_3^T = \frac{1}{9}\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}\begin{pmatrix} -2 & 2 & 1 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 4 & -4 & -2 \\ -4 & 4 & 2 \\ -2 & 2 & 1 \end{pmatrix}$$

3. (a) Let P_2 be the vector space of polynomials of degree less than or equal to 2. Suppose $L : P_2 \to \mathbb{R}^3$ is the linear transformation defined by

$$L[p(t)] = \begin{pmatrix} p(1) \\ p(0) \\ p(-1) \end{pmatrix}, \quad p(t) \in P_2.$$

Find the matrix representation of L relative to the standard bases of P_2 and $\mathbb{R}^3$.

Solution:

We know that $p_1 = 1, p_2 = t, p_3 = t^2$ is the standard basis of P_2 and

$$e_1^T = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, e_2^T = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, e_3^T = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

is the standard basis of $\mathbb{R}^3$. Then

$L[p_1] = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = e_1+e_2+e_3, \quad L[p_2] = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} = e_1-e_2, \quad L[p_3] = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} = e_1+e_3.$

Since $L[p_j] = \sum_{i=1}^{3} a_{ij} e_i$, where (a_{ij}) is the matrix representation, we have

$$A = (a_{ij}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

(b) Let P be an $n \times n$ matrix satisfying $P^2 = P$ and let $\lambda \neq 1$ be real. Prove that the matrix: $I - \lambda P$ is invertible and

$$(I - \lambda P)^{-1} = I + \frac{\lambda}{1-\lambda} P$$

Solution:

Consider the nullspace of $I - \lambda P$: $(I - \lambda P)x = 0$
$\Rightarrow \lambda P x = x$. Apply P: $\lambda P^2 x = Px = \lambda P x \Rightarrow (\lambda - 1)Px = 0 \Rightarrow Px = 0$
$\Rightarrow x = 0$. This means that the nullspace is trivial:
$dim \mathcal{N}(I - \lambda P) = 0$; $I - \lambda P$ is of rank $n \Leftrightarrow (I - \lambda P)^{-1}$ exists. Then

$$(I - \lambda P)(I - \lambda P)^{-1} = (I - \lambda P)[I + \frac{\lambda}{1-\lambda} P] = I - \lambda P - \frac{\lambda^2}{1-\lambda} P^2 + \frac{\lambda P}{1-\lambda}$$

$$= I + \lambda P(\frac{-\lambda}{1-\lambda} - 1 + \frac{1}{1-\lambda}) = I$$

Therefore,

$$(I - \lambda P)^{-1} = I + \frac{\lambda}{1-\lambda} P.$$

4. (a) Let $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$. How many of the 24 terms in $detA$ are nonzero? Justify your answer and find $detA$.

Solution:

There are 4 nonzero terms in $detA$. These are:

$$\begin{array}{ccccccc} + & a_{11} & a_{22} & a_{33} & a_{44} & = & -1 \\ - & a_{11} & a_{24} & a_{33} & a_{42} & = & -1 \\ - & a_{13} & a_{22} & a_{31} & a_{44} & = & -1 \\ + & a_{13} & a_{24} & a_{31} & a_{42} & = & -1 \end{array}$$

$\Rightarrow detA = -4.$

(b) Prove that if B is an $n \times n$ matrix of rank n, then the adjugate matrix B_{cof} must also have rank n.

Solution:

We have $BB_{cof} = (detB)I$, $detB \neq 0$.
$\Rightarrow det(B_{cof}) = (detB)^{n-1} \neq 0$
$\Rightarrow B_{cof}$ has rank n.

MATH 201
SECOND MIDTERM EXAM
December 21, 2002, 11:00-12:00

I.	/22
II.	/26
III.	/26
IV.	/26
Total:	/100

NO: --------

NAME: SOLUTIONS

SIGNATURE:----------------------------

Please write your name at the top of each page (**in ink**). Label all answers clearly and show all work. Calculators and cellular phones should be switched off.

I. (a) Find the equations (**do not solve**) for the coefficients C, D, E in: $b = C + Dt + Et^2$, the parabola which best fits the four points: $(t, b) = (0, 0)$, $(1, 1)$, $(1, 3)$ and $(2, 2)$.

$C = 0$
$C + D + E = 1$
$C + D + E = 3$
$C + 2D + 4E = 2$

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$, $x = \begin{bmatrix} C \\ D \\ E \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix}$.

We have the system $Ax = b$ which is inconsistent. We need to look for its least square solution and solve the system:

$A^T A \bar{x} = A^T b$, i.e.

$$\begin{bmatrix} 4 & 4 & 6 \\ 4 & 6 & 10 \\ 6 & 10 & 18 \end{bmatrix} \begin{bmatrix} \bar{C} \\ \bar{D} \\ \bar{E} \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 12 \end{bmatrix}.$$

(b) (Fill in the blanks)
In solving this problem you are projecting the vector: b
onto the subspace spanned by: the columns of A.

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II. Let $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$.

(a) Find orthonormal vectors e_1, e_2 and e_3 so that e_1 and e_2 form a basis for the column space of A.

(b) Find the projection matrix P which projects onto the left nullspace of A.

(a) Let $a = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$, $e_1 = \frac{a}{\|a\|} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

$b' = b - (e_1^T b)\, e_1$, $e_2 = \frac{b'}{\|b'\|} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

e_3 must be in the left nullspace $N(A^T)$:

$A^T y = \begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ $y_1 = -2y_3$, $y_2 = 2y_3$, y_3 is free

$\Rightarrow \dim N(A^T) = 1$

A basis for $N(A^T)$ is: $c = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$

$e_3 = \frac{c}{\|c\|} = \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$. $\{e_1, e_2, e_3\}$ is now an ON basis for $\mathbb{R}^3$.

(b) $P = e_3 (e_3^T e_3)^{-1} e_3^T = e_3 e_3^T$

$= \frac{1}{9}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} -2 & 2 & 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 4 & -4 & -2 \\ -4 & 4 & 2 \\ -2 & 2 & 1 \end{bmatrix}$

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III. (a) Let P_2 be the vector space of polynomials of degree less than or equal to 2. Suppose $L: P_2 \to R^3$ is the linear transformation defined by $L[p(t)] = \begin{bmatrix} p(1) \\ p(0) \\ p(-1) \end{bmatrix}$, $p(t) \in P_2$. Find the matrix representation of L relative to the standard bases of P_2 and R^3.

$p_1 = 1,\ p_2 = t,\ p_3 = t^2$ is the standard basis of P_2

$e_1^T = [1\ 0\ 0],\ e_2^T = [0\ 1\ 0],\ e_3^T = [0\ 0\ 1]$ is the standard basis of R^3

$L[p_1] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = e_1 + e_2 + e_3,\quad L[p_j] = \sum_{i=1}^{3} a_{ij} e_i$

$L[p_2] = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = e_1 - e_3,\quad A = (a_{ij}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}.$

$L[p_3] = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = e_1 + e_3,$

(b) Let P be an $n \times n$ matrix satisfying $P^2 = P$ and $\lambda \neq 1$ be real. Prove that the matrix: $I - \lambda P$ is invertible and $(I - \lambda P)^{-1} = I + \frac{\lambda}{1-\lambda} P$.

Consider the nullspace of $I - \lambda P$: $(I - \lambda P)x = 0$

$\Rightarrow \lambda P x = x$. Apply P: $\lambda P^2 x = Px = \lambda P x$

$\Rightarrow (\lambda - 1) P x = 0 \Rightarrow Px = 0 \Rightarrow x = 0$. This means that the nullspace is trivial: $\dim N(I - \lambda P) = 0$; $I - \lambda P$ is of rank $n \Leftrightarrow (I - \lambda P)^{-1}$ exists.

$(I - \lambda P)(I - \lambda P)^{-1} = (I - \lambda P)\left[I + \frac{\lambda}{1-\lambda} P\right]$

$= I - \lambda P - \frac{\lambda^2}{1-\lambda} P^2 + \frac{\lambda P}{1-\lambda} = I + \lambda P\left(\frac{-\lambda}{1-\lambda} - 1 + \frac{1}{1-\lambda}\right)$

$= I$. Therefore, $(I - \lambda P)^{-1} = I + \frac{\lambda}{1-\lambda} P$.

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IV. (a) Let $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$. How many of the 24 terms in det A are nonzero? Justify your answer and find det A .

There are 4 nonzero terms in det A. These are :

$$+ a_{11} a_{22} a_{33} a_{44} = -1$$
$$- a_{11} a_{24} a_{33} a_{42} = -1$$
$$- a_{13} a_{22} a_{31} a_{44} = -1$$
$$+ a_{13} a_{24} a_{31} a_{42} = -1$$

$$\Rightarrow \det A = -4.$$

(b) Prove that if B is an n x n matrix of rank n, then the adjugate matrix B_{cof} must also have rank n.

$$B B_{cof} = (\det B) I \quad , \quad \det B \neq 0$$

$$\Rightarrow \det(B_{cof}) = (\det B)^{n-1} \neq 0$$

$$\Rightarrow B_{cof} \text{ has rank } n.$$

B U Department of Mathematics

Math 201 Matrix Theory

Fall 2004 Second Midterm

1. (a) Let $\boldsymbol{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$ and $\det \boldsymbol{A} = -3$. Find the value of the determinant:

$$\det \boldsymbol{B} = \begin{vmatrix} a & b & c \\ g+2a & h+2b & k+2c \\ 3d & 3e & 3f \end{vmatrix}.$$

Solution:

$$\begin{aligned} \det \boldsymbol{A} &= -\begin{vmatrix} a & b & c \\ g & h & k \\ d & e & f \end{vmatrix} \text{ (interchanging 2}^{\text{nd}}\text{ and 3}^{\text{rd}}\text{ rows)} \\ &= -\begin{vmatrix} a & b & c \\ g+2a & h+2b & k+2c \\ d & e & f \end{vmatrix} \text{ (adding twice the 1}^{\text{st}}\text{ row to the 2}^{\text{nd}}\text{ row)} \\ &= -\frac{1}{3}\begin{vmatrix} a & b & c \\ g+2a & h+2b & k+2c \\ 3d & 3e & 3f \end{vmatrix} \text{ (multiplying the 3}^{\text{rd}}\text{ row by 3)} \\ &= -\frac{1}{3}\det \boldsymbol{B}. \end{aligned}$$

Hence $\det \boldsymbol{B} = -3 \cdot (-3) = 9$.

(b) Find the $n \times n$ determinant: $\begin{vmatrix} & & & n \\ & 0 & n-1 & \\ & \iddots & & \\ & 2 & 0 & \\ 1 & & & \end{vmatrix}$.

Solution:

Interchange 1^{st} and n^{th} rows, 2^{nd} and $(n-1)^{\text{st}}$ rows, in general, k^{th} and $(n-k+1)^{\text{st}}$ rows to obtain the diagonal matrix

$$\begin{vmatrix} 1 & & & & \\ & 2 & & & \\ & & \ddots & & \\ & & & n-1 & \\ & & & & n \end{vmatrix}$$

provided $k < n-k+1$ i.e. $2k < n+1$. If n is odd then the number s of swappings is equal to $\frac{n+1}{2} - 1 = \frac{n-1}{2}$. If n is even then $s = \frac{n}{2}$. Hence,

$$\begin{vmatrix} & & & n \\ & 0 & n-1 & \\ & \iddots & & \\ & 2 & 0 & \\ 1 & & & \end{vmatrix} = (-1)^s \begin{vmatrix} 1 & & & & \\ & 2 & & & \\ & & \ddots & & \\ & & & n-1 & \\ & & & & n \end{vmatrix} = (-1)^s n!.$$

(c) Find $\det \boldsymbol{A}_n$ if $\boldsymbol{A}_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1+a_1 & 1 & \cdots & 1 \\ 1 & 1 & 1+a_2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1+a_{n-1} \end{bmatrix}_{n\times n}$, i.e. all off-diagonal entries are 1.

Solution:

Subtract 1^{st} row from 2^{nd}, 3^{rd}, $\cdots$, n^{th} rows. This does not change the determinant. Hence,

$$\det \boldsymbol{A}_n = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & a_1 & 0 & \dots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & a_{n-1} \end{vmatrix} = a_{n-1}a_{n-2}\cdots a_2 a_1.$$

2. Consider the following linear system:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 2 \\ x_1 + 4x_2 + x_3 + x_4 &= 5 \\ x_1 + x_2 - x_3 + x_4 &= 4 \\ x_1 + x_2 + x_3 + 3x_4 &= 2 \end{aligned}$$

Use the Cramer's rule to find x_1 and x_4 (Hint: Part **(c)** of question **1** might be helpful in computations).

Solution:

By Cramer's rule,

$$x_1 = \frac{\det \boldsymbol{B}_1}{\det \boldsymbol{A}} = \frac{\begin{vmatrix} 2 & 1 & 1 & 1 \\ 5 & 4 & 1 & 1 \\ 4 & 1 & -1 & 1 \\ 2 & 1 & 1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix}}$$

$\det \boldsymbol{A} = (4-1)\cdot(-1-1)\cdot(3-1) = -12$ by question (1c). As for $\det \boldsymbol{B}_1$, subtracting first row from other rows of $\boldsymbol{B}_1$ we get:

$$\det \boldsymbol{B}_1 = \begin{vmatrix} 2 & 1 & 1 & 1 \\ 3 & 3 & 0 & 0 \\ 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 2\cdot(2\cdot(-3) + (-2)\cdot(6-3)) = -24.$$

Hence $x_1 = \frac{-24}{-12} = 2$. Similarly,

$$x_1 = \frac{\det \boldsymbol{B}_4}{\det \boldsymbol{A}} = -\frac{1}{12}\begin{vmatrix} 1 & 1 & 1 & 2 \\ 1 & 4 & 1 & 5 \\ 1 & 1 & -1 & 4 \\ 1 & 1 & 1 & 2 \end{vmatrix} = 0$$

since first and last rows of $\boldsymbol{B}_4$ are identical.

3. Consider $\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$.

(a) Solve $\boldsymbol{Ax} = \boldsymbol{b}$ by the method of least squares.

Solution:

Since $\boldsymbol{b}$ is not in the column space of $\boldsymbol{A}$, there is no solution for $\boldsymbol{x}$. Yet, there is a vector $\boldsymbol{y}$ that minimizes the error $\boldsymbol{Ay} - \boldsymbol{b}$:

$$\begin{aligned}
\boldsymbol{y} &= (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{b} \\
&= \left(\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \\
&= \left(\begin{bmatrix} 14 & 2 \\ 2 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 10 \\ -2 \end{bmatrix} \\
&= \frac{1}{14 \cdot 2 - 2 \cdot 2} \begin{bmatrix} 2 & -2 \\ -2 & 14 \end{bmatrix} \begin{bmatrix} 10 \\ -2 \end{bmatrix} \\
&= \frac{1}{24} \begin{bmatrix} 24 \\ -48 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ -2 \end{bmatrix}
\end{aligned}$$

(b) Find the projection of $\boldsymbol{b}$ onto the orthogonal complement of the column space of $\boldsymbol{A}$.

Solution:

Note that the projection of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$ equals $\boldsymbol{Ay}$ where $\boldsymbol{y}$ is the vector found in part (a). Therefore, the projection of $\boldsymbol{b}$ onto $(\text{col}(\boldsymbol{A}))^{\perp}$:

$$\begin{aligned}
\boldsymbol{proj}_{(\text{col}(\boldsymbol{A}))^{\perp}}\boldsymbol{b} &= \boldsymbol{b} - \boldsymbol{Ay} \\
&= \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}
\end{aligned}$$

One can also use longer ways such as finding a basis for the orthogonal complement (which is in fact the left null space of $\boldsymbol{A}$) and projecting $\boldsymbol{b}$ onto that subspace.

4. Let $\boldsymbol{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 2 \\ 3 \\ -1 \\ 2 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} -1 \\ -6 \\ -1 \\ -1 \end{bmatrix}$. Let $V = \text{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\} \subset \mathbb{R}^4$.

(a) Find an orthonormal basis for V.

Solution:

The dimension of V is 2. One way to see this is to observe that $\boldsymbol{w} = 3\boldsymbol{u} - 2\boldsymbol{v}$. Alternatively, one can check that the matrix $[\boldsymbol{u} \vdots \boldsymbol{v} \vdots \boldsymbol{w}]$ has rank 2 (If you do not check this, then the computation gets tedious). Now, it is enough to perform Gram-Schmidt process to $\boldsymbol{w}$ and $\boldsymbol{v}$ which are linearly independent. Let $\boldsymbol{q_1'} = \boldsymbol{u}$. Then $||\boldsymbol{q_1'}||^2 = 3$. Now let:

$$\boldsymbol{q_2'} = \boldsymbol{v} - \frac{\boldsymbol{q_1'}^T \boldsymbol{v}}{\boldsymbol{q_1'}^T \boldsymbol{q_1'}} \boldsymbol{q_1'} = \begin{bmatrix} 2 \\ 3 \\ -1 \\ 2 \end{bmatrix} - \frac{5}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 9 \\ 2 \\ 1 \end{bmatrix}$$

Finally define

$$\boldsymbol{q_1} = \frac{\boldsymbol{q_1'}}{||\boldsymbol{q_1'}||} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \text{ and } \boldsymbol{q_2} = \frac{\boldsymbol{q_2'}}{||\boldsymbol{q_2'}||} = \frac{1}{\sqrt{87}} \begin{bmatrix} 1 \\ 9 \\ 2 \\ 1 \end{bmatrix}.$$

(b) Find an orthonormal basis for the orthogonal complement $V^\perp$ of V.

Solution:

Each vector of $V^\perp$ is by definition orthogonal to each vector of V. In particular, a vector $\boldsymbol{x}^T = (x_1, x_2, x_3, x_4) \in V^\perp$ must be orthogonal to $\boldsymbol{u}$ and $\boldsymbol{v}$ of part (a), i.e. $\boldsymbol{x}^T\boldsymbol{u} = 0$ and $\boldsymbol{x}^T\boldsymbol{v} = 0$. These give:

$$\boldsymbol{Ax} = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

or equivalently, $\boldsymbol{x}$ must be in nul($\boldsymbol{A}$). Since $\boldsymbol{A}$ is row equivalent to $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 3 & 1 & 0 \end{bmatrix}$, choosing x_3, x_4 as free variables, one gets $\boldsymbol{x}^T = (x_3 - x_4, -3x_3, x_3, x_4)$. Now choose for example $x_3 = 0, x_4 = 1$ and then $x_3 = 1, x_4 = 0$ to get

$$\boldsymbol{y_1} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{y_2} = \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix} \text{ so that } V^\perp = \text{span}(\boldsymbol{y_1}, \boldsymbol{y_2}).$$

Since $\boldsymbol{y_1}$ is not orthogonal to $\boldsymbol{y_2}$, apply Gram-Schmidt process. Let $\boldsymbol{q_3'} = \boldsymbol{y_1}$. Then $||\boldsymbol{q_3'}||^2 = 2$. Now let:

$$\boldsymbol{q_4'} = \boldsymbol{y_2} - \frac{\boldsymbol{q_3'}^T \boldsymbol{y_2}}{\boldsymbol{q_3'}^T \boldsymbol{q_3'}} \boldsymbol{q_3'} = \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -6 \\ 2 \\ 1 \end{bmatrix}$$

Finally define

$$\boldsymbol{q_3} = \frac{\boldsymbol{q_3'}}{||\boldsymbol{q_3'}||} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } \boldsymbol{q_4} = \frac{\boldsymbol{q_4'}}{||\boldsymbol{q_4'}||} = \frac{1}{\sqrt{42}}\begin{bmatrix} 1 \\ -6 \\ 2 \\ 1 \end{bmatrix}.$$

(c) Assume that you are given arbitrary real numbers x_1, x_2, x_3, x_4. What is the length of the vector $\boldsymbol{x} = x_1\boldsymbol{q}_1 + x_2\boldsymbol{q}_2 + x_3\boldsymbol{q}_3 + x_4\boldsymbol{q}_4$? Here $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3, \boldsymbol{q}_4$ are the vectors you found in part **(a)** and part **(b)** (Hint: Even if you cannot solve the previous parts you can solve part **(c)**).

Solution:

Observe that

$$\boldsymbol{x} = x_1\boldsymbol{q}_1 + x_2\boldsymbol{q}_2 + x_3\boldsymbol{q}_3 + x_4\boldsymbol{q}_4 = [\boldsymbol{q_1} \vdots \boldsymbol{q_2} \vdots \boldsymbol{q_3} \vdots \boldsymbol{q_4}]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \boldsymbol{Q}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Since Q is an orthogonal matrix, it preserves lengths. Hence

$$||\boldsymbol{x}|| = ||\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}|| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}.$$

The crucial remark in this part is that an orthonormal basis has basis vectors with inner product zero between distinct pairs. If you do not explicitly state this and show how it leads the result, you will not get more than half of the points of this part.

B U Department of Mathematics

Math 201 Matrix Theory

Spring 2001 Second Midterm

1. Using the method of least squares find the straight line which fits best to the data points: $(2,1)$, $(3,2)$, $(4,3)$, $(5,2)$.

Solution:

Let $y = C + Dt$. The data points give: $C + 2D = 1$, $C + 3D = 2$, $C + 4D = 3$, $C + 5D = 2$. Therefore $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}$, $x = \begin{bmatrix} C \\ D \end{bmatrix}$.

The least square solution is $\overline{x} = (A^TA)^{-1}A^Tb$.

$$A^TA = \begin{bmatrix} 4 & 14 \\ 14 & 54 \end{bmatrix}, \quad (A^TA)^{-1} = \frac{1}{20}\begin{bmatrix} 54 & -14 \\ -14 & 4 \end{bmatrix}, \quad A^Tb = \begin{bmatrix} 8 \\ 30 \end{bmatrix}$$

$$\overline{x} = \begin{bmatrix} \overline{C} \\ \overline{D} \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 54 & -14 \\ -14 & 4 \end{bmatrix}\begin{bmatrix} 8 \\ 30 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}$$

Therefore the line of best fit is:

$$y = \frac{2}{5}t + \frac{3}{5} = 0.4t + 0.6$$

2. **(a)** Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by:

$$L(x_1, x_2) = (-x_2, x_1), \quad (x_1, x_2) \in \mathbb{R}^2$$

Prove that if $A = (a_{ij})$ is the matrix representation of L with respect to any ordered basis of $\mathbb{R}^2$, then $a_{12}.a_{21} \neq 0$.

Solution:

Note that $(L \circ L)(x_1, x_2) = -(x_1, x_2)$. Let $\mathcal{B}$ be a basis of $\mathbb{R}^2$ and $A = [L]_{\mathcal{B}}$. Then $[L \circ L]_{\mathcal{B}} = A^2$ and $L \circ L = -$(identity transformation). Hence $A^2 = -I$.

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ where $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$. Then we obtain

$$a_{11}^2 + a_{12}.a_{21} = -1, \quad a_{22}^2 + a_{12}.a_{21} = -1.$$

By the way contradiction assume that $a_{12}.a_{21} = 0$. Then $a_{11}^2 = -1$ and $a_{22}^2 = -1$. But this is a contradiction since $a_{11}, a_{22} \in \mathbb{R}$. Therefore $a_{12}.a_{21} \neq 0$.

(b) Let a and b be arbitrary real numbers. Using Cauchy-Schwarz inequality show that

$$\left(\frac{a+b}{2}\right)^2 \leq \frac{a^2+b^2}{2}$$

(No credit will be given to other approaches.)

Solution:

Cauchy-Schwarz inequality : $|u^T v| \leq ||u||\ ||v||$.

Let $u = \frac{1}{2}\begin{bmatrix} a \\ b \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then,

$$u^T v = \frac{a+b}{2}, \quad ||u|| = \frac{1}{2}(a^2+b^2)^{\frac{1}{2}}, \quad ||v|| = \sqrt{2}.$$

Taking the square of the Cauchy-Schwarz inequality gives

$$\left(\frac{a+b}{2}\right)^2 \leq \frac{a^2+b^2}{2}$$

3. **(a)** Let V be the line: $x_1 = 2t,\ x_2 = -t,\ x_3 = 2t\ (t \in \mathbb{R})$ and $W = V^{\perp}$ be the orthogonal complement of V in $\mathbb{R}^3$. Find an orthonormal basis for W.

Solution:

Let $(x_1, x_2, x_3) \in \mathbb{R}^3$. $W = V^{\perp} = \{x \in \mathbb{R}^3 \mid 2x_1 - x_2 + 2x_3 = 0\}$.

$x_1 = \frac{1}{2}x_2 - x_3 \Longrightarrow a = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, b = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is a basis for W.

$e_1 = \frac{a}{||a||} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. $b' = b - (e_1^T b)e_1 = \begin{bmatrix} -\frac{4}{5} \\ \frac{2}{5} \\ 1 \end{bmatrix}$, $||b'|| = \frac{3}{\sqrt{5}} \Rightarrow e_2 = \frac{b'}{||b'||} = \frac{1}{\sqrt{5}}\begin{bmatrix} -\frac{4}{3} \\ \frac{2}{3} \\ \frac{5}{3} \end{bmatrix}$

$\{e_1, e_2\}$ is an orthonormal basis for W.

(b) Show that if A is an $n \times n$ orthogonal matrix and $I + A$ is non-singular, then the matrix: $B = (I - A)(I + A)^{-1}$ is skew-symmetric.

Solution:

A is orthogonal: $AA^T = A^T A = I$; $A^{-1} = A^T$.

$$\begin{aligned} B &= (I+A)^{-1} - A(I+A)^{-1} \\ &= (I+A)^{-1} - [(I+A)A^{-1}]^{-1} \\ &= (I+A)^{-1} - (A^{-1}+I)^{-1} \\ &= (I+A)^{-1} - (I+A^T)^{-1} \\ &= (I+A)^{-1} - [(I+A)^T]^{-1} \\ &= (I+A)^{-1} - [(I+A)^{-1}]^T \end{aligned}$$

$B^T = \left[(I+A)^{-1} - [(I+A)^{-1}]^T\right]^T = [(I+A)^{-1}]^T - (I+A)^{-1} = -B.$

Therefore $B^T = -B$.

4. **(a)** Without using cofactor expansion show that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)(c-a)(c-b).$$

(Show and justify all your steps)

Solution:

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 1 & c & c^2 \end{vmatrix} = (b-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix} = (b-a)(c-a)(c-b)\underbrace{\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & 1 \end{vmatrix}}_{1}$$

$$= (b-a)(c-a)(c-b)$$

Here we used:

- If a multiple of one row is added to another row, determinant is not changed.
- A common factor of a row can be taken out.
- The upper triangular matrix has unit determinant.

(b) Let $A = (a_{ij})$ be a 5×5 matrix. Determine the sign of the term: $a_{44}a_{15}a_{23}a_{52}a_{31}$ in the determinant of A. Justify your answer.

Solution:

The term in the question is $a_{44}a_{15}a_{23}a_{52}a_{31}$ which involves the permutation $\sigma = (5, 3, 1, 4, 2)$.

5 has 4 inversions
3 has 2 inversions
1 has 0 inversion
4 has 1 inversion

$\Longrightarrow$ The total number of inversions is 7, odd. $\Rightarrow \sigma$ is odd.

Therefore the sign of this term is $-$.

B U Department of Mathematics

Math 201 Matrix Theory

Spring 2002 Second Midterm

1. Let W be in the plane in $\mathbb{R}^3$ defined by
$2x_1 - x_2 + 4x_3 = 0,\ (x_1, x_2, x_3) \in \mathbb{R}^3$
(a) Find an orthonormal basis for W.
(b) Determine the orthonormal comlement $W^\perp$ of W.

Solution:

(a)

$$x \in W \Leftrightarrow x = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow a = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, b = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \text{ is basis for } W.$$

Let

$$q_1 = \frac{a}{\|a\|},\ b' = b - (q_1^T b)q_1,\ q_2 = \frac{b'}{\|b'\|}$$

$$q_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix},\ q_2 = \frac{1}{\sqrt{105}} \begin{bmatrix} -8 \\ 4 \\ 5 \end{bmatrix}$$

$\{q_1, q_2\}$ is an ON basis for W.Note that the answer is not unique; this just on of the possible ON basis.

(b)

$$\text{The normal two W is } n = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \Rightarrow y \in W^\perp \Leftrightarrow y = c \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, c \in \mathbb{R}$$

$$\text{Hence } W^\perp = \{\ y \in \mathbb{R}^3 | y = c \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix},\ c \in \mathbb{R}\}$$

2. **(a)**
Let Q be an $n \times n$ matrix. Find n if every in Q is either $\frac{1}{3}$ or $\frac{-1}{3}$.

Solution:

$QQ^T = Q^TQ = I$. Let x and y be two different columns

if $Q : X^TX = 1,\ X^TY = 0,\ Y^TY = 1. X^TX = Y^TY = 0$ can
hold if $n = q$. However, $X^TY = \sum_{i=1}^9 x_iy_i = 0 \Rightarrow \sum_{i=1}^9 m_i = 0$ where $m_i = \pm 1$.

It is not possible because q is odd.Hence n does not exist.
(See problem 3.15, p.208 of the textbook)

(b)
Find the least squares solution to the inconsistent system:

$$\mathbf{A}x = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix} = b$$

What is the projection p of b onto space **A**.

Solution:

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

The least square solution is $x = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T b$

$$x = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 18 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad x = \begin{bmatrix} 6 \\ \frac{5}{2} \end{bmatrix}$$

The projection of b onto the column space of **A** is

$$p = \mathbf{A}x = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^{\mathbf{T}}b$$

$$p = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ \frac{5}{2} \end{bmatrix} = \begin{bmatrix} \frac{7}{2} \\ 6 \\ \frac{17}{2} \end{bmatrix}$$

(see prob. 3.3.5; p.163 of the text book)

3. **(a)**Let $A : \mathbf{R^3} \to \mathbf{R^2}$ be the linear transformation defined by

$$A(x, y, z) = (2x + y, y - 3z).$$

Find the matrix representation of A relative to the standard bases of $\mathbf{R^3}$ and $\mathbf{R^2}$.
Solution:

(a)

Let $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. (Standard basis of $\mathbf{R^3}$)

and $q_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $q_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. (The matrix representation).

$[\mathbf{A}] = (a_{ij})$ is defined by $\mathbf{A}e_j = \sum_{i=1}^{2} a_{ij} q_i$

$\mathbf{A}e_i = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2q_1,\ \mathbf{A}e_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = q_1 + q_2,\ \mathbf{A}e_3 = \begin{bmatrix} 0 \\ -3 \end{bmatrix} = -3q_2.$

$\Rightarrow [\mathbf{A}] = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -3 \end{bmatrix}.$

(b) Suppose the permutation σ takes the values from $(1, 2, 3, 4, 5)$ to $(5, 4, 3, 2, 1)$. Determine whether σ is an even or an odd permutation.

$$P_\sigma = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},\ \det P_\sigma = 1.\text{ Hence } \sigma \text{ is an even permutation.}$$

Alternatively,
5 has 4 inversions,
4 has 3 inversions,
1 has 0 inversions,
3 has 1 inversions,
2 has 0 inversions.

Total number of inversions: $4 + 3 + 1 = 8$, .Since the total number of the inversions are even, σ is an even permutation.

4. **(a)** The adjugate matrix $\mathbf{A}_{cof}$ of a certain matrix A is singular or nonsingular. Justify your answer. $(\det \mathbf{A})(\det \mathbf{A_{cof}}) = (\det \mathbf{A})^3$
$\det(\mathbf{A}_{cof}) = 4\begin{vmatrix} 2 & 5 \\ 1 & 8 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 44 + 3 = 47.$

So $\mathbf{A}_{cof}$is invertible.If $\mathbf{A}$ is singular, $\mathbf{A}\mathbf{A}_{cof} = 0$.But when $\mathbf{A}_{cof}$ is invertible this implies $\mathbf{A} = 0$. Hence $\mathbf{A}$ must be nonsingular:$(\det \mathbf{A})^2 = (\det \mathbf{A_{cof}}) = 47$.
$\Rightarrow (\det \mathbf{A}) = \sqrt{47}$.

Solution:

(a) $\mathbf{A}\mathbf{A}_{cof} = (\det \mathbf{A})I$, $\mathbf{A}$ is a 3×3 matrix.

So,
$(\det \mathbf{A})(\det \mathbf{A_{cof}}) = (\det \mathbf{A})^3$
$\det(\mathbf{A}_{cof}) = 4\begin{vmatrix} 2 & 5 \\ 1 & 8 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 44 + 3 = 47.$

So $\mathbf{A}_{cof}$is invertible.If $\mathbf{A}$ is singular, $\mathbf{A}\mathbf{A}_{cof} = 0$.But when $\mathbf{A}_{cof}$ is invertible this implies $\mathbf{A} = 0$. Hence $\mathbf{A}$ must be nonsingular:$(\det \mathbf{A})^2 = (\det \mathbf{A_{cof}}) = 47$.
$\Rightarrow (\det \mathbf{A}) = \sqrt{47}$.

(b) Let $\mathbf{B}$ be the $n\times$ matrix in which every entry is 1. Prove that for every positive integer n, $\det(\mathbf{B} - \mathbf{nI}) = 0$, where I is the identity matrix.

Solution:

$$\det \mathbf{B}-\mathbf{nI} = \begin{bmatrix} 1-n & 1 & . & 1 \\ 1 & 1-n & . & 1 \\ . & . & . & . \\ 1 & 1 & . & 1-n \end{bmatrix}$$

If we add all other rows to the first row of $\mathbf{B}-\mathbf{nI}$, the first row becomes a zero row.Adding one row to another row does not change the value of the determinant. On the other hand , i f zero matrix has a zero row, its determinant is zero. Therefore $\det(\mathbf{B}-\mathbf{nI}) = 0$.

B U Department of Mathematics

Math 201 Matrix Theory

Spring 2004 Second Midterm

1. Let W be a subspace spanned by the following vectors

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1/3 \\ 1/3 \\ -2/3 \end{pmatrix} \right\}$$

Find an *orthonormal basis* for W

Solution:

Let $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1/3 \\ 1/3 \\ -2/3 \end{pmatrix}$.Then,

$$v_1^T v_2 = (1\ 1\ 1) \begin{pmatrix} 1/3 \\ 1/3 \\ -2/3 \end{pmatrix} = 1/3 + 1/3 - 2/3 = 0$$

Then, $v_1 \perp v_2$.

So, $u_1 = \frac{1}{\|v_1\|} = 1/\sqrt{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

and $u_2 = \frac{1}{\|v_2\|} = 3/\sqrt{6} \begin{pmatrix} 1/3 \\ 1/3 \\ -2/3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix}$

Therefore,$\{u_1, u_2\}$ is an *orthonormal basis* for W.

2. Let A= $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$. Find the QR factorization of A.

Solution:

$$A = QR = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} q_1^T\alpha_! & q_1^T\alpha_2 & q_1^T\alpha_3 \\ 0 & q_2^T\alpha_2 & q_2^T\alpha_3 \\ 0 & 0 & q_3^T\alpha_3 \end{bmatrix}$$

where $\alpha_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\alpha_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

Then

$$x_1 = \alpha_1,\ q_1 = \tfrac{x_1}{\|x_1\|} = 1/2\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

And,

$x_2 = \alpha_2 - \frac{(\alpha_2^T x_1)}{(\|x_1\|)^2}x_1$

First,

$$\alpha_2^T x_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 1+1+1 = 3 \Rightarrow \tfrac{\alpha_2^T x_1}{\|x_1\|^2}x_1 = 3/4\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/4 \\ 3/4 \\ 3/4 \\ 3/4 \end{bmatrix}$$

Then

$$x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 3/4 \\ 3/4 \\ 3/4 \\ 3/4 \end{bmatrix} = \begin{bmatrix} -3/4 \\ 3/4 \\ 3/4 \\ 3/4 \end{bmatrix}.\ \text{So, } q_2 = \tfrac{x_2}{\|x_1\|} = \sqrt{3}/2\begin{bmatrix} -3/4 \\ 3/4 \\ 3/4 \\ 3/4 \end{bmatrix} = \begin{bmatrix} -\sqrt{3}/8 \\ \sqrt{3}/8 \\ \sqrt{3}/8 \\ \sqrt{3}/8 \end{bmatrix}$$

And,

$x_3 = \alpha_3 - \frac{(\alpha_3^T x_1)}{(\|x_1\|)^2}x_1 - \frac{(\alpha_3^T x_2)}{(\|x_2\|)^2}x_2$

First,

$$\alpha_3^T x_1 = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 1+1 = 2 \Rightarrow \tfrac{\alpha_3^T x_1}{\|x_1\|^2}x_1 = 2/4\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

and similarly,

$$\alpha_3^T x_2 = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = 1/4+1/4 = 1/2 \Rightarrow \tfrac{\alpha_3^T x_2}{\|x_2\|^2}x_2 = \tfrac{1/2}{12/16}\begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/6 \\ 1/6 \\ 1/6 \end{bmatrix}$$

Then,

$$x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 1/6 \\ 1/6 \\ 1/6 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}. \text{ So, } q_3 = \frac{x_3}{\|x_3\|} = \sqrt{2}/\sqrt{3} \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2\sqrt{2}/3\sqrt{3} \\ \sqrt{2}/3\sqrt{3} \\ \sqrt{2}/3\sqrt{3} \end{bmatrix}$$

$$\Rightarrow Q = \begin{bmatrix} 1/2 & -\sqrt{3}/8 & 0 \\ 1/2 & \sqrt{3}/8 & -2\sqrt{2}/3\sqrt{3} \\ 1/2 & \sqrt{3}/8 & \sqrt{2}/3\sqrt{3} \\ 1/2 & \sqrt{3}/8 & \sqrt{2}/3\sqrt{3} \end{bmatrix}$$

Next;

$$q_1^T \alpha_1 = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 4/2 = 2$$

$$q_1^T \alpha_2 = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 3/2$$

$$q_1^T \alpha_3 = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 2/2 = 1$$

$$q_2^T \alpha_2 = \begin{bmatrix} -\sqrt{3}/8 & \sqrt{3}/8 & \sqrt{3}/8 & \sqrt{3}/8 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 3\sqrt{3}/8$$

$$q_2^T \alpha_3 = \begin{bmatrix} -\sqrt{3}/8 & \sqrt{3}/8 & \sqrt{3}/8 & \sqrt{3}/8 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 2\sqrt{3}/8 = \sqrt{3}/4$$

$$q_3^T \alpha_3 = \begin{bmatrix} 0 & -2\sqrt{2}/3\sqrt{3} & \sqrt{2}/3\sqrt{3}/8 & \sqrt{2}/3\sqrt{3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 2\sqrt{2}/3\sqrt{3}$$

$$\Rightarrow R = \begin{bmatrix} 2 & 3/2 & 1 \\ 0 & 3\sqrt{3}/8 & \sqrt{3}/4 \\ 0 & 0 & 2\sqrt{2}/3\sqrt{3} \end{bmatrix}$$

$Q^T Q = I$ because the columns of Q are *orthonormal*. Then

$$Q^T A = Q^T (QR) = IR = R$$

3. Use determinant to decide if v_1, v_2, v_3 are linearly independent where $v_1 = \begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix}$, $\begin{bmatrix} -3 \\ 3 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ -7 \\ 5 \end{bmatrix}$.

(b) Determine whether the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 3 \end{bmatrix}$ is singular or non-singular.

Solution:

(a) $| \, v_1 \;\; v_2 \;\; v_3 \, | = \begin{vmatrix} 5 & -3 & 7 \\ -7 & 3 & -7 \\ 9 & 5 & 5 \end{vmatrix} = 0$

So, the matrix $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$ is not invertible. The columns are linearly dependent.

(b) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 3 \end{bmatrix} \xrightarrow{-2r_1+r_2 \to r_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 3 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 7 & 3 \end{bmatrix} \xrightarrow{3r_2+r_3 \to r_3} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 3 \\ 0 & 0 & 7 & 3 \end{bmatrix}$

$\xrightarrow{-r_3+r_4 \to r_4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

So, $det(A) = 0$. Hence, A is singular.

4. Let $A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(a) Find the *eigenvalues* of A,

(b) For each *eigenvalue*, determine a *basis* for the associated *eigenspace*.

Solution:

(a)$\mid A - \lambda I \mid = \begin{vmatrix} -1-\lambda & 0 & 0 \\ 1 & -2-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = -(1+\lambda)(-(2+\lambda)((-\lambda)) = -(1+\lambda)(2\lambda+\lambda^2) =$
$-(2\lambda + \lambda^2 + 2\lambda^2 + \lambda^3) = -\lambda(\lambda^2 + 3\lambda + 2) = -\lambda(\lambda+2)(\lambda+1)0$

$\Rightarrow \lambda_1 = 0,\ \lambda_2 = -2,\ \lambda_3 = -1$

For $\lambda = 0$ we have $Ax = 0$, that is, $\begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\Rightarrow -x_1 = 0,\ x_1 - 2x_2 = 0$ i.e $x_1 = 2x_2$ i.e $x_2 = 0$ and x_3 is $free$.

$\Rightarrow$ The *eigenvector* for λ_1 is $u_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

For $\lambda = -2$ we have $Ax = -2x$, that is,$(A + 2I)x = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\Rightarrow x_1 = 0,\ x_2$ is $free$ and $x_3 = 0$.

$\Rightarrow$ The *eigenvector* for λ_2 is $u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

For $\lambda = -1$ we have $Ax = -1x$, that is,$(A + I)x = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\Rightarrow x_1$ is free $x_1 = -x_2$ and $x_3 = 0$.

$\Rightarrow$ The *eigenvector* for λ_3 is $u_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

B U Department of Mathematics

Math 201 Matrix Theory

Spring 2005 Second Midterm

1.) Let W be the plane in R^3 defined by $x - y + z = 0$.
a) Find an orthonormal basis for W.
b) Determine the orthogonal complement $W^\perp$ of W.

Solution:

a)The vectors $a = (1,1,0)$ and $b = (0,1,1)$ form a basis for W since they are linearly independent. To form an orthonormal basis we will use Gram-Schmidt method. As $\|a\| = \sqrt{2}$ take $q_1 = (1/\sqrt{2}, 1/\sqrt{2}, 0)$. Then $q_2' = b - (q_1^T b)q_1 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}$.

Hence $\|q_2'\| = \frac{\sqrt{3}}{\sqrt{2}}$ implies $q_2 = (-1/\sqrt{6}, 1/\sqrt{6}, 2/\sqrt{6})$.

b)To find the orthogonal complement $W^\perp$ of W it suffices to determine the Null-space of

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}.$$

But this then gives that $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 2/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, where x, y are basic and z is a free variable. Solving these equations simultaneously yields $y = -z$ and $x = z$. Thus

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -z \\ z \end{bmatrix} = -z \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

and so we get that $W^\perp = \text{Span}\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \}$.

2.) Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 0 \end{bmatrix}$

a) Find the least squares solution to $Ax = b$
b) Find the matrix that projects a given vector to the column space of A. Find the projection of b onto this space.
c) Find a vector which is orthogonal to the column space of A.

Solution:

a) Noting $A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$

and $A^Tb = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$ we get that

$$\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix},$$

which yields $\bar{x} = 25/11$ and $\bar{y} = 10/11$.

b) The projection matrix is given by $P = A(A^TA)^{-1}A^T$, where $A^TA = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$ and hence $(A^TA)^{-1} = \frac{1}{\det A^TA} \begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 3/11 & -1/11 \\ -1/11 & 4/11 \end{bmatrix}$.

Thus

$$P = \frac{1}{11} \begin{bmatrix} 9 & 4 & -1 & -1 \\ 4 & 3 & 2 & 2 \\ -1 & 2 & 5 & 5 \\ -1 & 2 & 5 & 5 \end{bmatrix},$$

and $Pb = \frac{1}{11} \begin{bmatrix} 15 \\ 25 \\ 35 \\ 35 \end{bmatrix}$.

c) Since b is *not* in the column space of A, $(b - Pb) = \frac{1}{11} \begin{bmatrix} -4 \\ 8 \\ 31 \\ -35 \end{bmatrix}$ is orthogonal to the column space of A.

3.) a) Calculate the determinant $\begin{vmatrix} 1 & 1 & 1 & 1 \\ r & 1 & 1 & 1 \\ r & r & 1 & 1 \\ r & r & r & 1 \end{vmatrix}$, where r is a real number:

Solution:

Note that, adding $-r$ times the first row to others, we get

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ r & 1 & 1 & 1 \\ r & r & 1 & 1 \\ r & r & r & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1-r & 1-r & 1-r \\ 0 & 0 & 1-r & 1-r \\ 0 & 0 & 0 & 1-r \end{vmatrix} = (1-r)^3.$$

b) Using determinant find all real numbers k such that the system below has a unique solution:

$$\begin{aligned} x + y - z &= 1 \\ 2x + 3y + kz &= 3 \\ x + ky + 3z &= 2 \end{aligned}$$

Solution:

Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & k \\ 1 & k & 3 \end{bmatrix}$ be the coefficient matrix. If $\det A \neq 0$ then the given system has a unique solution. By triangulating A we get

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & k \\ 1 & k & 3 \end{bmatrix} \xrightarrow[r_2 - 2r_1 \to r_2]{r_3 - r_1 \to r_3} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & k+2 \\ 0 & k-1 & 4 \end{bmatrix} \xrightarrow{r_3 - (k-1)r_2 \to r_3} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & k+2 \\ 0 & 0 & 4-(k+2)(k-1) \end{bmatrix}.$$

Hence $\det A = 4 - (k+2)(k-1) \neq 0$ implies that for $k \in \mathbb{R} \setminus \{-3, 2\}$ the system has a unique solution.

c) Find the solution of the above system for $k = 0$, by finding the inverse of the coefficient matrix via the cofactor expansion.

Solution:

For $k = 0$, $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix}$, hence applying the cofactor method to the second row we get,

$$\det A = 2(-1)^{2+1} \begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} + 3(-1)^{2+2} \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} + 0(-1)^{2+3} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 6.$$

Also, $A_{\text{cof}} = \text{adj}A = \begin{bmatrix} 9 & -3 & 3 \\ -6 & 4 & -2 \\ -3 & 1 & 1 \end{bmatrix}$. Hence, $A^{-1} = \frac{1}{\det A} \text{adj}A$ implies that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/3 \\ 1/3 \end{bmatrix}.$$

4.) Let A be an $n \times n$, invertible matrix and let adj$A$ denote the cofactor matrix of A (also denoted as A_{cof}).
a) Calculate $\det(\text{adj}A)$ in terms of $\det A$.
b) Find $\text{adj}(A^{-1})$ in terms of adj$A$.
c) For this part let A also be an orthogonal matrix with $\det A = 1$ and let n be an odd number. Calculate $\det(A - I)$, where I is the $n \times n$ identity matrix.

Solution:

a) Recalling $A^{-1} = \frac{1}{\det A}\text{adj}A$, multiplying both sides with A we get, $I = AA^{-1} = \frac{1}{\det A} A(\text{adj}A)$ and hence

$$(\det A)I = A(\text{adj}A)$$

Thus, taking the determinant of both sides yields $(\det A)^n(\det I) = \det((\det A)I) = (\det A)(\det(\text{adj}A))$ implies that

$$\det(\text{adj}A) = (\det A)^{n-1}.$$

b) $(\det A)I = A(\text{adj}A)$ entails $(\det A^{-1})I = A^{-1}(\text{adj}A^{-1})$ which then gives $\frac{1}{\det A}I = A^{-1}(\text{adj}A^{-1})$,i.e.,

$$\text{adj}A^{-1} = \frac{A}{\det A}.$$

Also knowing $\text{adj}A = A^{-1}(\det A)$ we get that $(\text{adj}A)(\text{adj}A^{-1}) = I$; so

$$(\text{adj}A^{-1}) = (\text{adj}A)^{-1}$$

c) Note that

$$\begin{aligned}\det(I - A^T) &= \det(A^T(A - I)) \\ &= \det(A^T)\det(A - I) \\ &= \det(A)\det(A - I) = \det(A - I).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\det(I - A^T) &= \det((I - A)^T) \\ &= \det(I - A)\end{aligned}$$

Thus we have

$$\det(A - I) = \det(I - A).$$

But $\det(I-A) = \det(-(A-I)) = (-1)^n \det(A-I)$. From this we see that $\det(A-I) = 0$ since n is odd.

B U Department of Mathematics

Math 201 Matrix Theory

Spring 2006 Second Midterm

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation satisfying;
$T(1,0,0) = (3-k, -1, 0)$
$T(0,1,0) = (-1, 2-k, -1)$
$T(0,0,1) = (0, -1, 3-k)$, where k is a real number.
What should k be so that the dimension of the range of T is 2? (15 points)

Solution:

For the matrix $A = \begin{bmatrix} 3-k & -1 & 0 \\ -1 & 2-k & -1 \\ 0 & -1 & 3-k \end{bmatrix}$ of the given transformation, R(T), the range of T is spanned by the column vectors of T. Since its dimension is 2, the column vectors are linearly dependent and so $det(A) = 0$.
Using cofactor expansion along the first row gives

$$\begin{aligned} detA &= (3-k)\begin{vmatrix} 2-k & -1 \\ -1 & 3-k \end{vmatrix} + 1\begin{vmatrix} -1 & -1 \\ 0 & 3-k \end{vmatrix} \\ &= (3-k)(k^2-5k+5) - (3-k) \\ &= (3-k)(k^2-5k+4) \\ &= (3-k)(k-4)(k-1) \end{aligned}$$

and $detA = 0$ if $k = 1, 3$, or 4.

2. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & b \end{bmatrix}$.

a. Find b so that A has zero as an eigenvalue;(6 points)

For $\lambda = 0$ to be an eigenvalue $det(A - \lambda I) = detA = 0$ must hold. Expanding along the first column we get
$detA = \begin{vmatrix} 1 & 1 \\ 1 & b \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = b - 1 - 1 = b - 2 = 0$ if $b = 2$

b. Find the other eigenvalues and all the corresponding eigenvectors when b has the value found in part (a) above. (15 points)

For $b = 2$, $det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$
if $(1-\lambda)[(1-\lambda)(2-\lambda) - 1] - (1-\lambda) = 0$
if $(1-\lambda)(\lambda^2 - 3\lambda) = \lambda(1-\lambda)(\lambda-3) = 0$
So $\lambda = 0, 1$ or 3 are eigenvalues.

For $\underline{\lambda = 0}$:

$$A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ if } \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

if $\begin{matrix} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \end{matrix}$

if $x_1 = x_2 = -x_3$

$$\Rightarrow x = a\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, a \in \mathbb{R}, a \neq 0 \text{ (eigenvectors corresponding to } \lambda = 0).$$

For $\underline{\lambda = 1}$:

$$(A - I)x = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ if } x_3 = 0 \text{ and } x_1 + x_2 = 0$$

$$\Rightarrow x = b\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, b \in \mathbb{R}, b \neq 0 \text{ (eigenvectors corresponding to } \lambda = 1).$$

For $\lambda = 3$:

$$(A - 3I)x = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{if } \begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & 1 \\ 0 & 1 & -\frac{1}{2} \end{bmatrix} \rightarrow \begin{bmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

if $\begin{matrix} -2x_1 + x_3 = 0 \\ x_2 - \frac{1}{2}x_3 = 0 \end{matrix}$

if $x_1 = \frac{1}{2}x_3 = x_2$

$$\Rightarrow x = c\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}, c \in \mathbb{R}, c \neq 0 \text{ (eigenvectors corresponding to } \lambda = 3).$$

3. Decide whether the followings are TRUE or FALSE. If true prove; if false, give a counter example or explain.(15 points)

i. For A and B be both invertible nxn matrices $(A + B)^{-1} = A^{-1} + B^{-1}$;

FALSE: For $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, A and B are invertible since $detA = detB = 1 \neq 0$

But $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible since $det(A + B) = 0$.

So $(A + B)^{-1}$ is meaningless since A+B is not invertible.

ii. Transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x, 2)$ is linear.

FALSE: Since $T(0, 0, 0) = (0, 2)$, ie $T(0) \neq 0$, hence T is not linear.

iii. The orthogonal projection of $u = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ on the subspace of $\mathbb{R}^3$ spanned by $v = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is u.

FALSE: Since $u^T v = 1 - 1 + 0 = 0$ we have $u \perp v$. So the orthogonal projection of u on the subspace of $\mathbb{R}^3$ spanned by v (ie on v line) must be the zero vector.

4. Let $A = [a_{ij}]$ be an nxn matrix such that $a_{1,n} = 1, a_{2,n-1} = 1, \ldots, a_{n,1} = 1$ and $a_{i,j} = 0$ otherwise;

i. Write down the matrix A.

$$A = \begin{bmatrix} 0 & \cdots & & 1 \\ \vdots & & & \\ & 1 & & \vdots \\ 1 & & \cdots & 0 \end{bmatrix}$$

ii. Evaluate the determinant of A in the case that n=7.

$$\begin{aligned} detA &= (-1)(-1)(-1)detI \\ &= -1 \end{aligned}$$

since we exchange $r_1 \leftrightarrow r_7$, $r_2 \leftrightarrow r_6$, $r_3 \leftrightarrow r_5$ in order to obtain the identity matrix I from A.

iii. Find a formula for the determinant of A in case that n is an arbitrary positive integer.

Consider the number of row exchanges to reduce A to I.
For $n > 2$, if n is even, then $\frac{n}{2}$ row exchanges are needed. If n is odd, then $\frac{n-1}{2}$ row exchanges are needed.
If the number of row exchanges is even then detA=1. If it is odd, then detA=-1.
So for $k \in \mathbb{Z}^+$
if $n = 4k$ then $\frac{n}{2} = \frac{4k}{2} = 2k$ even $\Rightarrow detA = 1$
if $n = 4k+1$, then $\frac{n-1}{2} = \frac{4k}{2} = 2k$ even $\Rightarrow detA = 1$
if $n = 4k+2$, then $\frac{n}{2} = 2k+1$ odd $\Rightarrow detA = -1$
if $n = 4k+3$, then $\frac{n-1}{2} = 2k+1$ odd $\Rightarrow detA = -1$
In fact, if $n \equiv 0$ or $1 \pmod 4 \Rightarrow detA = 1$ and if $n \equiv 2$ or $3 \pmod 4 \Rightarrow detA = -1$.

5. Let W be the subspace of $\mathbb{R}^4$ containing all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ satisfying $x_1 + x_2 + x_3 + x_4 = 0$. Find a basis for $W^\perp$, the orthogonal complement of W. (16 points)

Solution:

$$W = \{\begin{bmatrix} -x_2 - x_3 - x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_2, x_3, x_4 \in \mathbb{R}\}$$

$$= \mathcal{S}(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \text{R(A)}, column space of A = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

We know that $R(A)^{\perp} = \mathcal{N}(A^{\perp)}$.

$$A^T = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[(-1)r_1 \to r_1, r_1 + r_2 \to r_2, r_1 + r_3 \to r_3]{} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[(-1)r_2 \to r_2, r_2 + r_3 \to r_3]{} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

So $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathcal{N}(A^T)$ if $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

if $x_1 - x_2 = x_2 - x_3 = -x_3 + x_4 = 0$

if $x_3 = x_2 = x_1 = x_4$

Hence $\mathcal{N}(A^T) = \{x_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} : x_4 \in \mathbb{R}\} = \mathcal{S}(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}) = W^{\perp}$.

6.

i. Find QR-decomposition of A= $\begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$, where Q is an orthogonal matrix and R is an upper triangular matrix.(17 points)

Solution:

The vectors a= $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, b= $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, c= $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ are linearly independent.

Take $q_1 = a$ and $q_2 = b$, since a and b have unit lengths and $a \perp b$.

$$\begin{aligned} c' &= c - (q_1^T c)q_1 - (q_2^T c)q_2 \\ &= c - (a^T c)a - (b^T c)b \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} since a^T c = 0, b^T c = \frac{1}{\sqrt{2}} \end{aligned}$$

$$\Rightarrow q_3 = \frac{c'}{\|c'\|} = \sqrt{2} \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Now
$$\begin{array}{lll} q_1^T a = 1 & q_1^T b = 0 & q_1^T c = 0 \\ & q_2^T b = 1 & q_2^T c = \frac{1}{\sqrt{2}} \\ & & q_3^T c = \frac{1}{\sqrt{2}} \end{array}$$

Hence A= $\begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$ = QR

ii. Find the inverse of the matrix Q found in part i) above. (4 points)

Solution:

$$Q^{-1} = Q^T \Rightarrow Q^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = Q$$

B U Department of Mathematics

Math 201 Matrix Theory

Summer 2005 Second Midterm

1.) [5] Find an *orthonormal* set of vectors q_1, q_2, q_3 in R^3 for which q_1, q_2 span the column space of

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & -4 \end{bmatrix}$$

Solution:

$v_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix}$. Apply Gram-Schmidt process to v_1, v_2 to get q_1, q_2.

$q_1 = \frac{1}{\|v_1\|} v_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$. $\quad x_2 = v_2 - (v_2^T q_1) q_1 = \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix} - \frac{8}{\sqrt{6}} \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = (-\frac{1}{3}) \begin{pmatrix} 1 \\ 7 \\ 4 \end{pmatrix}$.

Note that $x_2 \perp q_1$.

$$q_2 = \frac{1}{\|x_2\|} x_2 = \frac{-1}{\sqrt{66}} \begin{pmatrix} 1 \\ 7 \\ 4 \end{pmatrix}.$$

To find q_3 first find a vector x_3 which is both perpendicular to $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 7 \\ 4 \end{pmatrix}$:

$$q_1^T x_3 = 0 \quad \Rightarrow \quad a + b - 2c = 0$$

$$q_2^T x_3 = 0 \quad \Rightarrow \quad a + 7b + 4c = 0$$

$\Rightarrow b = -c$ and $a = 3c$. So $x_3 = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$. $q_3 = \frac{1}{\|x_3\|} x_3 = \frac{1}{\sqrt{11}} \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$.

2.) Let A be an $m \times n$ matrix.

a) [3] Prove that $R(A) \perp N(A^T)$, where $R(A)$ is the range of A and $N(A^T)$ is the null space of A^T.

Solution:

$x \in R(A) \Rightarrow x = Az$ for some z.

$y \in N(A^T) \Rightarrow A^T y = 0$.

$x^T y = (Az)^T y = z^T A^T y = z^T 0 = 0 \Rightarrow R(A) \perp N(A^T)$.

b) [3] Prove that $R(A^T) \perp N(A)$ where $R(A^T)$ is the range of A^T and $N(A)$ is the null space of A.

Solution:

$x \in R(A^T) \Rightarrow x = A^T z$ for some z.

$y \in N(A) \Rightarrow Ay = 0$.

$x^T y = (A^T z)^T y = z^T (A^T)^T y = z^T A y = z^T 0 = 0 \Rightarrow R(A^T) \perp N(A)$.

c) [3] Prove that a set of non-zero orthogonal vectors is linearly indepedent.

Solution:

$\alpha_1, \alpha_2, \ldots, \alpha_k$: nonzero orthogonal! $\alpha_i^T \alpha_j = 0$ if $i \neq j$.

Set $c_1\alpha_1 + c_2\alpha_2 + \ldots + c_k\alpha_k = 0$ where $c_1, c_2, \ldots, c_k \in \mathbb{R}$. Multiply both sides with α_i^T:

$\alpha_i^T [c_1\alpha_1 + c_2\alpha_2 + \ldots + c_k\alpha_k] = \alpha_i^T 0 = 0 \Rightarrow c_i \alpha_i^T \alpha_i = 0 \Rightarrow c_i = 0$ since $\alpha_i^T \alpha_i \neq 0$, because α_i: non-zero!
$\Rightarrow \alpha_1, \alpha_2, \ldots, \alpha_k$ are linearly independent.

3.) a) [5] Use the cofactor matrix to invert $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ (No partial credits, check your answer!)

Solution:

$$c_{11} = 3 \quad c_{12} = 2 \quad c_{13} = 1$$

$$c_{21} = 2 \quad c_{22} = 4 \quad c_{23} = 2$$

$$c_{31} = 1 \quad c_{32} = 2 \quad c_{33} = 3$$

$$C = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

$DetA = 2c_{11} + (-1)c_{12} + 0c_{13} = 4$ (using first row)

$$AdjA = C^T = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

$$A^{-1} = \frac{1}{DetA} AdjA = \frac{1}{4} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

b) [3] How are det($2A$), det(-A) and det(A^2) related to det(A), when A is $n \times n$?

Solution:

$det(2A) = 2^n det(A)$

$det(-A) = (-1)^n det(A)$

$det(A^2) = [det(A)]^2$

4.) a) [4] In R^2, find the projection of $b = (b_1, b_2)$ onto the column space of $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

Solution:

Column space is spanned by $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Projection onto v: $P(b) = \dfrac{v^T b}{v^T v} v = \dfrac{(b_1 + 2b_2)}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

b) [4] In R^3, construct the matrix P which projects onto the plane $x - y + z = 0$.

Solution:

$$x - y + z = 0 \Rightarrow y = x + z \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z011.$$

$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$ Find projection onto the column space of A: $P = A(A^T A)^{-1} A^T$

$$A^T A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad (A^T A)^{-1} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -1 \\ 1 & 2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}.$$