B U Department of Mathematics

Math 201 Matrix Theory

Fall 2003 Final Exam

1.) Suppose A is a 4×3 matrix, and the complete solution to

$$Ax = \begin{bmatrix} 1 \\ 4 \\ 1 \\ 1 \end{bmatrix} \quad \text{is} \quad x = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, c \in \mathbb{R}$$

(a) Find the second and the third columns of A.
(b) Determine the ranks of the coefficient matrix and the augmented matrix. Give all the known information about the first column of A.

Solution:

(a)

Let $b^T = [1\ 4\ 4\ 1]$. From the particular solution when $c = 0$ it follows that

$$\text{column}_2 + \text{column}_3 = b.$$

The homogenous solution says that

$$2\text{column}_2 + \text{column}_3 = 0.$$

From these we obtain

$$\text{column}_2 = -b, \quad \text{column}_3 = b$$

(b)

We have $\dim \text{Null}(A) = 1$, which means $\text{rank}(A) = 2$. Since the system is consistent, we have $\text{rank}([A \mid b]) = \text{rank}(A) = 2$. Since the matrix must have two linearly independent columns, and all the remaining columns are multiples of b we can infer that the first column of A is not a multiple of b.

2.) Let $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, $w_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $w_2 = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$ and $A = \begin{bmatrix} -2 & 3 \\ 4 & 5 \end{bmatrix}$.

(a) Show that $\beta_I = \{v_1, v_2\}$ is a basis for $\mathbb{R}^2$.

Solution:

Since $\dim \mathbb{R}^2 = 2$, any pair of linearly independent vectors form a basis for $\mathbb{R}^2$. The vectors v_1 and v_2 are linearly independent because

$$\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = 3$$

and therefore β_I is a basis for $\mathbb{R}^2$.

(b) Suppose $A = [T]_{\beta_I}$ is the matrix representation of the linear transform $T : \mathbb{R}^2 \to \mathbb{R}^2$ relative to the basis β_I. Find the matrix representation of the same linear transformation $B = [T]_{\beta_{II}}$ with respect to the basis $\beta_{II} = \{w_1, w_2\}$.

Solution:

If we can find a matrix M such that $[w_1 \ w_2] = [v_1 \ v_2]M$ then B is related to A by $B = M^{-1}AM$.

We note that $w_1 = v_1 + v_2$ and $w_2 = 2v_2$, so $M = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ is the desired matrix.

$$M^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$B = M^{-1}AM = \begin{bmatrix} 1 & 6 \\ 4 & 2 \end{bmatrix}$$

.

3.)

Let $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & 0 \end{bmatrix}$.

(a) Using the Gram-Schmidt process, find the $A = QR$ factorization of A.

(b) Find the projection matrix which projects onto the column space of A.

Solution:

(a)

Let $a = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $b = \begin{bmatrix} \sin\theta \\ 0 \end{bmatrix}$. Since $\|a\| = 1$ we set $q_1 = a$. Then we have $b' = b - ({q_1}^T b)q_1$ and $q_2 = b'/\|b'\|$. We compute

$$b' = \sin^2\theta \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}, \quad q_2 = \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}.$$

So $Q = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $R = \begin{bmatrix} q_1^T a & q_1^T b \\ 0 & q_2^T b \end{bmatrix} = \begin{bmatrix} 1 & \cos\theta\sin\theta \\ 0 & \sin^2\theta \end{bmatrix}$

(b) If θ is not an integer multiple of π, then R has linearly independent columns, hence $R^T R$ is invertible, and $P = A(A^T A)^{-1}A^T = Q^T Q = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

If θ is an integer multiple of π, then the second column of A is zero, and the first column is $a = [1\ 0]^T$, so the column space is the one dimensional space spanned by a. The projection matrix onto this space is:

$$P = \frac{aa^T}{a^T a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

4.)

(a) Using the cofactor matrix, find the inverse A^{-1} of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Solution:

Since A is symmetric, A_{cof} is also symmetric.

$$A_{11} = \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} = 2, \quad A_{12} = -\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = -1, \quad A_{13} = 0, \quad A_{22} = 2 \quad A_{23} = -1, \quad A_{33} = 1$$

.

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$A^{-1} = \frac{1}{\det A} A_{\text{cof}} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

(b) Let a, b and c be nonzero real numbers. Without using the cofactor expansion, prove that

$$\begin{vmatrix} bc & a^2 & a^2 \\ b^2 & ca & b^2 \\ c^2 & c^2 & ab \end{vmatrix} = \begin{vmatrix} bc & ab & ca \\ ab & ca & bc \\ ca & bc & ab \end{vmatrix}$$

Solution:

We first multiply the columns and then divide the rows by a, b and c respectively:

$$\begin{vmatrix} bc & a^2 & a^2 \\ b^2 & ca & b^2 \\ c^2 & c^2 & ab \end{vmatrix} = \frac{1}{abc} \begin{vmatrix} abc & ba^2 & ca^2 \\ ab^2 & bca & cb^2 \\ ac^2 & bc^2 & cab \end{vmatrix} = \frac{abc}{abc} \begin{vmatrix} bc & ba & ca \\ ab & ca & cb \\ ac & bc & ab \end{vmatrix} = \begin{vmatrix} bc & ab & ca \\ ab & ca & bc \\ ca & bc & ab \end{vmatrix}$$

5.) Let $A = \begin{bmatrix} 3 & 4 & 6 \\ 0 & 1 & 0 \\ -1 & -2 & -2 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(a) Find all the eigenvalues and the eigenvectors of the singular matrix A. Is A diagonalizable? Explain.

(b) Compute $A^{99}b$.

Solution:

(a) $p(\lambda) = \det A - I\lambda = -(1-\lambda)^2\lambda$. So the eigenvalues are 1, 1, and 0.

The eigenspace for $\lambda = 0$ is spanned by $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

The eigenspace for $\lambda = 1$ is spanned by the vectors $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

Since these three vectors form a basis for $\mathbb{R}^3$, A is diagonalizable.

(b) $A = S\Lambda S^{-1}$, $A^{99} = S\Lambda^{99}S^{-1}$

$\Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, hence $\Lambda^{99} = \Lambda$, which means $A^{99} = A$. So $A^{99}b = Ab = \begin{bmatrix} 13 \\ 1 \\ -5 \end{bmatrix}$

6.)

(a) Given that $A = \begin{bmatrix} 7 & 0 \\ 4 & 7 \end{bmatrix}$ find $\exp At$.

Solution:

$$A = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix} = D + E$$

Because $DN = ND$, $\exp D + N = \exp D \exp N$. Also note that $D = 7I$ and $N^2 = I$. So we have $\exp Nt = I + Nt + 0$ and $\exp Dt = e^{7t} I$

So we have $B = \exp At = e^{7t}(I + Nt) = \begin{bmatrix} e^{7t} & 0 \\ 4te^{7t} & e^{7t} \end{bmatrix}$

(b) A 4×4 matrix is known to have the eigenvalues $\lambda_1 = 2$, $\lambda_2 = \lambda_2 = -3$ and $\lambda_4 = 5/2$. Find

(i) $\det(I + C)$
(ii) $\text{trace}(I + C)$
(iii) $\det 2C^{-1}$

Solution:

(i) $I + C$ has the eigenvalues 3, -2, -2, 7/2. So its determinant is 42, which is the product of these.

(ii) Trace equals the sum of the eigenvalues: -3 -2 -2 +7/2 = 5/2.

(iii) $\det 2C^{-1} = 16 \det C^{-1} = \frac{16}{\det C} = \frac{16}{45}$. ($\det C$ can be evaluated by multiplying its eigenvalues)

B U Department of Mathematics

Math 201 Matrix Theory

Fall 2004 Final

1. (a) Show that $\boldsymbol{v} = \begin{bmatrix} 7 \\ -1 \\ 4 \\ 4 \end{bmatrix}$ is an eigenvector of $\boldsymbol{A} = \begin{bmatrix} 4 & 2 & 0 & 4 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 4 & 0 & 7 \end{bmatrix}$ by finding the corresponding eigenvalue.

Solution:

Observe that $\boldsymbol{Av} = \begin{bmatrix} 42 \\ -6 \\ 24 \\ 24 \end{bmatrix} = 6 \begin{bmatrix} 7 \\ -1 \\ 6 \\ 6 \end{bmatrix}$. Hence $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$ corresponding to the eigenvalue 6.

(b) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two 4×4 matrices. Given that $\boldsymbol{B}^2(\boldsymbol{AB} - \boldsymbol{B}^2)\boldsymbol{B}^{-1}(\boldsymbol{A} - \boldsymbol{B})^2 = \boldsymbol{I}$ and $\det \boldsymbol{B} = 3$, find $\det(\boldsymbol{A} - \boldsymbol{B})$.

Solution:

$\boldsymbol{I} = \boldsymbol{B}^2(\boldsymbol{AB} - \boldsymbol{B}^2)\boldsymbol{B}^{-1}(\boldsymbol{A} - \boldsymbol{B})^2 = \boldsymbol{B}^2(\boldsymbol{A} - \boldsymbol{B})\boldsymbol{B}\boldsymbol{B}^{-1}(\boldsymbol{A} - \boldsymbol{B})^2 = \boldsymbol{B}^2(\boldsymbol{A} - \boldsymbol{B})^3$.
Therefore, $\det(\boldsymbol{A} - \boldsymbol{B}) = 3^{-\frac{2}{3}}$.

(c) Let P_3 be the vector space of all polynomials in x of degree <u>at most</u> 3 with real coefficients. Consider the differentiation transformation $D : P_3 \to P_3$ taking a polynomial to its derivative with respect to x. Is the matrix corresponding to D in standard basis $\{1, x, x^2, x^3\}$ of P_3 diagonalisable?

Solution:

The only eigenvalue of D is 0 with multiplicity 4. The eigenvectors of D are constant polynomials. Therefore the eigenspace of D corresponding to 0 is 1-dimensional. There is shortage of eigenvectors; D is not diagonalisable.

2. Assume that $\boldsymbol{A}$ is a diagonalisable matrix and $\boldsymbol{A}^{2005} = \boldsymbol{I}$. Show that $\boldsymbol{A} = \boldsymbol{I}$.

Solution:

Since $\boldsymbol{A}^{2005} = S^{-1}\boldsymbol{\Lambda}^{2005}S = \boldsymbol{I}$, we have $\boldsymbol{\Lambda}^{2005} = S\boldsymbol{I}S^{-1} = \boldsymbol{I}$. Therefore for each eigenvalue λ, we have $\lambda^{2005} = 1$ and hence every eigenvalue is 1 since 2005 is odd. Then $\boldsymbol{\Lambda} = \boldsymbol{I}$ and $\boldsymbol{A} = S^{-1}\boldsymbol{\Lambda}S = S^{-1}\boldsymbol{I}S = \boldsymbol{I}$.

3. Find the solution of the following system of first order linear differential equations which satisfies the given initial values:

$$\frac{d\boldsymbol{x}}{dt} = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \boldsymbol{x}, \quad \boldsymbol{x}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

Solution:

The solution is $\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0)$. To calculate $e^{\boldsymbol{A}t}$ we find the eigenvalues and corresponding eigenvectors:

$$0 = \det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{vmatrix} -3-\lambda & 1 \\ 2 & -2-\lambda \end{vmatrix} = (3+\lambda)(2+\lambda) - 2 = \lambda^2 + 5\lambda + 4 \quad \text{gives} \quad \lambda = -4 \text{ or } -1.$$

$$\text{Eigenspace for -4} = \text{null} \begin{bmatrix} -3-(-4) & 1 \\ 2 & -2-(-4) \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \{a \begin{bmatrix} 1 \\ -1 \end{bmatrix} | a \in \mathbb{R}\}.$$

$$\text{Eigenspace for -1} = \text{null} \begin{bmatrix} -3-(-1) & 1 \\ 2 & -2-(-1) \end{bmatrix} = \text{null} \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} = \{b \begin{bmatrix} 1 \\ 2 \end{bmatrix} | b \in \mathbb{R}\}.$$

Then,

$$\begin{aligned} \boldsymbol{x}(t) &= e^{\boldsymbol{A}t}\boldsymbol{x}(0) = Se^{\boldsymbol{\Lambda}t}S^{-1}\boldsymbol{x}(0) \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} e^{-4t} & 0 \\ 0 & e^{-t} \end{bmatrix} \left(\frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\right) \begin{bmatrix} 3 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} e^{-4t} & e^{-t} \\ -e^{-4t} & 2e^{-t} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2e^{-4t} + e^{-t} \\ -2e^{-4t} + 2e^{-t} \end{bmatrix}. \end{aligned}$$

4. Let $\boldsymbol{u}, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$ be column vectors in $\mathbb{R}^n$. Suppose that:

$$\boldsymbol{u}^T\boldsymbol{\gamma}_1 = 3, \quad \boldsymbol{u}^T\boldsymbol{\gamma}_2 = -5, \quad \boldsymbol{u}^T\boldsymbol{u} = 43, \quad \boldsymbol{\gamma}_1^T\boldsymbol{\gamma}_2 = 1 \quad \text{and} \quad \|\boldsymbol{\gamma}_1\| = \|\boldsymbol{\gamma}_2\| = 2.$$

(a) Show that $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ are linearly independent.

Solution:

One way to see this is that the angle between $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ is different than 0 since:

$$\cos\theta_{\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2} = \frac{\boldsymbol{\gamma}_1^T\boldsymbol{\gamma}_2}{\|\boldsymbol{\gamma}_1\|\|\boldsymbol{\gamma}_2\|} = \frac{1}{4}$$

Alternatively, assume that $c_1\boldsymbol{\gamma}_1 + c_2\boldsymbol{\gamma}_2 = 0$ for some $c_1, c_2 \in \mathbb{R}$. Then:

$$\boldsymbol{\gamma}_1^T(c_1\boldsymbol{\gamma}_1 + c_2\boldsymbol{\gamma}_2) = 4c_1 + c_2 = 0,$$

$$\boldsymbol{\gamma}_2^T(c_1\boldsymbol{\gamma}_1 + c_2\boldsymbol{\gamma}_2) = c_1 + 4c_2 = 0.$$

This is only possible when $c_1 = c_2 = 0$. Why?

(b) Express the projection vector of $\boldsymbol{u}$ onto the subspace $S = \text{span}(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)$ as a linear combination of $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$.

Solution:

Let $\boldsymbol{u} = a\boldsymbol{\gamma_1} + b\boldsymbol{\gamma_2} + \boldsymbol{u}^\perp$ where $a, b \in \mathbb{R}$ and $\boldsymbol{u}^\perp$ is the component of $\boldsymbol{u}$ orthogonal to S.
Then, $\boldsymbol{\gamma_1}^T\boldsymbol{u} = \boldsymbol{\gamma_1}^T(a\boldsymbol{\gamma_1} + b\boldsymbol{\gamma_2} + \boldsymbol{u}^\perp)$ gives $3 = 4a + b$.
Similarly: $\boldsymbol{\gamma_2}^T\boldsymbol{u} = \boldsymbol{\gamma_2}^T(a\boldsymbol{\gamma_1} + b\boldsymbol{\gamma_2} + \boldsymbol{u}^\perp)$ gives $-5 = a + 4b$.
Hence we get $a = \dfrac{17}{15}$ and $b = -\dfrac{23}{15}$.

5. Consider the system of linear equations:

$$\begin{array}{rcrcrcl} & & kx_2 & + & x_3 & = & 1 \\ kx_1 & & & + & x_3 & = & 1 \\ kx_1 & + & kx_2 & + & 2x_3 & = & 2 \end{array}$$

Find values of k for which the system has:
(a) no solution;
(b) a unique solution (if such a k exists, write down the solution);
(c) infinitely many solutions (if such a k exists, write down all possible solutions).

Solution:

Reduce the augmented matrix:

$$\begin{bmatrix} 0 & k & 1 & 1 \\ k & 0 & 1 & 1 \\ k & k & 2 & 2 \end{bmatrix} \to \begin{bmatrix} k & k & 2 & 2 \\ 0 & k & 1 & 1 \\ k & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} k & k & 2 & 2 \\ 0 & k & 1 & 1 \\ 0 & -k & -1 & -1 \end{bmatrix} \to \begin{bmatrix} k & 0 & 1 & 1 \\ 0 & k & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are 2 pivots if $k \neq 0$. If $k = 0$ then the augmented matrix is row equivalent to $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and there is one pivot.

(a) Whether or not k is 0, there is no inconsistency in the system. Therefore the system has always a solution.

(b) The coefficient matrix is singular. Therefore, for every value of k there are infinitely many solutions.

(c) If $k = 0$ then the solutions are $\{(x_1, x_2, 1)\}$. If $k \neq 0$ then the solutions are $\{(\dfrac{x_3 - 1}{k}, \dfrac{x_3 - 1}{k}, x_3)\}$.

6. Consider the subspace $V = \{4x_1 + 3x_2 + x_3 - x_4 = 0\}$ in $\mathbb{R}^4$.
(a) Find a 3×4 matrix $\boldsymbol{A}$ with <u>entries all nonzero</u> such that $\text{null}(\boldsymbol{A}) = V$. What is rank $\boldsymbol{A}$?

Solution:

Observe that for any $(x_1, x_2, x_3, x_4) \in V$, $4x_1 + 3x_2 + x_3 - x_4 = [4\ \ 3\ \ 1\ {-1}] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0.$

Hence the matrix $\boldsymbol{A} = \begin{bmatrix} 4 & 3 & 1 & -1 \\ 4 & 3 & 1 & -1 \\ 4 & 3 & 1 & -1 \end{bmatrix}$ works. rank $\boldsymbol{A} = 1$.

(b) Find a 4×4 matrix $\boldsymbol{B}$ with $\mathrm{col}(\boldsymbol{B}) = V$. What is rank $\boldsymbol{B}$?

Solution:

Observe that $\dim V = 3$. Choose a basis for V; for example, $\{(1,0,0,4),(0,1,0,3),(0,0,1,1)\}$.

Then let $\boldsymbol{B} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 4 & 3 & 1 \end{bmatrix}$. rank $\boldsymbol{B} = 3$.

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Math 201 Matrix Theory

Spring 2002 Final Exam

1. Consider in $\mathbb{R}^3$ the vectors: $u = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, w = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$, and $b = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$.

(a) Determine whether $\{u, v, w\}$ is a linearly independent set.
(b) Determine whether b is in the subspace spanned by $\{u, v, w\}$.

Solution:
(a) Clearly not linearly independent, since $v = 2u + \frac{1}{2}w$.

(b) Consider

$$\begin{bmatrix} A & \vdots & b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & \vdots & 3 \\ 1 & 2 & 0 & \vdots & 4 \\ 0 & 1 & 2 & \vdots & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & \vdots & 3 \\ 0 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 2 & \vdots & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & \vdots & 3 \\ 0 & 1 & 2 & \vdots & 5 \\ 0 & 0 & 0 & \vdots & 1 \end{bmatrix}$$

so $Rank(A \vdots b) = 3$ and $Rank(A) = 2$.
$\Rightarrow b$ is not in the $Columnspace(A)$. Therefore, b is not in the subspace of $\mathbb{R}^3$ which is spanned by $\{u, v, w\}$.

2.(a) Let V be the vector space of 3×3 skew-symmetric, real matrices. Find a basis and the dimension of V.

Solution:

$A \in V \Leftrightarrow A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ where $a, b, c \in \mathbb{R}$.

Then, $A = a \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

$$\Rightarrow \left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\} \text{ is a basis of } V.$$

Hence, $dimV = 3$.

(a) Let $A : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformations such that $A(2,0,0) = (2,8), A(0,1,0) = (2,5)$ and $A(0,0,1) = (3,6)$. Find $A(1,5,-3)$.

Solution:

First, $(1,5,-3) = \frac{1}{2}(2,0,0) + 5(0,1,0) - 3(0,0,1)$.

Then,

$A(1,5,-3) = A\left[\frac{1}{2}(2,0,0) + 5(0,1,0) - 3(0,0,1)\right]$.

$A(1,5,-3) = \frac{1}{2}A(2,0,0) + 5A(0,1,0) - 3A(0,0,1)$ since A is linear.

Therefore, $A(1,5,-3) = (1,4) + (10,25) + (-9,-18) = (2,11)$.

3.(a) Show that if x and y are orthogonal unit vectors in $\mathbb{R}^n$, then $\|x - y\| = \sqrt{2}$.

Solution:

$\|x - y\|^2 = (x-y)^T(x-y) = x^Tx + y^Ty - 2y^Tx$

$y^Tx = 0$,$x^Tx = y^Ty = 1$ since x and y are orthogonal and each of them is a unit vector.

So, $\|x - y\|^2 = 2$. Therefore, $\|x - y\| = \sqrt{2}$.

(b) Let P_1 be the projection matrix that projects every point of $\mathbb{R}^2$ onto the line through $a = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Let P_2 the matrix that projects onto the line perpendicular to a. Determine P_1 and P_2. Compute $P_1 + P_2$ and P_1P_2 and interpret your results.

Solution:

Let b be perpendicular to a: $b^Ta = 0$.

e.g. $b = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. $P_1 = \frac{aa^T}{a^Ta}, P_2 = \frac{bb^T}{b^Tb}$.

Then, $a^Ta = b^Tb = 0$.

$$P_1 = 1/10 \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix} = \begin{bmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{bmatrix}$$

$$P_2 = 1/10 \begin{bmatrix} -3 \\ 1 \end{bmatrix} \begin{bmatrix} -3 & 1 \end{bmatrix} = \begin{bmatrix} 9/10 & -3/10 \\ -3/10 & 1/10 \end{bmatrix}$$

So, $P_1 + P_2 = I$; the sum of the projections onto two perpendicular lines gives the original vector.

$P_1P_2 = 0 \equiv \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ means that projection onto one line and then to the line which is perpendicular to the first one always gives 0.

4.(a) Let A and B be the 3×3 real matrices with the entries:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, B = \begin{bmatrix} a & g & d \\ b & h & e \\ c & i & f \end{bmatrix}.$$

Given that $det A = 5$, find: (i) $det(2A^{-1})$, (ii) $det B$. Justify your answers.

Solution:

(i) $det\,(2A^{-1}) = 2^3\, det A^{-1} = 2^3\,(det)^{-1} = 8/5$

(ii) Interchanging the last two columns of B gives A^T. So, $det\,B = -det\,A^T = -det\,A = -5$.

(b) Let C be the $n \times n$ matrix obtained from the identity matrix I by replacing the jth column of I with a vector x. Let $x^T = (x_1, x_2, \ldots, x_n)$. Find $det\,C$ and justify your answer.

Solution:

$$det\,C = \begin{vmatrix} 1 & & x_1 & & \\ & 1 & \cdot & & \\ & & x_j & & \\ & & \cdot & 1 & \\ & & x_n & & 1 \end{vmatrix} = x_j.$$

Expand with respect to the jth column of $det\,C$. The cofactors of $x_1, x_2, \ldots, x_j, \ldots, x_n$ are all zero because each cofactor has a zero row. The cofactor of x_j is, however,

$$(-1)^2 j det\,(I_{n-1}) = 1.$$

Hence, $det\,C = x_j$.

5.(a) Let $A = (a_{ij})$ be an arbitrary 4×4 matrix. Determine the sign of the term: $a_{14}a_{22}a_{33}a_{41}$ in the determinant of A. Justify your answer.

Solution:

(4,2,3,1) is an odd permutation of (1,2,3,4). Therefore $det\,A$ will contain: $-a_{14}a_{22}a_{33}a_{41}$, i.e. the sign of this term is negative.

(b) The characteristic polynomial of a 5×5, symmetric matrix S is known to be

$$f(\lambda) = -(\lambda - 4)^2(\lambda - 2)(\lambda - 1)^2.$$

Find: (i) $det\,S$, (ii) trace(S), (iii) rank(S), (iv) nullity(S). Is S diagonalizable? Explain.

Solution:

Eigenvalues: $\lambda_1 = \lambda_2 = 4$, $\lambda_3 = 2$, $\lambda_4 = \lambda_5 = 1$.

(i) $det\, S = \lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 = 32$.

(ii) $trace(S) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 12$.

(iii)$rank(S) = 5$; $detS \neq 0$, S is nonsingular.

(iv)$nullity(S) = 0$; N(S) is trivial.

Yes, S is diagonalizable. Every symmetric real matrix is diagonalizable. Since $S = S^T$, the geometric multiplicity of each eigenvalue is the same as its algebraic multiplicity.

6. Consider the Markov process: $u_{k+1} = Au_k$, k=0,1,2,..., where

$$u_k = \begin{bmatrix} y_k \\ z_k \end{bmatrix}, A = \begin{bmatrix} 6/10 & 3/10 \\ 4/10 & 7/10 \end{bmatrix}.$$

Given that $y_0 = 7, z_0 = 14$, determine u_k and find $u_\infty = \lim u_k$.

Solution:

Characteristic equation for A: $\lambda^2 - \frac{13}{10}\lambda + \frac{3}{10} = 0$

So, eigenvalues: $\lambda_1 = 1, \lambda_2 = \frac{3}{10}$.

eigenvectors: $x^{(1)} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, x^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Let $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, A = S\Lambda S^{-1}, A^k = S\Lambda^k S^{-1}$

$S = \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}, S^{-1} = \begin{bmatrix} 1/7 & 1/7 \\ 4/7 & -3/7 \end{bmatrix}, \Lambda^k = \begin{bmatrix} 1 & 0 \\ 0 & \lambda_2^k \end{bmatrix}$.

$u_{k+1} = Au_k \Rightarrow u_k = A^k u_0$ with $u_0 = \begin{bmatrix} 7 \\ 14 \end{bmatrix}$.

$A^k = S\Lambda^k S^{-1} = \frac{1}{7}\begin{bmatrix} 3 + 4\lambda_2^k & 3 - 3\lambda_2^k \\ 4 - 4\lambda_2^k & 4 + 3\lambda_2^k \end{bmatrix}$.

$u_k = A^k u_0 = \begin{bmatrix} 9 - 2\lambda_2^k \\ 12 + 2\lambda_2^k \end{bmatrix}$, i.e. $y_k = 9 - 2\lambda_2^k, z_k = 12 + 2\lambda_2^k$.

$u_\infty = \lim u_k = \begin{bmatrix} 9 \\ 12 \end{bmatrix}$ since $\lim \lambda_2^k = \lim (3/10)^k = 0$.

Notice that u_∞ is an eigenvector for $\lambda_1 = 1, y_k + z_k = y_0 + z_0 = 21$.

B U Department of Mathematics

Math 201 Matrix Theory

Spring 2004 Final Exam

1.) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 4 \end{bmatrix}$.

(a) Find a basis for the left nullspace of A.

Solution:

$$A^T y = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} y \in N(A^T) &\text{ iff } y_1 + 2y_3 = 0 \text{ and } y_2 + 2y_3 = 0 \\ &\text{ iff } y_1 = -2y_3 \text{ and } y_2 = -2y_3 = y_1. \end{aligned}$$

Hence, $\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ is a basis for $N(A^T)$.

(b) Verify that its left nullspace is orthogonal to the column space.

Solution:

$$\begin{bmatrix} 2 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 2 + 0 - 2 = 0.$$

$$\begin{bmatrix} 2 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = 2 + 2 - 4 = 0.$$

Hence, $N(A^T) \perp C(A)$.

2.) Let $A = \begin{bmatrix} 2 & -1 & 2 \\ -6 & 0 & -2 \\ 8 & -1 & 5 \end{bmatrix}$.

(a) Give a LU-decomposition of A.

Solution:

$$\begin{bmatrix} 2 & -1 & 2 \\ -6 & 0 & -2 \\ 8 & -1 & 5 \end{bmatrix} \xrightarrow{3R_1+R_2\to R_2} \begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 8 & -1 & 5 \end{bmatrix}$$

$$\xrightarrow{-4R_1+R_3\to R_3} \begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 3 & -3 \end{bmatrix}$$

$$\xrightarrow{R_2+R_3\to R_3} \begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

So, $U = \begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$, $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E_1E_2E_3A = U \Rightarrow A = E_3^{-1}E_2^{-1}E_1^{-1}U.$
Now, say, $E_3^{-1}E_2^{-1}E_1^{-1} = L.$

$$L = E_3^{-1}E_2^{-1}E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Using this decomposition, solve $Ax = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$.

Solution:

$Ax = b$ and $A = LU \Rightarrow LUx = b.$
Say, $Ux = y$, then
$Ly = b.$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \Rightarrow y_1 = 1,\ -3y_1 + y_2 = 0,\ 4y_1 - y_2 + y_3 = 4.$$

$$\Rightarrow y = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}.$$

We have $Ux = y$.

$$\Rightarrow \begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} \Rightarrow 2x_1 - x_2 + 2x_3 = 1,\ -3x_2 + 4x_3 = 3,\ x_3 = 3.$$

$$\Rightarrow x = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}.$$

(c) Find the rank of A.

Solution:

$Rank(A) = 3.$

3.) Let V be a vector space of $2x2$ matrices over $\mathbb{R}$, and let $T : V \to V$ be a linear transformation defined by $T(A) = MA$ where $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Let B be the matrix of T in the standard basis.

(a) Find the matrix B.

Solution:

Standard basis of V:

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$T(E_1) = ME_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = 1E_1 + 0E_2 + 3E_3 - 0E_4,$$

$$T(E_2) = ME_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} = 0E_1 + 1E_2 + 0E_3 + 3E_4,$$

$$T(E_3) = ME_3 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix} = 2E_1 + 0E_2 + 4E_3 + 0E_4,$$

$$T(E_4) = ME_4 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix} = 0E_1 + 2E_2 + 0E_3 + 4E_4,$$

$$B = [\ T(E_1)\ \ T(E_2)\ \ T(E_3)\ \ T(E_4)\ ] = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}.$$

(b) Find the trace of B.

Solution:

$$tr(B) = b_{11} + b_{22} + b_{33} + b_{44} = 1 + 1 + 4 + 4 = 10.$$

(c) Is T invertible? Justify your answer.

Solution:

$$B = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix} \begin{matrix} -3R_1{+}R_3{\to}R_3 \\ -3R_2{+}R_4{\to}R_4 \end{matrix} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$\Rightarrow detB = 1.1.(-2).(-2) = 4 \neq 0.$
$\Rightarrow B$ is invertible. Hence, T is invertible.

4.) Solve the system of differential equations $\frac{du}{dt} = Au$ where $A = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix}$ with the initial condition $u_0 = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$.

Solution:

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = (3-\lambda)\ .\ (2-\lambda) - 3\ .\ 4 = 0$$
$$\Rightarrow 6 - 3\lambda - 2\lambda + \lambda^2 - 12 = 0$$
$$\Rightarrow \lambda^2 - 5\lambda - 6 = 0$$
$$\Rightarrow (\lambda - 6)\ .\ (\lambda + 1) = 0$$
$$\Rightarrow \lambda_1 = 6, \lambda_2 = -1 \text{ are the eigenvalues.}$$

for $\lambda_1 = 6 : (A - 6I)x = 0$

$$\Leftrightarrow A - 6I = \begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix} R_1{+}R_2{\to}R_2 \begin{bmatrix} -3 & 4 \\ 0 & 0 \end{bmatrix} \Rightarrow -3x_1 + 4x_2 = 0 \Rightarrow x_1 = \frac{4}{3}x_2$$

$$\Rightarrow p_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

for $\lambda_1 = -1 : (A + I)x = 0$

$$\Leftrightarrow A + I = \begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix} \tfrac{R_1}{4}{\to}R_1 \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$$
$$-3R_1{+}R_2{\to}R_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$$

$$\Rightarrow p_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$$\Rightarrow S = \begin{bmatrix} 4 & 1 \\ 3 & -1 \end{bmatrix}, A = \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow e^{At} = \begin{bmatrix} e^{6t} & 4 \\ 0 & e^{-t} \end{bmatrix}$$

$$\Rightarrow A = S\Lambda S^{-1} = \begin{bmatrix} 4 & 1 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{3}{7} & -\frac{4}{7} \end{bmatrix}$$

$$\Rightarrow u = e^{At}u_0 = Se^{At}S^{-1}u_0 = \begin{bmatrix} 4 & 1 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} e^{6t} & 4 \\ 0 & e^{-t} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{3}{7} & -\frac{4}{7} \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4e^{6t} + 2e^{-t} \\ 3e^{6t} - 2e^{-t} \end{bmatrix} \qquad \text{OR}$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = c_1 e^{\lambda_1 t} p_1 + c_2 e^{\lambda_2 t} p_2, \quad S^{-1}u_0 = c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$S^{-1}u_0 = \begin{bmatrix} \frac{1}{7} & \frac{1}{7} \\ \frac{3}{7} & -\frac{4}{7} \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow u = e^{6t} \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} + 2e^{-t} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Hence, $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 4e^{6t} + 2e^{-t} \\ 3e^{6t} - 2e^{-t} \end{bmatrix}$.

5.) Let A be a $3x3$ matrix whose eigenvalues are -3, 4, and 4, with associated eigenvectors $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, respectively.

(a) Diagonalize the matrix A.

Solution:

$$S = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$|S| = (-1) \,.\, (-1) = 1 \neq 0$, so, S is invertible.

$$[\ S \mid I\ ] = \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1+R_3 \to R_3} \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3 \to R_3} \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

Then, $S^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

$$\Rightarrow A = S\Lambda S^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} . \begin{bmatrix} -3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} . \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(b) Find A^{15} (Leave it as a product).

Solution:

$$A^{15} = S\Lambda^{15}S^{-1} = S.\begin{bmatrix} -3^{15} & 0 & 0 \\ 0 & 4^{15} & 0 \\ 0 & 0 & 4^{15} \end{bmatrix}.S^{-1}$$

6.) Let A be a square matrix. Assume that $\lambda_1 = 2$ is a quadruple eigenvalue with three linearly independent eigenvectors and $\lambda_2 = 3$ is a triple eigenvalue with 2 linearly independent eigenvectors.

(a) Write down the chraracteristic polynomial of A.

Solution:

$$p(A) = (\lambda - 2)^4 \ . \ (\lambda - 3)^3$$

(b) Give a Jordan form J of A.

Solution:

$J = [\quad]_{7x7}$ and it has $2 + 3 = 5$ blocks.

$$J = \begin{bmatrix} 2 & 1 & & & & & \\ 0 & 2 & & & & & \\ & & 2 & & & & \\ & & & 2 & & & \\ & & & & 3 & 1 & \\ & & & & 0 & 3 & \\ & & & & & & 3 \end{bmatrix} \text{ with blocks } \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, [2], [2], \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \text{ and } [3].$$

(c) Is A invertible? Justify your answer.

Solution:

$det(A) = \lambda_1.\lambda_2...\lambda_7 = 2^4.3^3 = 432 \neq 0$. So, A is invertible.

B U Department of Mathematics

Math 201 Matrix Theory

Spring 2005 Final Exam

1.) Consider the set of 2×2, symmetric matrices.
a) Show that this is a subspace of all 2×2 matrices.
b) Write a basis for this space and find its dimension.
c) Consider the set of $n \times n$ symmetric matrices. What is the dimension of this space? (You should explain your answer.)

Solution:

a) Let $S_{2\times 2}$ denote the set of 2×2 symmetric matrices. Let $\begin{bmatrix} a_1 & b_1 \\ b_1 & d_1 \end{bmatrix}$ and $\begin{bmatrix} a_2 & b_2 \\ b_2 & d_2 \end{bmatrix}$ be any two elements of $S_{2\times 2}$. Then, for $c \in \mathbb{R}$, we have

$$\begin{bmatrix} a_1 & b_1 \\ b_1 & d_1 \end{bmatrix} + c \begin{bmatrix} a_2 & b_2 \\ b_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + ca_2 & b_1 + cb_2 \\ b_1 + cb_2 & d_1 + cd_2 \end{bmatrix} \in S_{2\times 2}.$$

Hence $S_{2\times 2}$ is a subspace of all 2×2 matrices.

b) Clearly, $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ is a basis for $S_{2\times 2}$ as its a linearly independent set spanning $S_{2\times 2}$. Thus, dimension of a vector space being equal to the number of elements in a basis, we get that $\dim S_{2\times 2} = 3$.

c)It is clear that to find the number of basis elements of an $n \times n$ symmetric matrix we need to count the number elements on the upper triangular part including the diagonal. (This is because for a symmetric matrix elements below the diagonal are determined by the elements above the diagonal.) This is given by the sum $n + \frac{(n-1)n}{2}$. Hence

$$\dim S_{n\times n} = n + \frac{(n-1)n}{2} = \frac{n(n+1)}{2}.$$

2.) a) Find the QR decomposition of $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 1 \end{bmatrix}$. (Constructions of matrices Q and R should be shown explicitly.)

b) Use the above QR decomposition to solve the least squares problem $Ax = b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution:

a) To find the QR decomposition of A we will apply Gram-Schmidt orthonormalization method to the column vectors of A, namely, $a_1 = (1, 2, 2)$ and $a_2 = (1, 3, 1)$.

Taking $a_1' = (1, 2, 2)$ with $\|a_1'\| = 3$ yields

$$q_1 = \frac{a_1'}{\|a_1'\|} = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right).$$

Now, $a_2' = a_2 - (q_1^T a_2)q_1 = (0, 1, -1)$ with $\|a_2'\| = \sqrt{2}$ implies

$$q_2 = \frac{a_2'}{\|a_2'\|} = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

Thus

$$Q = \begin{bmatrix} 1/3 & 0 \\ 2/3 & 1/\sqrt{2} \\ 2/3 & -1/\sqrt{2} \end{bmatrix}, \quad R = \begin{bmatrix} q_1^T a_1 & q_1^T a_2 \\ 0 & q_2^T a_2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix}.$$

b) Since Q is orthogonal $Q^T Q = I$ and $A^T A = R^T Q^T Q R = R^T R$, hence, $A^T A\bar{x} = A^T b$ implies $R\bar{x} = Q^T b$. Since

$$Q^T b = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 0 \end{bmatrix}$$

from $R\bar{x} = Q^T b$, setting $\bar{x} = \begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix}$, we get that

$$\begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix} = \begin{bmatrix} 5/3 \\ 0 \end{bmatrix}.$$

Thus $\bar{u} = \frac{5}{9}$ and $\bar{v} = 0$.

3.) a) Find the unknown x using Cramer's rule:

$$\begin{aligned} -2x + 3y - z &= 1 \\ x + 2y - z &= 4 \\ -2x - y + z &= -3 \end{aligned}$$

Solution:

Cramer's rule implies that x is given by

$$x = \frac{\det B_1}{\det A},$$

where

$$B_1 = \begin{bmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{bmatrix}.$$

Hence calculating $\det B_1$ and $\det A$ as -4 and -2, respectively yields

$$x = \frac{\det B_1}{\det A} = \frac{-4}{-2} = 2.$$

b) Find all possible invertible matrices A, if $(A_{cof})^{-1} = (adj A)^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 5 \end{bmatrix}$.

Solution:

Since $A^{-1} = \frac{1}{\det A} A_{cof}$, we get that $AA_{cof} = \det AI$ and hence

$$A = \det(A)(A_{cof})^{-1}.$$

Noting that A is 3×3, $\det A \in \mathbb{R}$ and taking the determinant of both sides in the above equation entails

$$\det A = (\det A)^3 \det((A_{cof})^{-1}).$$

Since we can calculate the determinant of $(A_{cof})^{-1}$ as $\det((A_{cof})^{-1}) = 4$ we have

$$(\det A)^2 = \frac{1}{4},$$

i.e.,

$$\det A = \pm\frac{1}{2}.$$

Thus,

$$A = \det(A)(A_{cof})^{-1} = \pm\frac{1}{2}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 5 \end{bmatrix}$$

4.) a) What is the relationship, if any, between rank$(A + B)$, rank(B) and rank(A). Explain.

Solution:

The relationship between rank$(A + B)$, rank(B) and rank(A) can be given as

$$\text{rank}(A + B) \leqslant \text{rank}(A) + \text{rank}(B),$$

since adding A and B some linearly independent columns (or rows) may cancel each other (for example when $B = -A$). Therefore the rank of $A + B$ may be less than the sum of individual ranks. To prove the above statement more formally note that if V and W are subspaces of a vector space, then $\dim(V + W) + \dim(V \cap W) = \dim V + \dim W$. Now, $\text{rank}(A+B) = \dim(\text{Column}(A)+\text{Column}(B)) \leqslant \dim(\text{Column}(A))+\dim(\text{Column}(B)) = \text{rank}(A) + \text{rank}(B)$.

b) Let A be a matrix with linearly independent columns. Write down the projection matrix onto the **row space** of a matrix A. (You should give enough explanation.)

Solution:

Since the columns of A are linearly independent, A^TA is invertible, hence, so is AA^T. We know that the projection matrix onto the column space of a matrix A is given by

$$P = A(A^TA)^{-1}A^T.$$

Now, changing A to A^T we get that

$$P' = A^T(AA^T)^{-1}A$$

which is the matrix projecting onto the column space of A^T,i.e., onto the **row space** of A.

c) [5] Let $A^T = -A$. Is the matrix $M = (I - A)(I + A)^{-1}$ orthogonal?

Solution:

$$
\begin{aligned}
M^T M &= ((I-A)(I+A)^{-1})^T (I-A)(I+A)^{-1} \\
&= ((I+A)^{-1})^T (I-A)^T (I-A)(I+A)^{-1} \\
&= ((I+A)^T)^{-1} (I-A)^T (I-A)(I+A)^{-1} \\
&= (I-A)^{-1}(I+A)(I-A)(I+A)^{-1} \\
&= (I-A)^{-1}(I-A^2)(I+A)^{-1} \\
&= (I-A)^{-1}(I-A)(I+A)(I+A)^{-1} \\
&= I.
\end{aligned}
$$

Hence M is an orthogonal matrix.

5.) Consider the system of recurrence relations:(n=0,1,2,.....)

$$
\begin{aligned}
x_{n+1} &= 3x_n - y_n \\
y_{n+1} &= -x_n + 3y_n
\end{aligned}
$$

with initial values $x_0 = 1$ and $y_0 = 2$. Find x_n and y_n as functions of n. (You should first convert the problem into matrix notation and then use diagonalization method.)

Solution:

Converting the problem into matrix notation yields

$$
\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}.
$$

Hence x_n and y_n are given by

$$
\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}^n \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
$$

Now, to diagonalize the coefficient matrix we have to find eigenvalues and eigenvectors of it. Setting

$$
\begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = 0
$$

we have $\lambda_1 = 2$, $\lambda_2 = 4$ and corresponding eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, respectively. So,

$$
\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1},
$$

and hence

$$
\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}^n = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 4^n \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}.
$$

Thus multiplying by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ we get that

$$
\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 3 \cdot 2^{n-1} - 2^{2n-1} \\ 3 \cdot 2^{n-1} + 2^{2n-1} \end{bmatrix}
$$

6.) a) Let $(A - \lambda I)$ and B be similar matrices. Can A and $(B + \lambda I)$ also be similar?

Solution:

Since $(A - \lambda I)$ and B are similar, there exists a matrix C such that

$$(A - \lambda I) = C^{-1}BC$$

From this we see that

$$\begin{aligned} A &= \lambda I + C^{-1}BC \\ &= C^{-1}\lambda IC + C^{-1}BC \\ &= C^{-1}(B + \lambda I)C \end{aligned}$$

Thus A and $(B + \lambda I)$ are similar.

b) Let Q be a real, orthogonal matrix with real eigenvalues. What are the possible eigenvalues? Explain.

Solution:

Eigenvalues of Q satisfy

$$Qx = \lambda x.$$

Taking the norm of both sides yields

$$\|Qx\| = |\lambda|\|x\|.$$

But, since $\|Qx\| = \|x\|$, we get that $\|x\| = |\lambda|\|x\|$,i.e., $|\lambda| = 1$. Thus

$$\lambda = \pm 1.$$

c) Let A be a 5×5 matrix that has 2 as an eigenvalue with order 5. If there are 2 linearly independent eigenvectors, what are the possible Jordan forms?

Solution:

Since there are 2 eigenvectors, the Jordan form contains two Jordan blocks. Note that 5 can be written as a sum of two numbers in two different ways $5 = 3 + 2$ or $5 = 4 + 1$. With decreasing sizes we can write two possible Jordan forms, namely,

$$\begin{bmatrix} \begin{bmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{bmatrix} & \\ & \begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} \end{bmatrix} \text{ and } \begin{bmatrix} \begin{bmatrix} 2 & 1 & & \\ & 2 & 1 & \\ & & 2 & 1 \\ & & & 2 \end{bmatrix} & \\ & \begin{bmatrix} 2 \end{bmatrix} \end{bmatrix}.$$

B U Department of Mathematics

Math 201 Matrix Theory

Spring 2006 Final Exam

1. Prove: If the entries in each row of an nxn matrix A add up to zero, then $det(A) = 0$. (Hint: Consider the product AX where X is an nx1 matrix, each of whose entries is one) (20 points)

Solution:

For $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ and $X = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

$AX = \begin{bmatrix} a_{11} + a_{12} + \cdots + a_{1n} \\ a_{21} + a_{22} + \cdots + a_{2n} \\ \vdots \\ a_{n1} + a_{n2} + \cdots + a_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ (Since the entries in each row of A add up to zero)

So $X = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \neq 0$ is a solution of $AX = 0 \Rightarrow det(A) = 0$.

(Recall: If $det(A) \neq 0$, then $AX = 0$ has only the trivial solution, $X = 0$).

2. Find the equation of the best line through the points (-1,-2), (0,0), (1,1) and (2,3). (25 points)

Solution:

Let $y = C + Dt$ be the best line fitting the given data.

Then for $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$, $b = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 3 \end{bmatrix}$ and $X = \begin{bmatrix} C \\ D \end{bmatrix}$

$(A^T A)x = A^T b$ must hold.

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$$

$$\Rightarrow \begin{array}{l} 4C + 2D = 2 \\ \underline{2C + 6D = 9} \end{array}$$

$\Rightarrow D = 1.6, C = -0.3$
Therefore $y = -0.3 + 1.6t$.

3. Decide whether the followings are TRUE or FALSE. If true prove; if false, give a counter example or explain. (32 points)

i. If 1 and 2 are eigenvalues of a 2x2 matrix A and $f(x) = x^2 - 1$ is a polynomial then the matrix f(A) is invertible.

FALSE: 1 is an eigenvalue of A. So $f(1) = 0$ is an eigenvalue of f(A). Hence f(A) is not invertible.

ii. If W is the set of all differentiable functions on [0,1] whose derivative is $\frac{2}{x+1}$ then W is a subspace of $C[0, 1]$, the vector space of continuous real functions on [0,1].

FALSE: 0 is not an element of W since $0' = 0 \neq \frac{2}{x+1}$

iii. If A is a skew-symmetric matrix then $det(A) = 0$.

FALSE: $A^T = -A \Rightarrow det(A^T) = det(-A)$
If A is an nxn matrix $\Rightarrow det(-A) = (-1)^n det(A)$
Also $det(A) = (-1)^n det(A)$
Hence $det(A) = 0$ if n is odd.

iv. If A is a 2x2 matrix with $Tr(A) = 5$ and $det(A) = 4$ then 1 and 4 are eigenvalues of A.

TRUE: The characteristic polynomial of A:
$p(\lambda) = \lambda^2 - Tr(A)\lambda + detA$
$= \lambda^2 - 5\lambda + 4$
$= (\lambda - 4)(\lambda - 1)$
So $p(\lambda) = 0$ if $\lambda = 1$ or 4.

4. Prove that if A is a nilpotent matrix then 0 is the only eigenvalue of A. (15 points)
Solution:
For some $k \in \mathbb{Z}^+$, $A^k = 0$ and $A^{k-1} \neq 0$.
If λ is an eigenvalue of A with associated eigenvector x, then λ^k is an eigenvalue of A^k with associated eigenvector x.
i.e. $A^k x = \lambda^k x$ where $A^k = 0$
$\Rightarrow \lambda^k x = 0 \Rightarrow \lambda^k = 0$ since $x \neq 0$ (an eigenvector)
Hence $\lambda = 0$.

5. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and A be the matrix of this transformation. Complete the following equivalent statements. (18 points)

i. A is invertible.

ii. $Ax = 0$ has $\underline{\text{only the trivial}}$ solution.

iii. For every vector $b \in \mathbb{R}^n$, $Ax = b$ is $\underline{\text{consistent}}$.

iv. The range of T is $\underline{\mathbb{R}^n}$.

v. The column (or the row) vectors of A are $\underline{\text{linearly independent}}$.

vi. Nullity of A is $\underline{0}$.

vii. The orthogonal complement of the row space of A is $\underline{\{0\}}$.

6. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation $T(x, y, z) = (x, x + ay, 3x + 6y + bz)$, where a and b are real numbers;
i. Determine all possible values of a and b so that T has an inverse transformation T^{-1}.
ii. Give $T^{-1}(x, y, z)$ in terms of a and b.
Solution:
i. For $\{e_1, e_2, e_3\}$ the standart basis of $\mathbb{R}^3$, the matrix A of T:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & a & 0 \\ 3 & 6 & b \end{bmatrix}$$

T has inverse if $|A| \neq 0$ i.e. if $ab \neq 0$

ii. The matrix of T^{-1} is A^{-1}:

$$A^{-1} = \frac{1}{detA} A_{cof} = \frac{1}{detA} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \frac{1}{ab} \begin{bmatrix} ab & 0 & 0 \\ -b & b & 0 \\ 6-3a & -6 & a \end{bmatrix}$$

Since $T^{-1}(x, y, z) = A^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$T^{-1}(x, y, z) = \frac{1}{ab}((ab)x, b(-x + y), (6 - 3a)x - 6y + az)$

7. Find an ON basis for $\mathbb{C}^2$ (the vector space of complex column-2 vectors), consisting eigenvectors of the matrix $A = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$. (25 points)

Solution: $A^H = A \Rightarrow$ A is Hermitian,

$p(\lambda) = \lambda^2 - Tr(A)\lambda + det(A) = \lambda^2 - 3\lambda + 0 = \lambda(\lambda - 3)$

so eigenvalues: $\lambda_1 = 0$, $\lambda_2 = 3$.

For $\lambda_1 = 0$:

$$Ax = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

if $a = -(1+i)b$ then take $x_1 = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$ as an eigenvector for $\lambda_1 = 0$.

For $\lambda_2 = 3$:

$$(A - 3I)x = \begin{bmatrix} -2 & 1+i \\ 1-i & -1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

if $c = (\frac{1+i}{2})d$, then take $x_2 = \begin{bmatrix} \frac{1+i}{2} \\ 1 \end{bmatrix}$ as an eigenvalue for $\lambda_2 = 3$.

Here $x_1 \perp x_2$ (eigenvectors of a Hermitian matrix corresponding to distinct eigenvalues are orthogonal).

$\|x_1\|^2 = x_1^H x_1 = 3$ then take $u_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$

$\|x_2\|^2 = x_2^H x_2 = \frac{3}{2}$ then take $u_2 = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1+i}{2} \\ 1 \end{bmatrix}$

Then $\{u_1, u_2\}$ is an ON basis of $\mathbb{C}^2$.

8. A certain football team derives confidence from each win but gets demoralized after each loss. After winning a game, it has 90% chance of winning the next game, but after loosing a game it has 20% chance winning the next game. In the long run what fraction of the games will this team win? (Hint: Form a Markov matrix $A = [a_{ij}]$ so that a_{ij} is the probability of being in state i in the next game given that the team is in state j after the last game. Also recall that the steady state solution is an eigenvector corresponding to a special eigenvalue, so omit the unnecessary calculations.) (20 points)

Solution: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$

$p(\lambda) = det(A - \lambda I) = (\frac{9}{10} - \lambda)(\frac{8}{10} - \lambda) - \frac{2}{10} \cdot \frac{1}{10} = 0$

if $(9 - 10\lambda)(8 - 10\lambda) - 2 = 0$

if $100\lambda^2 - 170\lambda + 70 = 0$

if $(10\lambda - 7)(\lambda - 1) = 0$, then eigenvalues will be 1 and $\frac{7}{10}$.

Let y_0: initial state of win, z_0: initial state of loss and $u_0 = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$

Then $\begin{matrix} y_1 = 0.9y_0 + 0.2z_0 \\ z_1 = 0.1y_0 + 0.8z_0 \end{matrix}$.

Hence $u_1 = \begin{bmatrix} y_1 \\ z_1 \end{bmatrix} = A \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$.

Inductively, $u_k = Au_{k-1}$, $k = 1, 2, ..$ if $u_k = \begin{bmatrix} y_k \\ z_k \end{bmatrix}$

Now remark that A is a Markov matrix and $\lambda_1 = 1$ is an eigenvalue of A. So for the corresponding eigenvector x_1 we have $Ax_1 = x_1$ and x_1 is the steady state solution.

For $\lambda = 1$:
$(A - \lambda I)x = (A - I)x = \begin{bmatrix} 0.9 - 1 & 0.2 \\ 0.1 & 0.8 - 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

if $0.1a - 0.2b = 0 \Leftrightarrow a = 2b$ then $x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector.

Hence the team will win $\frac{2}{3}$ of its games in the long run, since the chance of winning is 2 times the chance of loosing.

9. Find the general solution of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$.

Solution:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \end{bmatrix}$$

So for the homogeneous system $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

we have $y = 0, x + z = 0$

Hence $x_h = z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $z \in \mathbb{R}$ (Solutions to the associated homogeneous equation)

For $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

$$\begin{array}{c} x + y + z = 0 \\ x - y + z = -2 \\ \hline x + z = -1 \end{array}$$

Hence for $y = 1$, $z = -1$, $x = 0$, $x_p = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ is a particular solution of the given equation.

Then the general solution:
$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, z \in \mathbb{R}.$$

B U Department of Mathematics

Math 201 Matrix Theory

Summer 2003 Final Exam

1. For what values of a does the system :

$$ax + y = 1$$
$$4x + ay = 2$$

have (i) a unique solution (ii) infinitely many solutions (iii) no solution? Find also the rank of the coefficient matrix in each case.

Solution:

To have a unique solution the coefficient matrix A= $\begin{pmatrix} a & 1 \\ 4 & a \end{pmatrix}$ must be non-singular since the system is square.

$detA = a^2 - 4 = 0 \Rightarrow a = 2, a = -2$

(i) unique solution if $a \neq \pm 2$, which is :

$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

In this case $rankA = 2$

(ii) if $a = 2$:

$$2x + 1 = 1$$
$$4x + 2y = 2$$

Aug $= \left(\begin{array}{cc|c} 2 & 1 & 1 \\ 4 & 2 & 2 \end{array}\right) - - - > \left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$

$2x + y = 1$. One equation two unknowns $\Rightarrow$infinitely many solutions.

In this case $rankA = 1$

(iii) if $a = -2$: Aug $= \left(\begin{array}{cc|c} -2 & 1 & 1 \\ 4 & -2 & 2 \end{array}\right) - - - > \left(\begin{array}{cc|c} -2 & 1 & 1 \\ 0 & 0 & 4 \end{array}\right)$. Inconsistent and hence no solution.

In this case $rankA = 1$

2. Let A= $\begin{pmatrix} 0 & 0 & 1 \\ 3 & 2 & 2 \\ 4 & 1 & 3 \end{pmatrix}$.

(a) Find QR-decomposition of A, where Q is an orthogonal matrix.

Solution:

Q is the orthogonal matrix obtained from A by Gram-Schmidt process
$\alpha_1 =< 0, 3, 4 >$, $\alpha_2 =< 0, 2, 1 >$, $\alpha_3 =< 1, 2, 3 >$ labelling the columns
$x_1 = \alpha_1$
$x_2 = \alpha_2 - \frac{\alpha_2^T x_1}{||x_1||^2} x_1 =< 0, \frac{4}{5}, \frac{-3}{5} >$
$x_3 = \alpha_3 - \frac{\alpha_3^T x_1}{||x_1||^2} x_1 - \frac{\alpha_3^T x_2}{||x_2||^2} x_2 =< 1, 0, 0 >$
x_1, x_2, x_3 is an orthogonal set. Then

$$q_1 = \frac{x_1}{||x_1||} =< 0, \frac{3}{5}, \frac{4}{5} >$$

$$q_2 = \frac{x_2}{||x_2||} =< 0, \frac{4}{5}, \frac{-3}{5} >$$

$$q_3 = \frac{x_3}{||x_3||} =< 1, 0, 0 >$$

form an orthonormal set.

$$Q = [q_1|q_2|q_3] = \begin{pmatrix} 0 & 0 & 1 \\ 3/5 & 4/5 & 0 \\ 4/5 & -3/5 & 0 \end{pmatrix}.$$

$$R = \begin{pmatrix} q_1^T\alpha_1 & q_1^T\alpha_2 & q_1^T\alpha_3 \\ & q_2^T\alpha_2 & q_2^T\alpha_3 \\ & & q_3^T\alpha_3 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 18/5 \\ 0 & 1 & -1/5 \\ 0 & 0 & 1 \end{pmatrix}.$$

So that $A = QR$.

(b) Find the inverse of Q, if it exists.

Solution:

Since Q is orthogonal, we have $Q^TQ = QQ^T = I$. Hence $Q^{-1} = Q^T$.

3. (a) Show that if A is similar to B, then A^k is similar to B^k

Solution:

Given that there exists an invertible M such that $M^{-1}AM = B$ or $A = MBM^{-1}$, compute A^k

$A^k = (MBM^{-1})(MBM^{-1})...(MBM^{-1})[ktimes] \Rightarrow A^k = MB^kM^{-1}$.
Hence A^k is similar to B^k , via the same matrix M.

(b) Let A be an $m \times n$ matrix. Prove that if $\text{tr}(A^TA) = 0$ then $A = 0$.

Solution:

Let $A = [q_1|q_2|...|q_3]_{mxn}$

Hence, $A^TA = \begin{pmatrix} q_1^Tq_1 & q_1^Tq_2 & ... & q_1^Tq_n \\ q_2^Tq_1 & q_2^Tq_2 & ... & q_2^Tq_n \\ \vdots & & & \vdots \\ q_n^Tq_1 & q_n^Tq_2 & ... & q_n^Tq_n \end{pmatrix} \Rightarrow (A^TA)_{ii} = q_i^Tq_i = ||q_i||^2$

$tr(A^TA) = q_1^Tq_1 + q_2^Tq_2 + ... + q_n^Tq_n = ||q_1||^2 + ... + ||q_n||^2$

$tr(A^TA) = 0 \Rightarrow q_i = 0$ for every $i = 1, 2, ..., n$
So each column of A is zero. Therefore $A = 0$.

4. The matrix $A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 0 & 6 \\ 0 & 3 & -7 \end{pmatrix}$ has an eigenvector $v_1 = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$. It is also known that $\lambda = 2$ is an eigenvalue of A.

(a) Using the information, diagonalize A.

Solution:

If v_1 is an eigenvector, then there is an eigenvalue λ such that

$$Av_1 = \lambda v_1 \Rightarrow \begin{pmatrix} 2 & -1 & 3 \\ 0 & 0 & 6 \\ 0 & 3 & -7 \end{pmatrix}\begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} = \lambda \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 9 \\ 18 \\ -27 \end{pmatrix} = \lambda \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$$

Therefore another eigenvalue is $\lambda = -9$, with an eigenvector v_1

$\lambda = 2$ is an eigenvalue(given). Let us find the eigenvector(s) for $\lambda = 2$:

$$A - 2I = \begin{pmatrix} 0 & -1 & 3 \\ 0 & -2 & 6 \\ 0 & 3 & -9 \end{pmatrix} - - - > \begin{pmatrix} 0 & -1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow -x_2 + 3x_3 = 0$$

x_1 is free.$\Rightarrow (A - 2I)x = 0$ is satisfied if $x = \begin{pmatrix} x_1 \\ 3x_3 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$

There are two eigenvectors for $\lambda = 2$. Taking these two eigenvectors to be :
$v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $v_3 \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$, We can diagonalize A

$S^{-1}AS = \Lambda$ where $S = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ -2 & 0 & 3 \\ 3 & 0 & 1 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} -9 & & \\ & 2 & \\ & & 2 \end{pmatrix}$

(b) Find A^{2003} . (Leave it as a product.)

Solution:

$A = S\Lambda S^{-1} \Rightarrow A^{2003} = S\Lambda^{2003}S^{-1}$

$$\Rightarrow A^{2003} = \begin{pmatrix} -1 & 1 & 0 \\ -2 & 0 & 3 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} -9^{2003} & & \\ & 2^{2003} & \\ & & 2^{2003} \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ -2 & 0 & 3 \\ 3 & 0 & 1 \end{pmatrix}^{-1}$$

5. Let $T : R^2 -> R^2$ be a linear transformation satisfying $T(<1,0>) =< -4,3 >$ and $T(<1,1>) =< -10,8>$. Let A be a matrix of T in the standart basis.

(a) Find A.

Solution:

We need $T(<0,1>)$. But $<0,1>=<1,1> - <1,0>$
$\Rightarrow T(<0,1>) = T(<1,1>) - T(<1,0>)$
$\Rightarrow T(<0,1>) =< -10,8> - < -4,3>=< -6,5>$
$\Rightarrow A = [Te_1|Te_2] = \begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix}$

(b) What is the matrix B representing T in the basis that consist of eigenvectors of A ?

Solution:

Eigenvalues of $A : |A - \lambda I| = \begin{vmatrix} -4-\lambda & -6 \\ 3 & 5-\lambda \end{vmatrix} = (5-\lambda)(-4-\lambda) + 18 = 0$
$\Rightarrow \lambda = 2, \lambda = -1$

$\lambda = 2 : A - 2\lambda = \begin{pmatrix} -6 & -6 \\ 3 & 3 \end{pmatrix} \Rightarrow x_1 = -x_2 \Rightarrow$ an eigenvector is $p_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda = -1 : A + \lambda = \begin{pmatrix} -3 & -6 \\ 3 & 6 \end{pmatrix} \Rightarrow x_1 = -2x_2 \Rightarrow$ an eigenvector is $p_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$T(p_1) = \begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} = 2p_1 + 0p_2$

$T(p_2) = \begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = 0p_1 - 1p_2$

$B = [Tp_1|Tp_2] = \begin{pmatrix} 2 & 2 \\ -2 & -1 \end{pmatrix}$

(c) Solve the system of differential equations $\frac{du}{dt} = Au$ with the initial condition $u_0 = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

Solution:

$\frac{du}{dt} = Au$ has the general solution : $u = Se^{\Lambda t}S^{-1}u_0$

$S = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}$ (found above, eigenvectors of A.) $\Lambda = \begin{pmatrix} 2 & \\ & -1 \end{pmatrix} \Rightarrow e^{\Lambda t} = \begin{pmatrix} e^{2t} & \\ & e^{-t} \end{pmatrix}$

$$S^{-1} = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}, \; c = S^{-1}u_0 = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$\Rightarrow u = c_1 e^{\lambda_1 t} p_1 + c_2 e^{\lambda_2 t} p_2$

$u = -e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} - e^{-t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$$u_1 = -e^{2t} + 2e^{-t}$$
$$u_2 = +e^{2t} - e^{-t}$$

6. Let A be square matrix with eigenvalues $\lambda_1 = 2, \lambda_2 = 1$ and $\lambda_3 = 5$ with multiplicities 3, 2 and 2 respectively. Let E_i be the eigenspace associated with the eigenvalue λ_i, $i = 1, 2, 3$. Assume that $dimE_1 = 2, dimE_2 = 1$ and $dimE_3 = 2$

(a) Is A diagonalizable? Explain.

Solution:

Counting the multiplicities, we understand that A is $7x7$.
$dimE_1 + dimE_2 + dimE_3 = 2 + 1 + 2 = 5 \neq 7$. Hence A is not diagonalizable.

(b) Give a Jordan form J of A.

Solution:

$\lambda = (0), (0, 0)$ 2 eigenvectors.
$\lambda = (1, 1)$ 1 eigenvector.
$\lambda = (5), (5)$ 2 eigenvectors.
J contains 5 blocks.

$$J = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} & & & & \\ & [0] & & & \\ & & \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} & & \\ & & & [5] & \\ & & & & [5] \end{pmatrix}_{7x7}$$

(c) Write down the characteristic polynomial of A.

Solution:

Using the fact that eigenvalues are roots of the characteristic polynomial
$p(\lambda) = -\lambda^3(\lambda - 1)^2(\lambda - 5)^2$ (the front minus sign comes from the order of A)

(d) Is A invertible?Justify your answer fully.

Solution:

To be invertible, $detA \neq 0$ has to be satisfied. But A has $\lambda = 0$ as an eigenvalue and $detA = \lambda_1 \lambda_2 ... \lambda_7$ which yields that $detA = 0$. The answer is no.

(e) Find the trace of A.

Solution:

$$trA = \lambda_1 + \lambda_2 + ... + \lambda_7 = 0 + 0 + 0 + 1 + 1 + 5 + 5 = 12$$

B U Department of Mathematics

Math 201 Matrix Theory

Summer 2005 Final Exam

1.(a) [3] A 4×4 matrix C is known to have eigenvalues $\lambda_1 = 2, \lambda_2 = \lambda_3 = -3$ and $\lambda_4 = 4$.Find $\det(I + C)$, $Trace(I + C)$ and $\det(\exp C)$

Solution:

If C has eigenvalues 2,-3,-3,4 than (I+C) has eigenvalues 3,-2,-2,5 and $\exp(C)$ has eigenvalues e^2, e^{-3}, e^{-3}, e^4 then we have

$\det(I + C) = 3.(-2).(-2).5 = 60$

$Trace(I + C) = 3 + (-2) + (-2) + 5 = 4$

$\det(e^C) = e^2.e^{-3}.e^{-3}.e^4 = 1$

(b) [3] If K is a skew-symmetric square matrix show that $Q = (I - K)(I + K)^{-1}$ is an orthogonal matrix.

Solution:

Compute Q^TQ. $Q^T = [(I + K)^{-1}]^T[I - K]^T = (I + K^T)^{-1}(I - K^T) = (I - K)^{-1}(I + K)$

$Q^TQ = (I - K)^{-1}(I + K)(I - K)(I + K)^{-1} = I$

since $(I + K)(I - K) = (I - K)(I + K)$ therefore Q is orthogonal.

2.(a) [4] Suppose that $n \times n$ square matrices C and D obey $CD = -DC$. Find the flaw in the following argument and correct it: Taking determinants gives $\det(C)\det(D) = -\det(D)\det(C)$ so either C or D must have zero determinant. Thus $CD = -DC$ is only possible if C or D is singular.

Solution:

The flaw is that since $\det(-DC) = (-1)^n \det(D)\det(C)$

$\det(-DC) = -\det(D)\det(C)$ when n is odd.

The correct statement : C or D is singular when n is odd

b) [3] If A is a skew-symmetric matrix what can you sat about e^A(justify your answer).

Solution:

e^A is orthogonal since

$e^A(e^A)^T = e^Ae^{A^T} = e^Ae^{-A} = I$

3) [7] If linearly independent vectors x_1 and x_2 are in the columns of S, what are the eigenvalues and eigenvectors of $B = S\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} S^{-1}$

Solution:

Diagonalize $\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$. $\begin{vmatrix} 2-\lambda & 3 \\ 0 & 1-\lambda \end{vmatrix} = 0$ implies $(2 - \lambda)(1 - \lambda) = 0$.

Eigenvalues are $\lambda = 2$ and $\lambda = 1$.

Eigenvectors for $\lambda = 2$: $\begin{pmatrix} 0 & 3 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ implies $b = 0$. $x = a \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Eigenvectors for $\lambda = 1$: $\begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ implies $a + 3b = 0$. $x = b \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$\tilde{S} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$, $\tilde{\Lambda} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = \tilde{S}\tilde{\Lambda}\tilde{S}^{-1}$

$$B = S\tilde{S}\tilde{\Lambda}\tilde{S}^{-1}S^{-1} = (S\tilde{S})\tilde{\Lambda}(S\tilde{S})^{-1}$$

$\Rightarrow S\tilde{S}$ diagonalizes B

$S\tilde{S} = \begin{pmatrix} x_1 & x_2 \\ \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_1 & -3x_1 + x_2 \\ \downarrow & \downarrow \end{pmatrix}$

$\tilde{\Lambda} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

Eigenvector x_1 with the eigenvalue 2.
Eigenvector $-3x_1 + x_2$ with the eigenvalue 1.

4) Let $A = \begin{pmatrix} 3 & 4 & 6 \\ 0 & 1 & 0 \\ -1 & -2 & -2 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

a) [4] Find all eigenvalues and eigenvectors of A.

Solution:

$$\begin{vmatrix} 3-\lambda & 4 & 6 \\ 0 & 1-\lambda & 0 \\ -1 & -2 & -2-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda(\lambda - 1)^2 = 0$
$\lambda = 0$ and $\lambda = 1$ are eigenvalues.

Eigenvectors for $\lambda = 0$: $\begin{pmatrix} 3 & 4 & 6 \\ 0 & 1 & 0 \\ -1 & -2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$$\begin{array}{ccccccc} 3x_1 & + & 4x_2 & + & 6x_3 & = & 0 \\ & & x_2 & & & = & 0 \\ -x_1 & - & 2x_2 & - & 3x_3 & = & 0 \end{array}$$

$\Rightarrow x_1 = -2x_2$, $x = x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$.

Eigenvectors for $\lambda = 1$: $\begin{pmatrix} 2 & 4 & 6 \\ 0 & 0 & 0 \\ -1 & -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 6x_3 & = & 0 \\ & & & & 0 & = & 0 \\ -x_1 & - & 2x_2 & - & 3x_3 & = & 0 \end{array}$$

$\Rightarrow x_1 + 2x_2 + 3x_3 = 0$, $x = \begin{pmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$.

b) [3] Compute $A^{111}b$.

Solution:
Since eigenvectors are linearly independent A is diagonalizable.

$A = S\Lambda S^{-1}$, $\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$A''' = S\Lambda''' S^{-1} = S\Lambda S{-}1 = A$ since $\Lambda''' = \Lambda$.

$A'''b = Ab = \begin{pmatrix} 3 & 4 & 6 \\ 0 & 1 & 0 \\ -1 & -2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$.

5) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation satisfying $T(1, -1) = (-1, 2)$ and $T(1, 1) = (3, 4)$.

a) [3] Find the matrix representation of T in the standard basis.

Solution:

$T(1, 0) = \frac{1}{2}[T(1, 1) + T(1, -1)] = \frac{1}{2}[(3, 4) + (-1, 2)] = (1, 3)$.

$T(0, 1) = \frac{1}{2}[T(1, 1) - T(1, -1)] = \frac{1}{2}[(3, 4) - (-1, 2)] = (2, 1)$.

$$T = \begin{pmatrix} T(1,0) & T(1,0) \\ \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$$

b) [3] Compute $T(x, y)$.

Solution:

$T(x, y) = xT(1, 0) + yT(0, 1) = x(1, 3) + y(2, 1) = (x + 2y, 3x + y)$

6) a) [3] Find a basis for the vector space of $3x3$ skew-symmetric matrices.

Solution:

The most general $3x3$ skew-symmetric matrix

$$K = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \qquad a, b, c \in \mathbb{R}$$

$$= a \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{K_1} + b \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}}_{K_2} + c \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}}_{K_3}$$

$\{K_1, K_2, K_3\}$ form a basis.

b) [4] What can you say about a subset of a linearly independent set of vectors? (prove your claim).

Solution:
The subset should be linearly independent.

Proof: Let $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ be linearly independent vectors, $\{\alpha_1, ..., \alpha_k\} k \leq n$ be our subset. Assume that the subset is linearly dependent. Then if $c_1\alpha_1 + ... + c_k\alpha_k = 0$ there exists at least one $c_i \neq 0$. But then the numbers $(c_1, ..., c_k, 0, 0, ..., 0)$ gives $c_1\alpha_1 + ... + c_k\alpha_k + 0\alpha_{k+1} + ... + 0\alpha_n = 0$ with some non-zero c_i. This contradicts with the fact that $\{\alpha_1, ..., \alpha_n\}$ is linearly independent. Thus $\{\alpha_1, ..., \alpha_k\}$ should be linearly independent.