 NAME:----------------

II. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 0 \end{bmatrix}$ and $PA = \begin{bmatrix} 3 & 0 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}$.

(a) Find the permutation matrix P for the PA given above.

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

(b) Find the matrix K which only adds five times the third row of A to the first row of A when KA is considered.

$$K = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) Let $B = A^T A$. Find the inverse of B if it exists.

$$B = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 2 & 5 \end{bmatrix}$$

rank (B) = 2 (2 pivots); B is invertible

$[B \mid I] \to [I \mid B^{-1}]$ gives

$$B^{-1} = \frac{1}{46} \begin{bmatrix} 5 & -2 \\ -2 & 10 \end{bmatrix}.$$

MATH 201 NAME:--------------------

FIRST MIDTERM EXAM

November 5, 2002, 17:00-18:00

NO: -------

NAME:

SIGNATURE:----------------------------

I.	/25
II.	/25
III.	/25
IV.	/25
Total:	/100

Please write your name at the top of each page (**in ink**). Label all answers clearly and show all work. Calculators and cellular phones should be switched off.

I. Find the ranks of the coefficient and the augmented matrices of the system:

PAGE 4

IV. (a) The nullspace of a 4×5 matrix B is known to be a 3-dimensional subspace of $\mathbf{R}^5$. Find the rank of B. Justify your answer.

We know that nullity (B) = 3

SOLUTIONS

rank (B) + nullity (B) = 5

⇒ rank (B) = 2.

(b) Determine a basis for the left nullspace of the matrix:

-PAGE 3- NAME:

III. Given that

$$2u + 4v + 6w = -12,$$
$$2u - 3v - 4w = 15,$$
$$3u + 4v + 5w = -8,$$

and determine whether this system is consistent. If it is consistent, find all solutions.

Let $v_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $v_2 = \ldots$ and $y = \begin{bmatrix} 3 \\ 5 \\ k \end{bmatrix}$, $k \in \mathbf{R}$,

let W be the subspace of $\mathbf{R}^3$ spanned by $\{v_1, v_2, v_3\}$.

(a) Find all values of k for which y is in W.

(b) Find dim W.

$$A = \begin{bmatrix} 2 & 5 & 0 & -1 & 3 \\ 6 & 15 & -4 & -3 & 9 \\ 3 & & 3 & & 0 \end{bmatrix}$$

The augmented matrix [A|b] is

$$\begin{bmatrix} 2 & 4 & 6 & -12 \\ 0 & -7 & -10 & 27 \end{bmatrix}$$

Rank [A|b] = 3

Rank (A) = 3, moreover the system has a unique soln.

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 3 & -6 & k-8 \end{bmatrix}$$

No. of rows of A − rank (A) = 2

$$\begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad k = 5$$

$k - 5 = 0$; $k = 5$.

⇒ $y \in W$ iff

(b) Rank (A) = dim W = 2.

B U Department of Mathematics

Math 201 Matrix Theory

Fall 2004 First Midterm

1. Prove the following statements:

(a) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be symmetric matrices. If $\boldsymbol{AB}$ is also symmetric then $\boldsymbol{AB} = \boldsymbol{BA}$.

Solution:

$\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric means $\boldsymbol{A}^T = \boldsymbol{A}$ and $\boldsymbol{B}^T = \boldsymbol{B}$. Also $(\boldsymbol{AB})^T = \boldsymbol{AB}$. Now using all these:

$$\boldsymbol{AB} = (\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T = \boldsymbol{BA}.$$

Proof is done.

(b) If $\boldsymbol{AB} = \boldsymbol{BA}$ and $\boldsymbol{B}$ is invertible then $\boldsymbol{AB}^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}$.

Solution:

We are given $\boldsymbol{AB} = \boldsymbol{BA}$ and $\boldsymbol{B}$ is invertible. Multiply this identity from both sides by $\boldsymbol{B}^{-1}$ to obtain:

$$\boldsymbol{B}^{-1}\boldsymbol{ABB}^{-1} = \boldsymbol{B}^{-1}\boldsymbol{BAB}^{-1} \iff \boldsymbol{B}^{-1}\boldsymbol{A} = \boldsymbol{AB}^{-1}.$$

Proof is done.

2. Let $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.

(a) Show that $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible matrices by finding their inverses explicitly.

Solution:

We construct the augmented matrices: $[\boldsymbol{A} : \boldsymbol{I}]$ and $[\boldsymbol{B} : \boldsymbol{I}]$ and apply elementary row operations:

$$[\boldsymbol{A} : \boldsymbol{I}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{e_1:\ -r_1+r_3 \to r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{e_2:\ -2r_2+r_3 \to r_3}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -2 & 1 \end{array}\right] \xrightarrow[e_4:\ r_3+r_2 \to r_2]{e_3:\ -r_3+r_1 \to r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 2 & -1 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & -1 & -2 & 1 \end{array}\right].$$

Hence we have found that:

$$\boldsymbol{A}^{-1} = \begin{bmatrix} 2 & 2 & -1 \\ -1 & -1 & 1 \\ -1 & -2 & 1 \end{bmatrix}.$$

$$
\begin{aligned}
[\boldsymbol{B}:\boldsymbol{I}] &= \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{f_1:\ -r_1+r_2\to r_2} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{f_2:\ -r_2+r_3\to r_3} \\
&= \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \xrightarrow{f_3:\ r_3+r_2\to r_2} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right].
\end{aligned}
$$

Hence we have found that:

$$
\boldsymbol{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}.
$$

Since $\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent to the 3×3 identity matrix, they are invertible matrices.

(b) Express $\boldsymbol{A}$ and $\boldsymbol{B}$ as a product of elementary matrices (Do not perform explicit matrix multiplication, but perform inversions, transpositions etc. whenever necessary).

Solution:

Let $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, $\boldsymbol{E}_3$ and $\boldsymbol{E}_4$ be elementary matrices corresponding to the operations e_1, e_2, e_3 and e_4, respectively. Similarly let $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ be elementary matrices corresponding to the row operations f_1, f_2 and f_3, respectively. Then in part **(a)** we have shown that: $\boldsymbol{E}_4\boldsymbol{E}_3\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A} = \boldsymbol{I}$ and $\boldsymbol{F}_3\boldsymbol{F}_2\boldsymbol{F}_1\boldsymbol{B} = \boldsymbol{I}$. Elementary matrices are invertible and product of invertible matrices is invertible, which let us write:

$$
\begin{aligned}
\boldsymbol{A} &= (\boldsymbol{E}_4\boldsymbol{E}_3\boldsymbol{E}_2\boldsymbol{E}_1)^{-1} = \boldsymbol{E}_1^{-1}\boldsymbol{E}_2^{-1}\boldsymbol{E}_3^{-1}\boldsymbol{E}_4^{-1} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix},
\end{aligned}
$$

and also:

$$
\begin{aligned}
\boldsymbol{B} &= (\boldsymbol{F}_3\boldsymbol{F}_2\boldsymbol{F}_1)^{-1} = \boldsymbol{F}_1^{-1}\boldsymbol{F}_2^{-1}\boldsymbol{F}_3^{-1} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Now $\boldsymbol{A}$ and $\boldsymbol{B}$ are written as a product of elementary matrices.

In this solution we utilized the practical way of inverting elementary matrices.

(c) Express $(\boldsymbol{AB})^{-1}$ as a product of elementary matrices (Do not perform explicit matrix multiplication, but perform inversions, transpositions etc. whenever necessary).

Solution:

First we note $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$. But inverses of $\boldsymbol{A}$ and $\boldsymbol{B}$ are just product of elementary matrices in the application order:

$$\begin{aligned}(\boldsymbol{AB})^{-1} &= \boldsymbol{B}^{-1}\boldsymbol{A}^{-1} = \boldsymbol{F}_3\boldsymbol{F}_2\boldsymbol{F}_1\boldsymbol{E}_4\boldsymbol{E}_3\boldsymbol{E}_2\boldsymbol{E}_1 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &\times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.\end{aligned}$$

3. Consider the 4×5 matrix $\boldsymbol{A} = \begin{bmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{bmatrix}$.

(a) Find all solutions $\boldsymbol{x}$ of the homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{0}$ by obtaining the row-reduced echelon matrix $\boldsymbol{R}$ of $\boldsymbol{A}$. What is the dimension of this solution space?

Solution:

By elementary row operations we pass to the unique row-reduced echelon matrix $\boldsymbol{R}$ of $\boldsymbol{A}$:

$$\boldsymbol{A} = \begin{bmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{bmatrix} \xrightarrow{r_1/3 \to r_1} \begin{bmatrix} 1 & 7 & 0 & 3 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{bmatrix} \xrightarrow[\substack{-2r_1 + r_3 \to r_3 \\ -6r_1 + r_4 \to r_4}]{-r_1 + r_2 \to r_2}$$

$$\begin{bmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & -1 & -5 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -5 & 0 \end{bmatrix} \xrightarrow{-r_2 \to r_2} \begin{bmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -5 & 0 \end{bmatrix} \xrightarrow{r_2 + r_4 \to r_4} \begin{bmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[-r_3 + r_4 \to r_4]{-r_3 + r_2 \to r_2} \begin{bmatrix} 1^* & 7 & 0 & 3 & 0 \\ 0 & 0 & 1^* & 5 & 0 \\ 0 & 0 & 0 & 0 & 1^* \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \boldsymbol{R}.$$

There are 3 pivots (leading 1s). If we rewrite the linear system $\boldsymbol{Ax} = \boldsymbol{0}$ in its row equivalent form $\boldsymbol{Rx} = \boldsymbol{0}$ and back substitute the variables, we get:

$$\begin{aligned} x_1 + 7x_2 + 3x_4 &= 0, \\ x_3 + 5x_4 &= 0, \\ x_5 &= 0. \end{aligned}$$

Choosing $x_2 = s$ and $x_4 = t$ as free variables, the solutions of the homogeneous system are vectors of the form:

$$\begin{bmatrix} -7s - 3t \\ s \\ -5t \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} -7 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -5 \\ 1 \\ 0 \end{bmatrix}.$$

This shows dimension of the solution space of the homogeneous system associated with $\boldsymbol{A}$ has 2 free parameters and hence its dimension is $2 = \dim \mathrm{Null}(\boldsymbol{A})$.

(b) Find a basis for the column space of $\boldsymbol{A}$.

Solution:

In $\boldsymbol{R}$ we see that 1st, 3rd and 5th columns are linearly independent because they contain the pivot elements. Thus, the set of corresponding columns of $\boldsymbol{A}$:

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

constitutes a basis for the column space of $\boldsymbol{A}$.

(c) Find a basis for the row space of $\boldsymbol{A}$.

Solution:

The last row is a zero row. Hence the first three rows of $\boldsymbol{R}$ (or of $\boldsymbol{A}$) form a basis for the row space:

$$\{[1\ 7\ 0\ 3\ 0],\ [0\ 0\ 1\ 5\ 0],\ [0\ 0\ 0\ 0\ 1]\}.$$

(d) Regarding the matrix $\boldsymbol{A}$ given in this question, fill in the blanks in the following statements explicitly:

•Row($\boldsymbol{A}$) is a <u>3</u> dimensional subspace of the Euclidean space <u>$\mathbb{R}^5$</u>.

•Col($\boldsymbol{A}$) is a <u>3</u> dimensional subspace of the Euclidean space <u>$\mathbb{R}^4$</u>.

•Rank($\boldsymbol{A}$) equals <u>3</u>.

•$\boldsymbol{Ax} = \boldsymbol{b}$ has a solution if $\boldsymbol{b}$ is a linear combination of the basis vectors <u>of Col($\boldsymbol{A}$)</u>.

4. Let V_2 denote the vector space of polynomials of degree at most 2, and V_3 denote the vector space of polynomials of degree at most 3.
We define a transformation $T : V_2 \longrightarrow V_3$ by:

$$T\big(a_0 + a_1 x + a_2 x^2\big) = a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3.$$

(a) Show that T is a linear transformation.

Solution:

Take two polynomials in V_2: $p(x) = a_0 + a_1 x + a_2 x^2$ and $q(x) = b_0 + b_1 x + b_2 x^2$ and a constant $c \in \mathbb{R}$.

$$(i) \quad T(p(x) + q(x)) = (a_0 + b_0)x + \frac{a_1 + b_1}{2}x^2 + \frac{a_2 + b_2}{3}x^3$$

$$= a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + b_0 x + \frac{b_1}{2}x^2 + \frac{b_2}{3}x^3 = T(p(x)) + T(q(x)).$$

$$(ii) \quad T(cp(x)) = (ca_0)x + \frac{(ca_1)}{2}x^2 + \frac{(ca_2)}{3}x^3 = c\left(a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3\right) = cT(p(x)).$$

Hence T is linear.

Side Info: Note that T is an integration operator, but not indefinite. If it were so, i.e. $T(p(x)) = \displaystyle\int p(x)\,dx$, then $T(0) =$ constant, not necessarily zero. Instead, the correct integral form of T is: $T(p(x)) = \displaystyle\int_0^x p(t)\,dt$ so that this integration constant is forced to be zero.

(b)[5] By finding their elements, describe the sets $U = \{p(x) \in V_2 \text{ such that } T(p(x)) = 0\}$ and $W = \{p(x) \in V_2 \text{ such that } T(p(x)) = 1\}$.

Solution:

$T(p(x)) = a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 = 0$ entails $a_0 = a_1 = a_2 = 0$ by the polynomial identity. Hence $U = \{0\}$.

$T(p(x)) = a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 = 1$ cannot be satisfied for all x for any choice of the coefficients. Namely, no element in V_2 has the image 1: $W = \varnothing$.

(c) Let the set $\mathcal{B} = \{1 + x,\ x + x^2,\ 1 + x^2\}$ form a basis for V_2 (Do not show this). Find the matrix $\boldsymbol{A}$ of T with respect to the basis $\mathcal{B}$.

Solution:

We simply find the images of each basis element via the transformation rule:

$$\begin{aligned} T(1+x) &= x + x^2/2 \\ T(x+x^2) &= x^2/2 + x^3/3 \\ T(1+x^2) &= x + x^3/3 \end{aligned}$$

and express them in terms of the standard basis $\{1,\ x,\ x^2,\ x^3\}$ of V_3 as coordinate vectors:

$$\begin{aligned} T(1+x) &= [0\ 1\ 1/2\ 0], \\ T(x+x^2) &= [0\ 0\ 1/2\ 1/3], \\ T(1+x^2) &= [0\ 1\ 0\ 1/3]. \end{aligned}$$

We now place them column-wise to find $\boldsymbol{A}$ to be:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/3 & 1/3 \end{bmatrix}$$

as a 4×3 matrix.

(d) Find the image of $q(x) = 3 + 2x + x^2$ under T by using the transformation matrix $\boldsymbol{A}$.

Solution:

We first need to write $q(x) = 3 + 2x + x^2$ in the basis $\mathcal{B}$. This is to find numbers $c_1,\ c_2$ and c_3 so that:

$$3 + 2x + x^2 = c_1(1+x) + c_2(x+x^2) + c_3(1+x^2) = (c_1+c_3) + (c_1+c_2)x + (c_2+c_3)x^2.$$

This is true for all x, hence is a polynomial identity. Comparing the coefficients of left- and right-hand sides, we reach a system of three nonhomogeneous linear equations:

$$\begin{aligned} c_1 + c_3 &= 3, \\ c_1 + c_2 &= 2, \\ c_2 + c_3 &= 1. \end{aligned}$$

This system can be solved by any means, for instance subtracting 3rd equation from the 2nd equation: $c_1 - c_3 = 1$. Adding this up to the 1st equation: $c_1 = 2$. Then $c_2 = 0$ and $c_3 = 1$. Thus, $q(x)$ has the coordinate vector $[2\ 0\ 1]$ in the basis $\mathcal{B}$.

Now, since $T(q(x))$ is a matrix multiplication, we have:

$$T(q(x)) = \boldsymbol{A}[2\ 0\ 1]^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 1/3 \end{bmatrix}.$$

This is to say that:

$$T(q(x)) = 3x + x^2 + x^3/3.$$

B U Department of Mathematics

Math 201 Matrix Theory

Fall 2005 First Midterm

1.) Let V be the subspace of $\mathbb{R}^4$ spanned by the vectors:

$$x = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad z = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 3 \end{bmatrix}.$$

Determine the dimension and find a basis for V.

Solution:

Note that $x + y = z$ and y is not a multiple of x. It follows that $\{x, y\}$ is a basis for V and $\dim V = 2$.

2.)

(a) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 7 & -1 & 2 \end{bmatrix}$. Determine A^T and find A^{-1} if it exists.

Solution:

$$A^T = \begin{bmatrix} 1 & 1 & 7 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

To find the inverse, we apply Gauss-Jordan procedure

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 7 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} -R_1+R_2 \to R_2 \\ -7R_1+R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 2 & -7 & 0 & 1 \end{array}\right]$$

$$R_2+R_3 \to R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -8 & 1 & 1 \end{array}\right]$$

$$\frac{R_3}{2} \to R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -4 & \frac{1}{2} & \frac{1}{2} \end{array}\right],$$

and we find $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -4 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

(b) Let A be an $m \times n$ matrix and B be and $n \times m$ matrix and suppose $n < m$. Prove that the $m \times m$ matrix $C = AB$ is not invertible.

Solution:

If C is invertible, then $\operatorname{rank} C = m$. On the other hand, $\text{Row}(C) \subset \text{Row}\, B$, therefore $\dim \text{Row}\, C \leq \dim \text{Row}\, B \leq n$. This implies $m = \operatorname{rank} C = \dim \text{Row}\, C \leq n$, which is a contradiction.

3.) Suppose $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 7 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 4 & 5 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 10 \\ 15 \\ 85 \end{bmatrix}$.

(a) What is the rank of A? Justify your answer.

Solution:

The matrix A is given in the LU form. Since it has 3 pivots, the rank of A is 3.

(b) Find a basis for the nullspace of A.

Solution:

The nullspaces of A and U are the same, if U is obtained by performing elementary row operations on A. So we want to find the solution set of $U\mathbf{x} = 0$, where

$$U = \begin{bmatrix} 1 & 0 & 1 & 4 & 5 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$. Then x_3 and x_5 are the free variables. Let $x_3 = t$, $x_5 = w$. From the last row, we have $x_4 + w = 0$, so $x_4 = -w$. From the second row, we have $x_2 + 2t + 2x_4 + w = 0$. Replacing x_4 with $-w$, we get $x_2 = -2t + w$. From the first row, we have $x_1 + x_3 + 4x_4 + 5x_5 = 0$. It follows that $x_1 = -t - w$. So the solution set is

$$\mathbf{x} = t \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad t, w \in \mathbb{R}.$$

So, the set $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis for the nullspace of A.

(c) Find the complete solution to $Ax = b$.

Solution:

We have already found the homogenous solution in part (b). We only need to find a particular solution. We set all the free variables to 0. We first solve $Lc = b$, and then $Ux = c$.

$$Lc = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 7 & -1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \\ 85 \end{bmatrix}.$$

We can easily find by forward substitution that $\mathbf{c} = \begin{bmatrix} 10 \\ 5 \\ 10 \end{bmatrix}$.

We now solve for

$$Ux = \begin{bmatrix} 1 & 0 & 1 & 4 & 5 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 10 \end{bmatrix}$$

with $x_3 = x_5 = 0$. By back substitution, we find

$$\mathbf{x} = \begin{bmatrix} -30 \\ -15 \\ 0 \\ 10 \\ 0 \end{bmatrix}$$

as a particular solution, so the general solution is

$$\mathbf{x} = \begin{bmatrix} -30 \\ -15 \\ 0 \\ 10 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad t, w \in \mathbb{R}.$$

4.)
(a) Determine whether the following matrices have the same row spaces:

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 4 & -3 & -1 \\ 3 & -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 5 \\ 2 & 3 & 13 \end{bmatrix}.$$

Solution:

Let us perform row operations on the given matrices:

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 4 & -3 & -1 \\ 3 & -1 & 3 \end{bmatrix} \xrightarrow[-3R_1+R_3\to R_3]{-4R_1+R_2\to R_2} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_3\to R_3} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-R_2+R_1\to R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & 5 \\ 2 & 3 & 13 \end{bmatrix} \xrightarrow{-2R_1+R_2\to R_2} \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{-R_2+R_1\to R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

Hence we see that A and B have exactly the same row spaces.

(b) Let $A = \begin{bmatrix} 1 & -2 & 1 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 7 & -5 & -1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Determine whether b belongs to the column space of A. Does the linear system $Ax = b$ have at least one solution? Justify your answers.

Solution:

Let us perform Gaussian elimination on the augmented matrix $[A : b]$:

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 1 & 1 & -1 & 1 & 2 \\ 1 & 7 & -5 & -1 & 3 \end{array}\right] \xrightarrow[-R_1+R_3\to R_3]{-R_1+R_2\to R_2} \left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 0 & 3 & -2 & -1 & 1 \\ 0 & 9 & -6 & -3 & 2 \end{array}\right]$$

$$\xrightarrow{-3R_2+R_3\to R_3} \left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 0 & 3 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

Since the system is inconsistent, b is not in the column space of A, and there exists no solution.

B U Department of Mathematics

Math 201 Matrix Theory

Spring 2003 First Midterm

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (x + 2y + z, x + y, 2y + z)$.

a) (5pnts) Write down what we must show to prove that T is a linear transformation
b) (5pnts) What is the matrix representing this transformation in the standard basis for $\mathbb{R}^3$.
c) (10pnts) Show that T is non-singular and find its inverse transformation!

Solution:

a) Must show $T(\alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2)) = \alpha T(x_1, y_1, z_1) + \beta T(x_2, y_2, z_2)$.

b) $T\vec{i} = (1, 1, 0), T\vec{j} = (2, 1, 2), T\vec{k} = (1, 0, 1)$ implies matrix $M = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b-a \\ a-2b+c \end{pmatrix}$; 3 pivots imply non-singularity : T is invertible.

$$\vec{X} = \begin{pmatrix} -2a + 4b - c \\ 2a_3b + c \\ -a + 2b - c \end{pmatrix} = \begin{pmatrix} -2 & 4 & -1 \\ 2 & -3 & 1 \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ implies } M^{-1} = \begin{pmatrix} -2 & 4 & -1 \\ 2 & -3 & 1 \\ -1 & 2 & -1 \end{pmatrix}$$

and correspondingly $T^{-1}(x, y, z) = (-2x + 4y - z, 2x - 3y + z, -x + 2y - 1)$.

2. (30pnts) Show that the set $\{\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\}$ is a basis for the space spanned by the set $\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 8 \\ 7 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}\}$.

Solution:

Linear independence: $c_1 \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ implies $\begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ implies $\begin{pmatrix} 2 & 1 \\ 0 & -1/2 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ implies $c_1 = c_2 = 0$.

Linear independence of the other 3 vectors:

$$\begin{pmatrix} 1 & 5 & 3 \\ 2 & 8 & 4 \\ 3 & 7 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ implies } \begin{pmatrix} 1 & 5 & 3 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b-a \\ 5a-4b+c \end{pmatrix}, \text{ so } \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ and } \begin{pmatrix} 5 \\ 8 \\ 7 \end{pmatrix}$$

form a basis, i.e. dim $= 2$.

$$\begin{pmatrix} 1 & 5 \\ 2 & 8 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ implies } \begin{pmatrix} 1 & 5 \\ 0 & -2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b-a \\ 5a-4b+c \end{pmatrix}, \text{ so } \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ is in the range}$$

$\Leftrightarrow 5a - 4b + c = 0$.

Since $\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ satisfy $5a - 4b + c$ they are in the span and sine they are linearly independent and dim $= 2$ they must form a base for the same space.

3. Given the matrix $A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 4 & 8 & 2 & 6 \end{pmatrix}$

a) (20pnts) Find basis for the 4-fundamental subspaces associated with M.

b) (5pnts) Why does the system $Ax = (1,1,1)^T$ has no solution? Explain!

Solution:

$$\begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 4 & 8 & 2 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ implies } \begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b-2a \\ c-2b \end{pmatrix}$$

$x_4 = s, x_3 = -s, x_2 = t, x_1 = -3s + 2s - 2t = -s - 2t$ implies $x = s \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$N(A) = \langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} \rangle$ and $\eta = 2$.

Row Space $= \langle \vec{R_1}, \vec{R_2} \rangle = \langle (1,2,2,3), (2,4,1,3) \rangle$ and $r = 2$.

Column Space $= \langle \vec{C_1}, \vec{C_3} \rangle = \langle \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \rangle$ and $r = 2$.

Co-Kernel: $A\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ has solutions $\Leftrightarrow$ $c - 2b = 0$ $\Leftrightarrow 0a + 2b + 1c =$

$$0 \Leftrightarrow \begin{pmatrix} 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0.$$ So Co-Kernel $= \langle \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \rangle$ and Co-rank $= 1$ $(2+1=3)$.

You can also get this form $2R_2 - R_3 = 0$.

4. (25pnts) Use LU-decomposition to solve $\begin{pmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 24 \\ 12 \end{pmatrix}$

Solution:

$$\begin{pmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{pmatrix} \Rightarrow \ldots \Rightarrow = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix}; \text{ so } LUx = b$$

$Ly = b: \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 24 \\ 12 \end{pmatrix}$ implies $y_1 = 12$, $y_2 = 24 - 24 = 0$, $y_3 = -24$

$Ux = y$ implies $\begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ -24 \end{pmatrix}$ implies $z = -\frac{24}{4} = -6$, $y = -z = 6$, $x = 6 - y - z = 6 - 6 + 6 = 6$.

$$\vec{x} = \begin{pmatrix} 6 \\ 6 \\ -6 \end{pmatrix}$$

B U Department of Mathematics

Math 201 Matrix Theory

Spring 2004 Second Midterm

1. For which value(s) of the real number k, does the following linear system has:

$$\begin{array}{lcccccc} L_1 : x & + & y & - & z & = & 1 \\ L_2 : 2x & + & 3y & + & kz & = & 3 \\ L_3 : x & + & ky & + & 3z & = & 2 \end{array}$$

(a) a unique solution
(b) no solution

Solution:
First we find $-2L_1 + L_2$

$$\begin{array}{rcrcrcrcr} -2L_1 & : & -2x & - & 2y & + & 2z & = & 2 \\ L_2 & : & 2x & + & 3y & + & kz & = & 3 \end{array}$$

Adding side by side we get $-2L_1 + L_2 : y + (k+2)z = 1$
Next we find $-L_1 + L_3$

$$\begin{array}{rcrcrcrcr} -L_1 & : & -x & - & y & + & z & = & 1 \\ L_3 & : & x & + & ky & + & 3z & = & 2 \end{array}$$

Adding side by side we get $-L_1 + L_3 : (k-1)y + 4z = 1$
So we have

$$\begin{array}{rcccccccc} L_1 & : & x & + & y & - & z & = & 1 \\ 2L_1 + L_2 & : & & & y & + & (k+2)z & = & 1 \\ -L_1 + L_3 & : & & & (k-1)y & + & 4z & = & 1 \end{array}$$

Then we compute $(k-1)(2L_1 + L_2) + (-L_1 + L_3)$

$$\begin{array}{rcrcrcl} -(k-1)(2L_1 + L_2) & : & -(k-1)y & - & (k-1)(k+2)z & = & 1-k \\ (-L_1 + L_3) & : & (k-1)y & + & 4z & = & 1 \end{array}$$

Adding side by side we get $(k-1)(2L_1 + L_2) + (-L_1 + L_3) : (-k^2 - k + 6)z = 2 - k$
OR we have $(k-1)(2L_1 + L_2) + (-L_1 + L_3) : (6 - k - k^2)z = 2 - k$
So, $(k-1)(2L_1 + L_2) + (-L_1 + L_3) : (3+k)(2-k) = 2 - k$
Therefore,
(a) If $k \neq 2, k \neq -3$, then the linear system has a unique solution.
(b) If $k = -3$, linear system has no solution.

2. For the vectors v_1 and v_2 in a vector space V, let $W = Span\{v_1, v_2\}$. Show that W is a subspace of V.

Solution:
First we will show that $\forall u, w \in W$, $u + w \in W$
Let $u, w \in W$ be arbitrary, then $u = c_1v_1 + c_2v_2$ and $w = c_3v_1 + c_4v_2$ for some $c_1, c_2, c_3, c_4 \in \mathbb{R}$.

$$u + w = (c_1v_1 + c_2v_2) + (c_3v_1 + c_4v_2) = (c_1 + c_3)v_1 + (c_2 + c_4)v_2$$

by the axioms for vector spaces. So, $u + w \in W$.
Next we will show that $\forall u \in W$ and $\forall c \in \mathbb{R}$,we have $cu \in W$.
Let $u \in W$ and $c \in \mathbb{R}$ be arbitrary, then $u = c_1v_1 + c_2v_2$ for some $c_1, c_2 \in \mathbb{R}$.

$$cu = c(c_1v_1 + c_2v_2) = (cc_1)v_1 + (cc_2)v_2$$

by the axioms for vector spaces. So, $cu \in W$.
Hence, W is a subspace of V.

3. Let M_{2x2} be the vector space of all $2x2$ matrices, and define $T : M_{2x2} \mapsto M_{2x2}$ by $T(A) = A + A^T$. Show that T is a linear transformation.

Solution:
First we will show that $\forall A, B \in M_{2x2}$,$T(A + B) = T(A) + T(B)$
Let $A, B \in M_{2x2}$ be arbitrary. Then,

$$T(A + B) = (A + B) + (A + B)^T = A + B + A^T + B^T = (A + A^T) + (B + B^T) = T(A) + T(B)$$

Next we will show that $\forall c \in \mathbb{R}$ and $A \in M_{2x2}$, $T(cA) = cT(A)$
Let $c \in \mathbb{R}$ and $A \in M_{2x2}$ be arbitrary.

$$T(cA) = (cA) + (cA)^T = cA + cA^T = c(A + A^T) = cT(A)$$

Hence, T is a linear transformation.

4. Let A= $\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$ be 3x5 matrix
(a) Find a basis for the row space of A.

(b) Find a basis for the column space of A.
(c) Find the dimension of the null space of A.
(d) Find the rank of A.

Solution:

$$A \xrightarrow{-1/3r_1 \to r_1} \begin{bmatrix} 1 & -2 & 1/3 & -1/3 & 7/3 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \xrightarrow{-r_1+r_2 \to r_2} \begin{bmatrix} 1 & -2 & 1/3 & -1/3 & 7/3 \\ 0 & 0 & 5/3 & 10/3 & -10/3 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \xrightarrow{r_2+r_1 \to r_1}$$

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 5/3 & 10/3 & -10/3 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \xrightarrow{3/5r_2 \to r_2} \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \xrightarrow{2r_1+r_3 \to r_3} \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix}$$

$$\xrightarrow{-r_2+r_3 \to r_3} \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-2r_2+r_1 \to r_1} \begin{bmatrix} 1 & -2 & 0 & 1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a)A basis for row space R(A) is $\{ \begin{bmatrix} 1 & -2 & 0 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & -0 & 1 & 2 & -2 \end{bmatrix} \}$

(b)A basis for column space C(A) is $\{ \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \}$

(c) Dimension of Null space of A is $dim\mathcal{N}(A) = n - r = 5 - 2 = 3$

(d) The rank of A is $rank(A) = 2$.

B U Department of Mathematics

Math 201 Matrix Theory

Spring 2005 First Midterm

1.) a) Find c such that the following set of columns is a basis for R^3: $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ c \end{bmatrix} \right\}$.

Solution:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ -1 & 0 & c \end{bmatrix} \xrightarrow[r_1+r_3 \to r_3]{-r_1+r_2 \to r_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 2 & c+1 \end{bmatrix} \xrightarrow{2r_2+r_3 \to r_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & c+1 \end{bmatrix}.$$ Hence $c \neq -1$, i.e., $\forall c \in \mathbb{R} \setminus \{-1\}$ the given set of columns is a basis for $\mathbb{R}^3$.

b) Is the set of polynomials $S = \{1-x,\, 1+x,\, 1-x^2\}$ linearly independent?

Solution:

Consider

$$a(1-x) + b(1+x) + c(1-x^2) = 0$$

Then $-cx^2 = 0$ implies $c = 0$. So $a + b = 0$ and $-a + b = 0$ give that $a = 0$, $b = 0$. Thus S is linearly independent.

c) If a matrix A is $n \times (n-1)$ and its rank is $(n-2)$ what is the dimension of its null space?

Solution:

Since the dimension of the null space is the difference of the number of unknowns and the rank, we get

$$\dim(\text{Null}(A)) = (n-1) - (n-2) = 1$$

2.) Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ -1 & 3 & 0 \end{bmatrix}$

a) Find the LU decomposition of A.

Solution:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ -1 & 3 & 0 \end{bmatrix} \xrightarrow[r_1+r_3 \to r_3]{-2r_1+r_2 \to r_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \\ 0 & 5 & 1 \end{bmatrix} \xrightarrow{r_2+r_3 \to r_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \\ 0 & 0 & 0 \end{bmatrix} = U,$$ where $E_3E_2E_1A = U$, i.e., $A = E_1^{-1}E_2^{-1}E_3^{-1}U$. Writing explicitly

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

where $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

b) Find a basis for the column space and the null space of A. What is the rank of A?

Solution:

From U we see that pivots 1 and -5 appear in the first and second columns. Therefore $\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}2\\-1\\3\end{bmatrix}\right\}$ is a basis for the column space of A. To find a basis for the null space recall that $Ax = 0 \iff Ux = 0$. Then

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ -5x_2 - x_3 &= 0 \end{aligned}$$

implies $x_3 = -5x_2$ and $x_1 = 3x_2$, hence, $x = x_2\begin{bmatrix}3\\1\\-5\end{bmatrix}$. Thus $\left\{\begin{bmatrix}3\\1\\-5\end{bmatrix}\right\}$ is a basis for the null space of A. Since rank equals to the dimension of the column space, $\text{rank}(A) = 2$.

c) Using the LU decomposition of A find the complete solution to

$$Ax = \begin{bmatrix}4\\3\\1\end{bmatrix}$$

Solution:

Setting $y = Ux$, $Ax = \begin{bmatrix}4\\3\\1\end{bmatrix}$ implies $Ly = \begin{bmatrix}4\\3\\1\end{bmatrix}$, since $Ax = LUx$. Then using L from part (a),

$$\begin{array}{llll} y_1 & & & = 4 \\ 2y_1 & +y_2 & & = 3 \\ -y_1 & -y_2 & +y_3 & = 1 \end{array}$$

entails $y_1 = 4$, $y_2 = -5$ and $y_3 = 0$. Now, $Ux = \begin{bmatrix}4\\-5\\0\end{bmatrix}$ gives that

$$\begin{array}{llll} x_1 & +2x_2 & x_3 & = 4 \\ & -5x_2 & -x_3 & = -5 \end{array}$$

hence $x_1 = 3x_2 - 1$ and $x_3 = -5x_1 + 5$, i.e., $x = \begin{bmatrix}-1\\0\\5\end{bmatrix} + x_2\begin{bmatrix}3\\1\\-5\end{bmatrix}$.

3.) a) Let A be an $m \times n$ and B be an $n \times m$ matrix, and $m > n$. What can you say about the invertibility of AB?

Solution:

We claim that AB is singular. Given $m > n$ there exists a nonzero solution to $Bx = 0$, i.e., $\exists x_0 \neq 0$ such that $Bx_0 = 0$. Then $(AB)x_0 = A(Bx_0) = 0$. But AB being an $m \times m$ matrix and $(AB)x_0$ being zero with $x_0 \neq 0$ implies $\dim(\text{Null}(AB)) \neq 0$, hence $\text{rank}(AB) \neq m$. Thus AB is not invertible.

b) Let A and B be $n \times n$ matrices. Show that if A is singular then AB is also singular.

Solution:

Assume that A is singular. Then A^T is also singular, i.e., $A^T x$ has a non-trivial solution, say, $A^T x_0 = 0$ for some $x_0 \neq 0$. But then we get that $(AB)^T x_0 = B^T(A^T x_0) = 0$, so that $(AB)^T$ is singular. Thus AB is singular.

c) If A is an $n \times n$ matrix with $A^2 = A$ and $\text{rank}(A) = n$, find A.

Solution:

$\text{rank}(A) = n$ implies that A is invertible, i.e., A^{-1} exists. Then multiplying both sides of $A^2 = A$ by A^{-1} we get

$$A^2 A^{-1} = AA^{-1} = I$$

and so

$$A = I.$$

4.) Let $T : R^3 \to R^3$ be defined by $T(x, y, z) = (x + 2y + z, x + y, 2y + z)$.
a) Write down what we must show to prove that T is a linear transformation. (Do not carry out the computations).
b) What is the matrix representing this transformation in the standard basis for R^3.
c) Show that T is non-singular and find its inverse transformation.

Solution:

a) We have to show that given two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in $\mathbb{R}^3$

$$T((x_1, y_1, z_1) + (x_2, y_2, z_2)) = T(x_1, y_1, z_1) + T(x_2, y_2, z_2),$$
$$T(c(x_1, y_1, z_1)) = cT(x_1, y_1, z_1), \ \forall c \in \mathbb{R}.$$

b) Since $T(1, 0, 0) = (1, 1, 0)$, $T(0, 1, 0) = (2, 1, 2)$ and $T(0, 0, 1) = (1, 0, 1)$, we get that the matrix representing T is

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

c) To show that T is non-singular, it suffices to show that A is row equivalent to $I_{3\times3}$. Using Gauss-Jordan method we get that

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[2r_2 - r_3 \to r_3]{r_1 - r_2 \to r_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{array}\right]$$
$$\xrightarrow[r_1 - r_3 \to r_1]{r_2 - r_3 \to r_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{array}\right] \xrightarrow{r_1 - 2r_2 \to r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{array}\right] = [I|A^{-1}].$$

Since $A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 2 & -2 & -1 \end{bmatrix}$, we get that

$$T^{-1}(x, y, z) = (x - z, -x + y + z, 2x - 2y - z).$$

B U Department of Mathematics

Math 201 Matrix Theory

Spring 2006 First Midterm

1. By Gauss-Jordan method compute the inverse of A $= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, if exists. (12 points).

Solution:

$$\left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_1 \leftrightarrow r_2, (-1)r_2 + r_3 \to r_3]{} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[(-1)r_4 + r_1 \to r_1]{} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[r_3 \leftrightarrow r_4]{} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \end{array}\right]$$

So the second part of the last matrix is $A^{-1} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$

2. Given a 3×3 matrix $A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$, for which vectors X does there exist a scalar c such that $AX = cX$? (13 points)

Solution:

$$AX = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\Leftrightarrow 5x_1 = cx_1$
$x_1 + 5x_2 = cx_2$
$x_2 + 5x_3 = cx_3$

$\Leftrightarrow (5 - c)x_1 = 0$
$x_1 = (c - 5)x_2$
$x_2 = (c - 5)x_3$

Case 1: If $c = 0 \Rightarrow$ System is homogeneous and $rank(A) = 3 \Rightarrow X = 0$, unique solution.

Case 2: If $c \neq 0$ and $c \neq 5$, then $x_1 = x_2 = x_3 = 0$ implying $X = 0$, unique solution.

Case 3: If $c = 5 \Rightarrow x_1 = x_2 = 0 \Rightarrow X = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix}$, $x_3 \in \mathbb{R}$. There are infinitely many solutions.

Hence; for $X = 0$, c can be any real number; for $X = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix}$, $x_3 \neq 0$, c must be 5.

3. Decide whether the followings are TRUE or FALSE. If true prove; if false, give a counter example or explain. (Each 5 points)

i. Let A,X,Y be square matrices of the same size. Then $AX = AY$ implies $X = Y$.

If $A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ and $X = \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}$, then $AX = AY = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$

But $X \neq Y$, so FALSE.

ii. The system

$$\begin{aligned} x - y + z &= 3 \\ -x + 2y + kz &= 5 \\ 2x - y + z &= 4 \end{aligned}$$

has a unique solution for each value of k.

$$\begin{bmatrix} 1 & -1 & 1 & \vdots & 3 \\ -1 & 2 & k & \vdots & 5 \\ 2 & -1 & 1 & \vdots & 4 \end{bmatrix} \xrightarrow[r_1+r_2 \to r_2 \text{ and } (-2)r_1+r_3 \to r_3]{} \begin{bmatrix} 1 & -1 & 1 & \vdots & 3 \\ 0 & 1 & k+1 & \vdots & 8 \\ 0 & 1 & -1 & \vdots & -2 \end{bmatrix}$$

$$\xrightarrow[(-1)r_2+r_3 \to r_3]{} \begin{bmatrix} 1 & -1 & 1 & \vdots & 3 \\ 0 & 1 & k+1 & \vdots & 8 \\ 0 & 0 & -k-2 & \vdots & -10 \end{bmatrix}$$

So the system has a unique solution if $-k-2 \neq 0$, i.e. if $k \neq -2$, so FALSE.

iii. $W = \{x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} : x, y \in \mathbb{R}\}$ is the only subspace of $\mathbb{R}^3$ of dimension 2.

There are other 2-dimensional subspaces like xz or yz planes, so FALSE.

iv. The subset $W = \{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1^2 + x_2^2 + x_3^2 = 1\}$ of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not an element of W since $0^2 + 0^2 + 0^2 \neq 1$. So FALSE.

v. The nullity of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ is 1.

$Rank(A) = 3 \Rightarrow$ Nullity of $A = 3 - 3 = 0$, so FALSE.

vi. The left null space of a 4×6 matrix A with real entries is a subspace of $\mathbb{R}^6$.

A is 4×6 matrix, so A^T is 6×4 matrix. $\Rightarrow \mathcal{N}(A^T) = \{y \in \mathbb{R}^4 : A^T y = 0\} \subseteq \mathbb{R}^4$, so FALSE.

vii. Let A and B be invertible $n \times n$ matrices, then $(A+B)^{-1} = A^{-1} + B^{-1}$.

For $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ both are invertible matrices. But $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible, so FALSE.

viii. If $\{u, v, w\}$ is a linearly independent set of vectors in a vector space V then $\{u, u+v, u+v+w\}$ is also a linearly independent set of vectors.

Let α, β, γ be scalars so that $\alpha u + \beta(u+v) + \gamma(u+v+w) = 0 \Rightarrow (\alpha+\beta+\gamma)u + (\beta+\gamma)v + \gamma w = 0$ Then, since $\{u, v, w\}$ is a linearly independent set of vectors, each coefficient of the last equation should be zero, so

$\alpha + \beta + \gamma = 0$
$\quad \beta + \gamma = 0$
$\qquad \gamma = 0$
$\Rightarrow \alpha = \beta = \gamma = 0$
Hence $\{u, u+v, u+v+w\}$ is also linearly independent set of vectors, so the statement is TRUE.

4.(i) Prove that the polynomials 1, x, $\frac{3}{2}x^2 - \frac{1}{2}$, $\frac{5}{2}x^3 - \frac{3}{2}x$ form a basis for P_4, the vector space of all polynomials with degree at most 3. (10 points)

Solution:
For a, b, c, d reals let $a \cdot 1 + bx + c(\frac{3}{2}x^2 - \frac{1}{2}) + d(\frac{5}{2}x^3 - \frac{3}{2}x) = 0$

$\Rightarrow (a - \frac{c}{2}) + (b - \frac{3}{2}d)x + (\frac{3}{2}c)x^2 + (\frac{5}{2}d)x^3 = 0$

$\Rightarrow a - \frac{c}{2} = b - \frac{3}{2}d = \frac{3}{2}c = \frac{5}{2}d = 0$

$\Rightarrow a = b = c = d = 0$
So linear independence follows. Since $dimP_4 = 4$ and there are 4 linearly independent polynomials, they form a basis for P_4.

(ii) Can we find an $m \times n$ matrix A and vectors b and c so that $Ax = b$ has no solution and $A^T y = c$ has exactly one solution. (10 points)

Solution:
In order that $Ax = b$ has no solution, $r = dim\mathcal{R}(A) = rank(A) < m$ must hold. i.e the column space $\mathcal{R}(A)$ of A cannot span $\mathbb{R}^m$.
But, for A^T is an $n \times m$ matrix, $A^T y = c$ has exactly one solution if $r = rank(A^T) = rank(A) = m$ must hold.
Clearly, we can not have both $r < m$ and $r = m$. So we have a contradiction.

5. Find LDU factorization of the matrix $A = \begin{bmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & 2 & 0 & 2 \end{bmatrix}$, where L is a lower triangular, D is a diagonal and U is an echelon matrix. (15 points)

Solution:

$$\begin{bmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & 2 & 0 & 2 \end{bmatrix} \xrightarrow[E_{32}(1)E_{21}(-2)]{} \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 4 & -2 & 2 \end{bmatrix}$$

$$\xrightarrow[E_{32}(4)]{} \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix} = \text{DU}$$

So $E_{32}(4)E_{31}(1)E_{21}(-2)A = DU$

Hence $L = E_{21}(2)E_{31}(-1)E_{32}(-4) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -4 & 1 \end{bmatrix}$

Thus $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$

is the required factorization.

B U Department of Mathematics

Math 201 Matrix Theory

Summer 2003 First Midterm

1. Let $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 4 \\ 1 & 2 & 1 & 1 \end{pmatrix}$, and let R be its row-reduced echelon form.

a) Find all solutions of $Ax = 0$ by first finding R.

Solution:

$$A \xrightarrow{E_1: r_1+r_2\to r_2, E_2: -r_1+r_3\to r_3} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 4 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{E_3: \frac{1}{2}r_2\to r_2} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{E_4: -2r_2+r_1\to r_1} \begin{pmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{E_5: 4r_3+r_1\to r_1, E_6: -2r_3+r_2\to r_2} \begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= R.$$

$Ax = 0 \;\Leftrightarrow\; Rx = 0$, so

$$\begin{aligned} x_1 - 3x_3 &= 0 \\ x_2 + 2x_3 &= 0 \\ x_4 &= 0. \end{aligned}$$

Setting x_3 to be the free parameter, the complete solutions are

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} +3t \\ -2t \\ t \\ 0 \end{pmatrix} \quad t \in \mathbb{R}.$$

b) Find a $3x3$ matrix S such that $SA = R$. (Hint: Think of the elementary matrices corresponding to the operations performed in part (a).)

Solution:

The row operations applied above corresponds to the following matrices E_i in 3×3 dimensions:

$$
\begin{aligned}
R &= \begin{pmatrix} 1 & & \\ & 1 & -2 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & 4 \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1/2 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{pmatrix} \\
& \quad \begin{pmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{pmatrix} = A,
\end{aligned}
$$

so $E_6E_5E_4E_3E_2E_1 = S$.

Multiplying these elementary matrices we get S:

$$
\begin{aligned}
S &= \begin{pmatrix} 1 & & \\ & 1 & -2 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & 4 \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1/2 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ 1 & 1 & \\ -1 & & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & & \\ & 1 & -2 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & 4 \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ 1/2 & 1/2 & \\ -1 & & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & & \\ & 1 & -2 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & 4 \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} & -1 & \\ 1/2 & 1/2 & \\ -1 & & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & & \\ & 1 & -2 \\ & & 1 \end{pmatrix} \begin{pmatrix} -4 & -1 & 4 \\ 1/2 & 1/2 & \\ -1 & & 1 \end{pmatrix} \\
&= \begin{pmatrix} -4 & -1 & 4 \\ 5/2 & 1/2 & -2 \\ -1 & 0 & 1 \end{pmatrix} \\
&= S.
\end{aligned}
$$

You can always check your answer by explicit multiplication.

2. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{pmatrix}$.

a) Give an LU-decomposition of A.

Solution:

$$A \xrightarrow{E_1:-2r_1+r_2\to r_2, E_2:-3r_1+r_3\to r_3} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = U$$

where $E_1 = \begin{pmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & & \\ & 1 & \\ -3 & & 1 \end{pmatrix}$.

So $E_2E_1A = U$ implies $A = E_1^{-1}E_2^{-1}U = LU$.

$$\begin{aligned} A &= \begin{pmatrix} 1 & & \\ & 1 & \\ 3 & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{pmatrix} U \\ &= \begin{pmatrix} 1 & & \\ 2 & 1 & \\ 3 & & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

is the LU-decomposition.

b) Using the decomposition, solve $Ax = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$

Solution:

$Ax = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$ if and only if $LUx = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$, say $Ux = y$. Then $Ly = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$.

$$\begin{aligned} y_1 &= 1 \\ 2y_1 + y_2 &= 3 \Rightarrow y_2 = 1 \\ 3y_1 + y_3 &= 4 \Rightarrow y_3 = 1. \end{aligned}$$

$\Rightarrow y = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Then we solve $Ux = y = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ with the above U.

Back substitution gives:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ x_3 &= 1 \\ x_3 &= 1 \end{aligned}$$

$\Rightarrow x_1 + 2x_2 = -2\, x_1 = -2 - 2x_2$. Letting $x_2 = t$ we get $x = \begin{pmatrix} -2-2t \\ t \\ 1 \end{pmatrix}$.

c) Is A invertible? Justify your answer (do not find A^{-1}, in case it exists!).

Solution:

No, A is not invertible. If A, being square, were invertible, $Ax = b$ would have a unique solution for every b. There might be other reasoning as well: if we go one step further

$$U \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

is not row-equivalent to I_3, hence not invertible (There is a zero row).

3. Consider the matrix $A = \begin{pmatrix} 1 & 1 & 5 & 1 & 4 \\ 2 & -1 & 1 & 2 & 2 \\ 3 & 0 & 6 & 0 & -3 \end{pmatrix}$

a) Describe the null space $N(A)$ of A by giving a basis for it and finding its dimension.

Solution:

$$A \xrightarrow{-2r_1+r_2\to r_2,\,-3r_1+r_3\to r_3} \begin{pmatrix} 1 & 1 & 5 & 1 & 4 \\ 0 & -3 & -9 & 0 & -6 \\ 0 & -3 & -9 & -3 & -15 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{3}r_2\to r_2} \begin{pmatrix} 1 & 1 & 5 & 1 & 4 \\ 0 & 1 & 3 & 0 & 2 \\ 0 & -3 & -9 & -3 & -15 \end{pmatrix}$$

$$\xrightarrow{-r_2+r_1\to r_1,\,3r_2+r_3\to r_3} \begin{pmatrix} 1 & 0 & 2 & 1 & 2 \\ 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & -3 & -3 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{3}r_3\to r_3} \begin{pmatrix} 1 & 0 & 2 & 1 & 2 \\ 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{-r_3+r_1\to r_1} \begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$N(A)$ is the set x such that $Ax = 0 \Leftrightarrow Rx = 0 \Leftrightarrow$

$$\begin{array}{rclcrcl} x_1 + 2x_3 + x_5 & = & 0 & & x_1 & = & -2x_3 - x_5 \\ x_2 + 3x_3 + 2x_5 & = & 0 & \Rightarrow & x_2 & = & -3x_3 - 2x_5 \\ x_4 + x_5 & = & 0 & & x_1 & = & -x_5. \end{array}$$

If $x \in N(A)$ then $x = \begin{pmatrix} -2t-s \\ -3t-2s \\ t \\ -s \\ s \end{pmatrix}$. A basis for $N(A) = \{ \begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 0 \\ -1 \\ 1 \end{pmatrix} \}$,
then $dim\, N(A) = 2$.

b) Give bases for and dimensions of the column space $C(A)$ and the row space $R(A)$ of A. Tell also which vector spaces they sit in, respectively.

Solution:

First, second and fourth columns of R contain leading 1s, so we choose first, second and fourth columns of A as a basis for $C(A)$: a basis for $C(A) = \{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \}$

A basis for $R(A) = \{ \begin{pmatrix} 1 & 0 & 2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 3 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \end{pmatrix} \}$

$dim\, C(A) = dim\, R(A) = 3$. Lastly $C(A) \subseteq \mathbb{R}^3$ and $R(A) \subseteq \mathbb{R}^5$.

c) Find $rank\,(A)$. Is A of maximal rank? Explain.

Solution:

$rank\,(A) = dim\, C(A) = dim\, R(A) = 3$.
Yes, A ,s of maximal rank, for $rank\,(A)$ is at most the minimum of row number $(= 3)$ and column number $(= 5)$, which is in this case 3.

4. Let $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ be the standard basis for $\mathbb{R}^3$

a) Show that the set $B = \{\epsilon_1 + \epsilon_2, \epsilon_2 + \epsilon_3, \epsilon_1 + \epsilon_3\}$ is a basis for $\mathbb{R}^3$.

Solution:

Solve $c_1(\epsilon_1 + \epsilon_2) + c_2(\epsilon_2 + \epsilon_3) + c_3(\epsilon_1 + \epsilon_3) = 0$.

Coefficient matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{-r_1+r_2 \to r_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{-r_2+r_3 \to r_3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{pmatrix} \xrightarrow{-\frac{1}{2}r_3 \to r_3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{-r_3+r_1 \to r_1, r_3+r_2 \to r_2} I$$

$\Rightarrow Ax = 0$ has a unique solution $x = 0$.
$\Rightarrow c_1 = c_2 = c_3 = 0$ is the only solution.
B is a linearly independent set. Note that this suffices to say B is a basis for $\mathbb{R}^3$, as $dim\,(\mathbb{R}^3) =$ the number of vectors in $B = 3$.

b) If $A = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$ is the matrix of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ in the standard basis, what is the matrix of T in the basis B?

Solution:

We understand that

$$\begin{aligned} T(\epsilon_1) &= \langle 3, 1, 2\rangle \\ T(\epsilon_2) &= \langle 0, -1, 2\rangle \\ T(\epsilon_3) &= \langle 0, 0, 1\rangle. \end{aligned}$$

In the new basis :

$$\begin{aligned} T(\epsilon_1 + \epsilon_2) &= T(\epsilon_1) + T(\epsilon_2) &&= \langle 3, 0, 3\rangle \\ T(\epsilon_2 + \epsilon_3) &= T(\epsilon_2) + T(\epsilon_3) &&= \langle 0, -1, 2\rangle \\ T(\epsilon_1 + \epsilon_3) &= T(\epsilon_1) + T(\epsilon_3) &&= \langle 3, 1, 3\rangle \end{aligned}$$

Then the matrix in B is $\begin{pmatrix} 3 & 0 & 3 \\ 0 & -1 & -1 \\ 3 & 2 & 3 \end{pmatrix}$.

c) Find the image of the point $(2, -1, 0)$ under T.

Solution:

$$T(2, -1, 0) = A \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 3 \end{pmatrix}.$$

5. Let V be a 3-dimensional vector space and let $B = \{\alpha_1, \alpha_2, \alpha_3\}$ be a basis for V. Let T be a linear transformation such that:

$$T(\alpha_1) = \alpha_2, T(\alpha_2) = \alpha_3, T(\alpha_3) = 0.$$

Show that $T^2 \neq 0$ but $T^3 = 0$. (T^n means n successive applications of T.)

Solution:

First we check the images of the basis elements:

$T(\alpha_1) = \alpha_2$ implies $T^2(\alpha_1) = T(\alpha_2) = \alpha_3$ implies $T^3(\alpha_1) = T(\alpha_3) = 0$

$T(\alpha_2) = \alpha_3$ implies $T^2(\alpha_2) = T(\alpha_3) = 0$ implies $T^3(\alpha_2) = T(0) = 0$

$T(\alpha_3) = 0$ implies $T^2(\alpha_3) = T(0) = 0$ implies $T^3(\alpha_3) = 0$ since T is linear.

$T^2 \neq 0$ since $T^2(\alpha_1) = \alpha_3 \neq 0$.
$T^3 = 0$ means $T^3(x) = 0$ for all $x \in V$.

$x \in V \;\Rightarrow\; x = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 \;\Rightarrow\; T^3(x) = c_1T^3(\alpha_1) + c_2T^3(\alpha_2) + c_3T^3(\alpha_3) = 0 \;\Rightarrow\; T^3(x) = 0$ for all $x \in V \;\Rightarrow\; T^3 = 0$.

B U Department of Mathematics

Math 201 Matrix Theory

Summer 2005 First Midterm

1.) a)[4] Find c such that the following set of columns is a basis for R^3: $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ c \end{bmatrix} \right\}$.

Solution:

$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ -1 & 0 & c \end{bmatrix} \xrightarrow[r_1+r_3 \to r_3]{-r_1+r_2 \to r_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 2 & c+1 \end{bmatrix} \xrightarrow{2r_2+r_3 \to r_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & c+1 \end{bmatrix}$. Hence $c \neq -1$, i.e., $\forall c \in \mathbb{R} \setminus \{-1\}$ the given set of columns is a basis for $\mathbb{R}^3$.

b)[4] Is the set of polynomials $S = \{1-x,\, 1+x,\, 1-x^2\}$ linearly independent?

Solution:

Consider

$$a(1-x) + b(1+x) + c(1-x^2) = 0$$

Then $-cx^2 = 0$ implies $c = 0$. So $a + b = 0$ and $-a + b = 0$ give that $a = 0$, $b = 0$. Thus S is linearly independent.

c)[2] If a matrix A is $n \times (n-1)$ and its rank is $(n-2)$ what is the dimension of its null space?

Solution:

Since the dimension of the null space is the difference of the number of unknowns and the rank, we get

$$\dim(\text{Null}(A)) = (n-1) - (n-2) = 1$$

2.) Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ -1 & 3 & 0 \end{bmatrix}$

a) [10] Find the LU decomposition of A.

Solution:

$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ -1 & 3 & 0 \end{bmatrix} \xrightarrow[r_1+r_3 \to r_3]{-2r_1+r_2 \to r_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \\ 0 & 5 & 1 \end{bmatrix} \xrightarrow{r_2+r_3 \to r_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \\ 0 & 0 & 0 \end{bmatrix} = U$, where $E_3E_2E_1A = U$, i.e., $A = E_1^{-1}E_2^{-1}E_3^{-1}U$. Writing explicitly

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

where $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

b) [6] Find a basis for the column space and the null space of A. What is the rank of A?

Solution:

From U we see that pivots 1 and -5 appear in the first and second columns. Therefore $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\}$ is a basis for the column space of A. To find a basis for the null space recall that $Ax = 0 \iff Ux = 0$. Then

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ -5x_2 - x_3 &= 0 \end{aligned}$$

implies $x_3 = -5x_2$ and $x_1 = 3x_2$, hence, $x = x_2 \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}$. Thus $\left\{ \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix} \right\}$ is a basis for the null space of A. Since rank equals to the dimension of the column space, $rank(A) = 2$.

c) [4] Using the LU decomposition of A find the complete solution to

$$Ax = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

Solution:

Setting $y = Ux$, $Ax = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$ implies $Ly = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$, since $Ax = LUx$. Then using L from part (a),

$$\begin{array}{llll} y_1 & & & = 4 \\ 2y_1 & +y_2 & & = 3 \\ -y_1 & -y_2 & +y_3 & = 1 \end{array}$$

entails $y_1 = 4$, $y_2 = -5$ and $y_3 = 0$. Now, $Ux = \begin{bmatrix} 4 \\ -5 \\ 0 \end{bmatrix}$ gives that

$$\begin{array}{llll} x_1 & +2x_2 & x_3 & = 4 \\ & -5x_2 & -x_3 & = -5 \end{array}$$

hence $x_1 = 3x_2 - 1$ and $x_3 = -5x_1 + 5$, i.e., $x = \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}$.

3.) a) [6] Let A be an $m \times n$ and B be an $n \times m$ matrix, and $m > n$. What can you say about the invertibility of AB?

Solution:

We claim that AB is singular. Given $m > n$ there exists a nonzero solution to $Bx = 0$, i.e., $\exists x_0 \neq 0$ such that $Bx_0 = 0$. Then $(AB)x_0 = A(Bx_0) = 0$. But AB being an $m \times m$ matrix and $(AB)x_0$ being zero with $x_0 \neq 0$ implies $\dim(\text{Null}(AB)) \neq 0$, hence $rank(AB) \neq m$. Thus AB is not invertible.

b) [6] Let A and B be $n \times n$ matrices. Show that if A is singular then AB is also singular.

Solution:

Assume that A is singular. Then A^T is also singular, i.e., $A^T x$ has a non-trivial solution, say, $A^T x_0 = 0$ for some $x_0 \neq 0$. But then we get that $(AB)^T x_0 = B^T(A^T x_0) = 0$, so that $(AB)^T$ is singular. Thus AB is singular.

c) [3] If A is an $n \times n$ matrix with $A^2 = A$ and rank(A)=n, find A.

Solution:

$rank(A) = n$ implies that A is invertible, i.e., A^{-1} exists. Then multiplying both sides of $A^2 = A$ by A^{-1} we get

$$A^2 A^{-1} = AA^{-1} = I$$

and so

$$A = I.$$

4.) Let $T : R^3 \to R^3$ be defined by $T(x, y, z) = (x + 2y + z, x + y, 2y + z)$.
a) [2] Write down what we must show to prove that T is a linear transformation. (Do not carry out the computations).
b) [5] What is the matrix representing this transformation in the standard basis for R^3.
c) [8] Show that T is non-singular and find its inverse transformation.

Solution:

a) We have to show that given two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in $\mathbb{R}^3$

$$T((x_1, y_1, z_1) + (x_2, y_2, z_2)) = T(x_1, y_1, z_1) + T(x_2, y_2, z_2),$$
$$T(c(x_1, y_1, z_1)) = cT(x_1, y_1, z_1),\ \forall c \in \mathbb{R}.$$

b) Since $T(1, 0, 0) = (1, 1, 0)$, $T(0, 1, 0) = (2, 1, 2)$ and $T(0, 0, 1) = (1, 0, 1)$, we get that the matrix representing T is

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

c) To show that T is non-singular, it suffices to show that A is row equivalent to $I_{3\times3}$. Using Gauss-Jordan method we get that

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[2r_2 - r_3 \to r_3]{r_1 - r_2 \to r_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{array}\right]$$

$$\xrightarrow[r_1 - r_3 \to r_1]{r_2 - r_3 \to r_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{array}\right] \xrightarrow{r_1 - 2r_2 \to r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & -2 & -1 \end{array}\right] = [I|A^{-1}].$$

.

Knowing $A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 2 & -2 & -1 \end{bmatrix}$, we get that

$$T^{-1}(x, y, z) = (x - z, -x + y + z, 2x - 2y - z).$$