Math 170 - Worksheet 4

1. Find the domains and derivatives of the given vector functions:

(a) $\vec{r}(t) = \langle \arctan t^2, \ln(1+t), t\rangle$ (b) $\vec{r}(t) = \langle e^{t/2}, \sqrt{t+1}, \ln t\rangle$ (c) $\vec{r}(t) = \langle \ln(1-t^2), \frac{1}{t}, \cos t\rangle$

2. Do the curves below cross the given points or sets? If yes, where?

(a) $\vec{r}(t) = \langle 2t+1, t^2-3, \sin t\rangle, t \in \mathbb{R}$, the xy-plane (b) $\vec{r}(t) = \langle \ln(1-t), t, \sqrt{t}\rangle, t \in (0,1)$, the x-axis (c) $\vec{r}(t) = \langle t+5, t^3, \frac{t+2}{t-2}\rangle, t \in (-\infty, -2)$, the point $(1, -64, \frac{1}{3})$. (d) $\vec{r}(t) = \langle \frac{1}{1+t^2, \cos t, \sin t}, t \in \mathbb{R}$, the xy-plane.

3. Find the unit tangent vector at the given points:

(a) $\vec{r}(t) = \langle 2t, t^3/5, t-t^2\rangle,\ t = 5$ (b) $\vec{r}(t) = \langle \cos t, \sin t, 3t^2\rangle,\ t = 1/\sqrt{3}$ (c) $\vec{r}(t) = \langle \frac{1}{t}, t-3, \sqrt{t}\rangle,\ t = 1$.

4. Compute the vector $\boldsymbol{T}$ for $\vec{r}(t) = \langle \cos t, \sin t, \sqrt{3}t\rangle$ at $t = \pi$.

5. Find the lengths of the given arcs:

(a) $\vec{r}(t) = \langle t^2/2, \sqrt{2}t, \ln t\rangle,\ 1 \leqslant t \leqslant 2$. (b) $\vec{r}(t) = \langle \sin t, \cos t, t^2/2\rangle,\ 0 \leqslant t \leqslant 1$

(c) $\vec{r}(t) = \langle e^t \cos t, e^t, e^t \sin t\rangle,\ 0 \leqslant t \leqslant 2\pi$.

6. Show that if $\vec{r}(t)$ is a differentiable curve in $\mathbb{R}^3$, then $\frac{d}{dt}(\vec{r}(t) \times \vec{r}\,'(t)) = \vec{r}(t) \times \vec{r}\,''(t)$.

7*. Show that the tangent vectors of $\vec{r}(t) = \langle 3t-t^3, 3t^2, 3t+t^3\rangle$ make a constant angle with some fixed unit vector $\vec{u}$.

8. Show that if $\vec{r}(t)$ is always orthogonal to $\vec{r}\,'(t)$, then $\vec{r}(t)$ is moving on a sphere whose center is the origin.

9. Evaluate $\int_0^2 \langle 6t^2, -4t, 3\rangle\, dt$.

10. Find a curve $\vec{r}(t)$ which satisfies $\vec{r}\,''(t) = \langle 6t, -12t^2, 1\rangle$, $\vec{r}\,'(0) = \langle 1, 2, -3\rangle$ and $\vec{r}(0) = \langle 7, 0, 1\rangle$.

11. Find the curvature of the curve $\vec{r}(t) = \langle 5t+2, -t^2, t^3\rangle$ at $t = 1$.

12. Show that if a plane curve is parametrized as $x = f(t)$ and $y = g(t)$, then its curvature is given by

$$\kappa(t) = \frac{|f'(t)g''(t) - g'(t)f''(t)|}{[(f'(t))^2 + (g'(t)^2)]^{3/2}}$$

Answers:

1. (a) $(-1, \infty), \vec{r}\,'(t) = \langle \frac{2t}{1+t^4}, \frac{1}{1+t}, 1\rangle$ (b) $(0, \infty), \vec{r}\,'(t) = \left\langle \frac{1}{2}e^{t/2}, \frac{1}{2\sqrt{t+1}}, \frac{1}{t}\right\rangle$

(c) $(-1, 0) \cup (0, 1), \vec{r}\,'(t) = \left\langle -\frac{2}{1-t^2}, -\frac{1}{t^2}, -\sin t\right\rangle$. **2.** (a) Yes, at $\vec{r}(\sqrt{3}) = \langle 2\sqrt{3}+1, 0, \sin\sqrt{3}\rangle$ and $\vec{r}(-\sqrt{3}) = \langle -2\sqrt{3}+1, 0, -\sin\sqrt{3}\rangle$ (b) No (c) Yes, at $t = -4$ (d) Yes, at $t = k\pi, k \in \mathbb{Z}$.

3. (a) $\left\langle \frac{2}{\sqrt{310}}, \frac{15}{\sqrt{310}}, -\frac{9}{\sqrt{310}}\right\rangle$ (b) $\frac{\langle -\sin(1/\sqrt{3}), \cos(1/\sqrt{3}), 2\sqrt{3}\rangle}{\sqrt{13}}$ (c) $\left\langle -\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right\rangle$.

4. $\boldsymbol{t} = \left\langle 0, -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle$. **5.** (a) $3/2 + \ln 2$ (b) $\frac{\sqrt{2}+\ln(1+\sqrt{2})}{2}$ (c) $\sqrt{3}(e^{2\pi}-1)$ **9.** $\langle 16, -8, 6\rangle$

10. $\left\langle t^3 + t + 7, 2t - t^4, \frac{t^2}{2} - 3t + 1\right\rangle$. **11.** $\frac{\sqrt{1036}}{38\sqrt{38}}$.