MATHEMATICS 102E FINAL 10 January 2012

QUESTION 1 The blanks below will be filled by students. (Except the score)

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For the solution of this question please use only the front face and if necessary the back face of this page.

[12pt] a) Find the equations of the tangent plane and the normal line of the surface $z = \sin^2(x - 3y)$ at the point $P(\pi, \frac{\pi}{3}, 0)$.

[13pt] b) Find the absolute maximum and the absolute minimum values of the function

$$f(x, y) = 2y^3 + 2x^2y - x^2 - y$$

on the closed region $D = \{(x, y) : x^2 + y^2 \leq 1\}$.

a) Let $F(x,y,z) = \sin^2(x-3y) - z$. Then, $\nabla F|_P$ will be a normal vector of the tangent plane, and $\nabla F|_P$ will be a direction vector of the normal line

$\nabla F|_P = \langle F_x, F_y, F_z\rangle|_P = \langle 2\sin(x-3y)\cos(x-3y), 2\sin(x-3y)\cos(x-3y)(-3), -1\rangle_P = \langle 0, 0, -1\rangle$

Hence, $0(x-\pi) + 0(y - \frac{\pi}{3}) - (z-0) = 0 \iff \underline{z = 0}$ is the tangent plane

And, $\underline{x = \pi, \ y = \frac{\pi}{3}, \ z = -t}$ $(t \in \mathbb{R})$ is the normal line.

b) $\left.\begin{array}{l} f_x = 4xy - 2x = 0 \\ f_y = 6y^2 + 2x^2 - 1 = 0 \end{array}\right\} \iff (x,y) = (0, \mp\frac{1}{\sqrt{6}}) \in D \leftarrow$ Critical points of $f(x,y)$

On the boundary of D, $x^2+y^2 = 1$ so that $x^2 = 1 - y^2$, and hence

$\phi(y) = f(x,y)|_{x^2+y^2=1} = 2y^3 + 2(1-y^2)y - (1-y^2) - y = y^2 + y - 1, \ y \in [-1, 1]$

$\phi'(y) = 2y + 1 = 0 \iff y = -\frac{1}{2}$ (so, $x = \mp\frac{\sqrt{3}}{2}$)

And if $y = -1$ then $x = 0$; and if $y = 1$ then $x = 0$.

We need to calculate $f(x,y)$ at the points $(0, \mp\frac{1}{\sqrt{6}}), (\mp\frac{\sqrt{3}}{2}, \frac{-1}{2}), (0, \mp 1)$:

$f(0, \frac{1}{\sqrt{6}}) = \frac{-2}{3\sqrt{6}}$, $f(0, \frac{-1}{\sqrt{6}}) = \frac{2}{3\sqrt{6}}$, $\underbrace{f(\mp\frac{\sqrt{3}}{2}, \frac{-1}{2}) = \frac{-5}{4}}_{\text{Absolute Min}}$, $f(0,-1) = -1$, $\underbrace{f(0,1) = 1}_{\text{Absolute Max}}$

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QUESTION 2 The blanks below will be filled by students. (Except the score)

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[12pt] a) Let $\mathbf{f(x,y)}$ be a continuous real valued function. If the functions $\mathbf{A(x)}$ and $\mathbf{B(y)}$ are defined as

$$A(x)=\int_0^{x+1} f(x,y)dy \quad \text{and} \quad B(y)=\int_{y-1}^{0} f(x,y)dx,$$

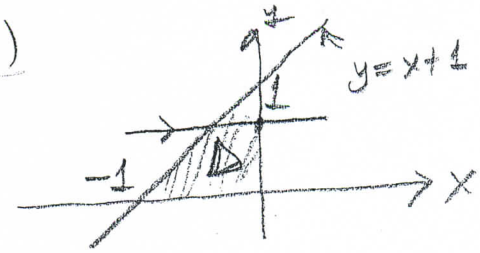
prove that

$$\int_{-1}^{0} A(x)dx=\int_0^1 B(y)dy.$$

[13pt] b) Let $\mathbf{R}$ be the region in the $\mathbf{xy}$-plane bounded by the lines $\mathbf{y=x}$, $\mathbf{y=0}$, and $\mathbf{x=1}$. Let $\mathbf{C}$ be the boundary of $\mathbf{R}$ oriented in the counterclockwise way. Calculate the line integral

$$\oint_C (y^2x-ye^{x^2})dx+x^2ydy$$

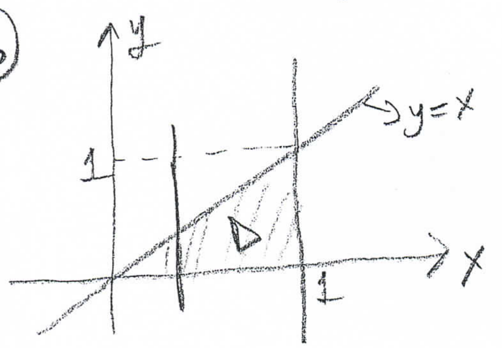
by converting it to a double integral. (use the Green's theorem)



a) $$\int_{-1}^{0} A(x)dx=\int_{-1}^{0}\left(\int_0^{x+1} f(x,y)dy\right)dx=\iint_D f(x,y)dA$$

$$=\int_0^1 dy\left(\int_{y-1}^{0} f(x,y)dx\right)=\int_0^1 B(y)dy$$

b)



$$\oint_C (y^2x-ye^{x^2})dx+x^2y\,dy \overset{\text{Green's thm}}{=} \iint_D\left(\frac{\partial}{\partial x}(x^2y)-\frac{\partial}{\partial y}(y^2x-ye^{x^2})\right)dA=\iint_D e^{x^2}dA$$

$$=\int_0^1 dx\int_0^x e^{x^2}dy=\int_0^1\left(ye^{x^2}\Big|_{y=0}^{y=x}\right)dx=\int_0^1 xe^{x^2}dx$$

$$=\frac{1}{2}e^{x^2}\Big|_{x=0}^{x=1}=\frac{1}{2}(e-1)$$

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[25pt] Let D be the solid that is bounded above by the cone $z = 2 - \sqrt{x^2 + y^2}$ and bounded below by the paraboloid $z = x^2 + y^2$. Write the triple integral expressing the volume of D as an iterated integral in terms of

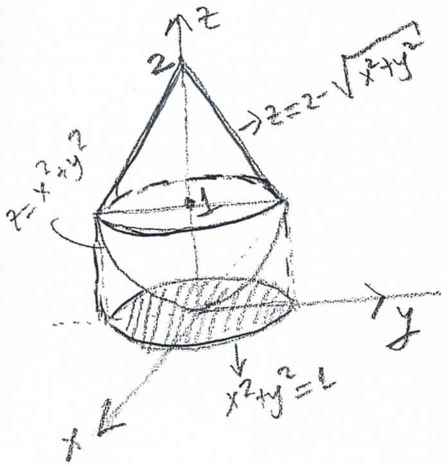
a) Cartesian coordinates,

b) Spherical coordinates,

c) Cylindrical coordinates,

Do not calculate the integrals you wrote in a), b), c).

d) Find the volume of D by calculating one of the integrals constructed in a), b), or c).



(a) $V = \int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_{x^2+y^2}^{2-\sqrt{x^2+y^2}} dz$

(c) $V = \int_0^{2\pi} d\theta \int_0^1 dr \int_{r^2}^{2-r} r\, dz$

(b) $z = 2-\sqrt{x^2+y^2}$ becomes $\rho\cos\phi = 2-\sqrt{\rho^2\sin^2\phi} \Rightarrow \rho = \dfrac{2}{\cos\phi+\sin\phi}$

$z = x^2+y^2$ becomes $\rho\cos\phi = \rho^2\sin^2\phi \Rightarrow \rho = \dfrac{\cos\phi}{\sin^2\phi}$

Note that  =  + 

$V = \int_0^{2\pi} d\theta \int_0^{\pi/4} d\phi \int_0^{2/(\cos\phi+\sin\phi)} \rho^2\sin\phi\, d\rho + \int_0^{2\pi} d\theta \int_{\pi/4}^{\pi/2} d\phi \int_0^{\frac{\cos\phi}{\sin^2\phi}} \rho^2 \sin\phi\, d\rho$

(d) We use part (c) to find V,

$V = \int_0^{2\pi} d\theta \int_0^1 dr \int_{r^2}^{2-r} r\, dz = \int_0^{2\pi} d\theta \int_0^1 dr \left(rz \Big|_{z=r^2}^{z=2-r} \right) = \int_0^{2\pi} d\theta \int_0^1 (2r - r^2 - r^3)$

$= \int_0^{2\pi} d\theta \left(r^2 - \frac{r^3}{3} - \frac{r^4}{4} \Big|_{r=0}^{r=1} \right) = \int_0^{2\pi} d\theta \left(\frac{5}{12}\right) = 2\pi \frac{5}{12} = \frac{5\pi}{6}$

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Let $\overrightarrow{\mathbf{F}}$ be the vector field given by

$$\overrightarrow{\mathbf{F}}(x, y, z) = x\,\overrightarrow{\mathbf{i}} + y\,\overrightarrow{\mathbf{j}} + z\,\overrightarrow{\mathbf{k}}.$$

Let $\mathbf{D}$ be a solid bounded by the closed oriented surface $\mathbf{S}$ with a parametrization that gives outer unit normal $\overrightarrow{\mathbf{n}}$ of $\mathbf{S}$. If the volume of $\mathbf{D}$ is 2π, find the surface integral $\iint_{\mathbf{S}} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{n}}\, d\sigma$.

$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma \underset{\text{Divergence Thm}}{=} \iiint_D \vec{\nabla}\cdot\vec{F}\,dV$$

$$= \iiint_D 3\,dV = 3\left(\begin{matrix}\text{The volume}\\ \text{of } D\end{matrix}\right) = 3(2\pi)$$

$$= 6\pi$$