CALCULUS III

Three Dimensional Space

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Table of Contents

Preface

Here are my online notes for my Calculus III course that I teach here at Lamar University. Despite the fact that these are my "class notes", they should be accessible to anyone wanting to learn Calculus III or needing a refresher in some of the topics from the class.

These notes do assume that the reader has a good working knowledge of Calculus I topics including limits, derivatives and integration. It also assumes that the reader has a good knowledge of several Calculus II topics including some integration techniques, parametric equations, vectors, and knowledge of three dimensional space.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn't covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Calculus III many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don't have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren't worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can't anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I've not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.

Three Dimensional Space

Introduction

In this chapter we will start taking a more detailed look at three dimensional space (3-D space or $\mathbb{R}^3$). This is a very important topic in Calculus III since a good portion of Calculus III is done in three (or higher) dimensional space.

We will be looking at the equations of graphs in 3-D space as well as vector valued functions and how we do calculus with them. We will also be taking a look at a couple of new coordinate systems for 3-D space.

This is the only chapter that exists in two places in my notes. When I originally wrote these notes all of these topics were covered in Calculus II however, we have since moved several of them into Calculus III. So, rather than split the chapter up I have kept it in the Calculus II notes and also put a copy in the Calculus III notes. Many of the sections not covered in Calculus III will be used on occasion there anyway and so they serve as a quick reference for when we need them.

Here is a list of topics in this chapter.

The 3-D Coordinate System – We will introduce the concepts and notation for the three dimensional coordinate system in this section.

Equations of Lines – In this section we will develop the various forms for the equation of lines in three dimensional space.

Equations of Planes – Here we will develop the equation of a plane.

Quadric Surfaces – In this section we will be looking at some examples of quadric surfaces.

Functions of Several Variables – A quick review of some important topics about functions of several variables.

Vector Functions – We introduce the concept of vector functions in this section. We concentrate primarily on curves in three dimensional space. We will however, touch briefly on surfaces as well.

Calculus with Vector Functions – Here we will take a quick look at limits, derivatives, and integrals with vector functions.

Tangent, Normal and Binormal Vectors – We will define the tangent, normal and binormal vectors in this section.

Arc Length with Vector Functions – In this section we will find the arc length of a vector function.

Curvature – We will determine the curvature of a function in this section.

Velocity and Acceleration – In this section we will revisit a standard application of derivatives. We will look at the velocity and acceleration of an object whose position function is given by a vector function.

Cylindrical Coordinates – We will define the cylindrical coordinate system in this section. The cylindrical coordinate system is an alternate coordinate system for the three dimensional coordinate system.

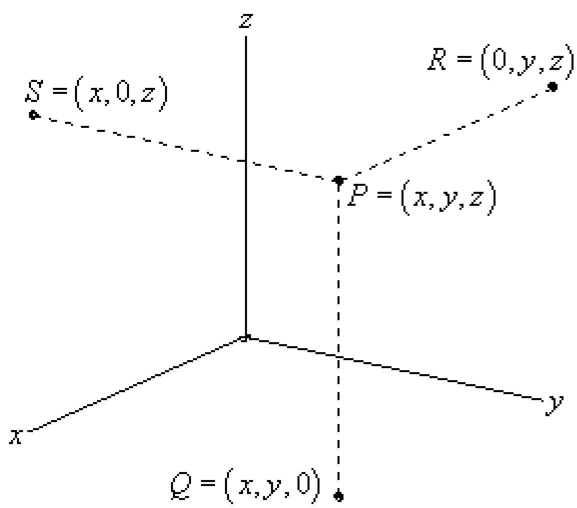
Spherical Coordinates – In this section we will define the spherical coordinate system. The spherical coordinate system is yet another alternate coordinate system for the three dimensional coordinate system.

The 3-D Coordinate System

We'll start the chapter off with a fairly short discussion introducing the 3-D coordinate system and the conventions that we'll be using. We will also take a brief look at how the different coordinate systems can change the graph of an equation.

Let's first get some basic notation out of the way. The 3-D coordinate system is often denoted by $\mathbb{R}^3$. Likewise the 2-D coordinate system is often denoted by $\mathbb{R}^2$ and the 1-D coordinate system is denoted by $\mathbb{R}$. Also, as you might have guessed then a general *n* dimensional coordinate system is often denoted by $\mathbb{R}^n$.

Next, let's take a quick look at the basic coordinate system.



This is the standard placement of the axes in this class. It is assumed that only the positive directions are shown by the axes. If we need the negative axes for any reason we will put them in as needed.

Also note the various points on this sketch. The point *P* is the general point sitting out in 3-D space. If we start at *P* and drop straight down until we reach a *z*-coordinate of zero we arrive at the point *Q*. We say that *Q* sits in the *xy*-plane. The *xy*-plane corresponds to all the points which have a zero *z*-coordinate. We can also start at *P* and move in the other two directions as shown to get points in the *xz*-plane (this is *S* with a *y*-coordinate of zero) and the *yz*-plane (this is *R* with an *x*-coordinate of zero).

Collectively, the *xy*, *xz*, and *yz*-planes are sometimes called the coordinate planes. In the remainder of this class you will need to be able to deal with the various coordinate planes so make sure that you can.

Also, the point *Q* is often referred to as the projection of *P* in the *xy*-plane. Likewise, *R* is the projection of *P* in the *yz*-plane and *S* is the projection of *P* in the *xz*-plane.

Many of the formulas that you are used to working with in $\mathbb{R}^2$ have natural extensions in $\mathbb{R}^3$. For instance the distance between two points in $\mathbb{R}^2$ is given by,

$$d\left(P_1, P_2\right) = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$$

While the distance between any two points in $\mathbb{R}^3$ is given by,

$$d\left(P_1, P_2\right) = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2}$$

Likewise, the general equation for a circle with center $\left(h,k\right)$ and radius *r* is given by,

$$\left(x-h\right)^2 + \left(y-k\right)^2 = r^2$$

and the general equation for a sphere with center $\left(h,k,l\right)$ and radius *r* is given by,

$$\left(x-h\right)^2 + \left(y-k\right)^2 + \left(z-l\right)^2 = r^2$$

With that said we do need to be careful about just translating everything we know about $\mathbb{R}^2$ into $\mathbb{R}^3$ and assuming that it will work the same way. A good example of this is in graphing to some extent. Consider the following example.

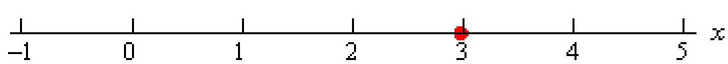
Example 1 Graph $x = 3$ in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$.

Solution
In $\mathbb{R}$ we have a single coordinate system and so $x = 3$ is a point in a 1-D coordinate system.

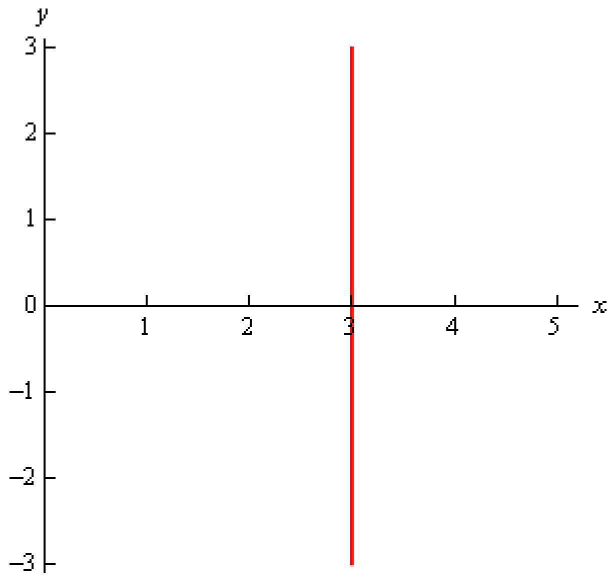
In $\mathbb{R}^2$ the equation $x = 3$ tells us to graph all the points that are in the form $\left(3, y\right)$. This is a vertical line in a 2-D coordinate system.

In $\mathbb{R}^3$ the equation $x = 3$ tells us to graph all the points that are in the form $\left(3, y, z\right)$. If you go back and look at the coordinate plane points this is very similar to the coordinates for the *yz*-plane except this time we have $x = 3$ instead of $x = 0$. So, in a 3-D coordinate system this is a plane that will be parallel to the *yz*-plane and pass through the *x*-axis at $x = 3$.

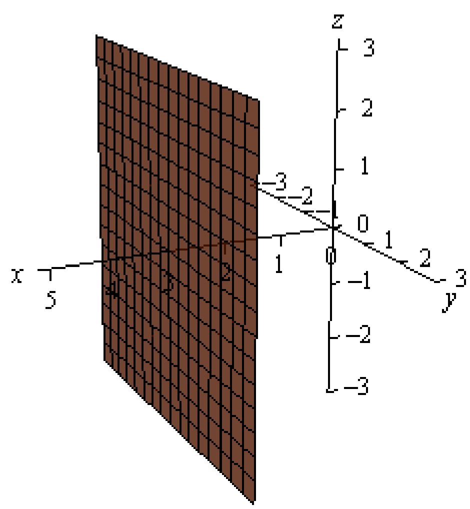
Here is the graph of $x = 3$ in $\mathbb{R}$.

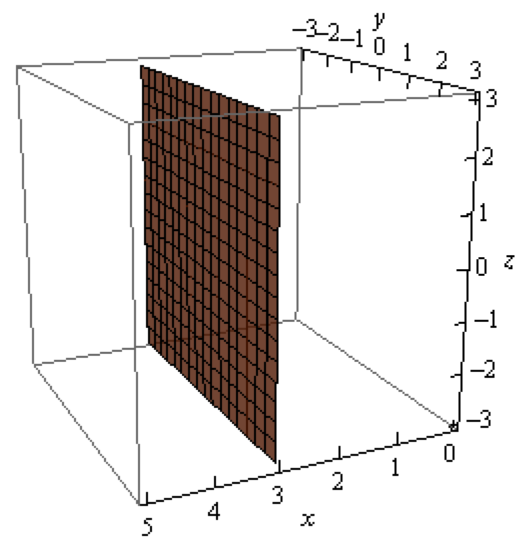


Here is the graph of $x = 3$ in $\mathbb{R}^2$.



Finally, here is the graph of $x = 3$ in $\mathbb{R}^3$. Note that we've presented this graph in two different styles. On the left we've got the traditional axis system and we're used to seeing and on the right we've put the graph in a box. Both views can be convenient on occasion to help with perspective and so we'll often do this with 3D graphs and sketches.





Note that at this point we can now write down the equations for each of the coordinate planes as well using this idea.

$$
\begin{array}{ll}
z = 0 & \qquad xy - \text{plane} \\
y = 0 & \qquad xz - \text{plane} \\
x = 0 & \qquad yz - \text{plane}
\end{array}
$$

Let's take a look at a slightly more general example.

Example 2 Graph $y = 2x - 3$ in $\mathbb{R}^2$ and $\mathbb{R}^3$.

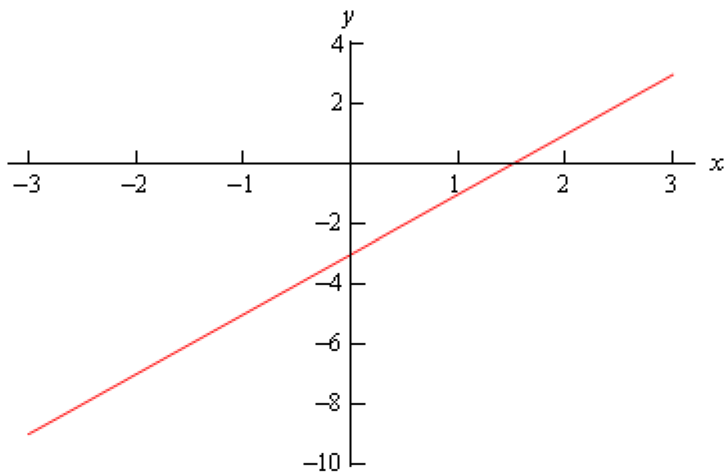
Solution

Of course we had to throw out $\mathbb{R}$ for this example since there are two variables which means that we can't be in a 1-D space.

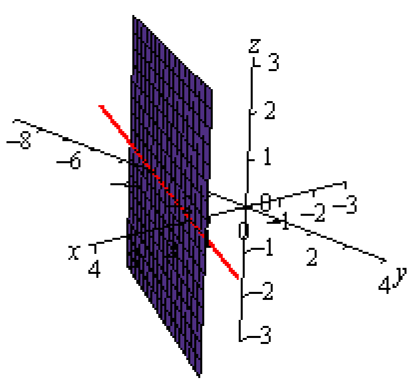
In $\mathbb{R}^2$ this is a line with slope 2 and a *y* intercept of -3.

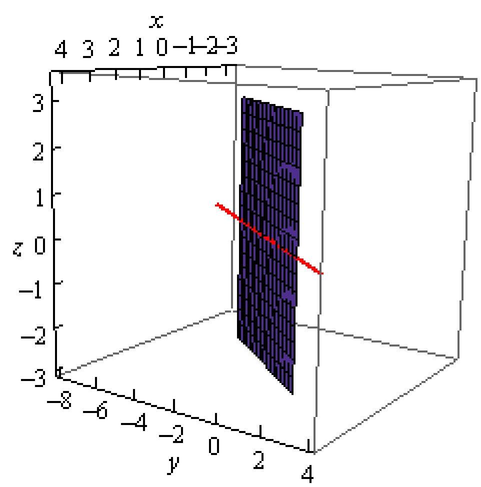
However, in $\mathbb{R}^3$ this is not necessarily a line. Because we have not specified a value of *z* we are forced to let *z* take any value. This means that at any particular value of *z* we will get a copy of this line. So, the graph is then a vertical plane that lies over the line given by $y = 2x - 3$ in the *xy*-plane.

Here is the graph in $\mathbb{R}^2$.



here is the graph in $\mathbb{R}^3$.





Notice that if we look to where the plane intersects the *xy*-plane we will get the graph of the line in $\mathbb{R}^2$ as noted in the above graph by the red line through the plane.

Let's take a look at one more example of the difference between graphs in the different coordinate systems.

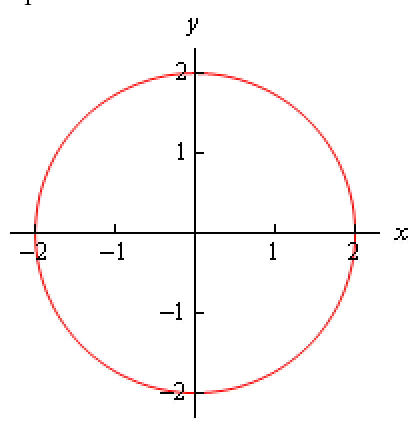
Example 3 Graph $x^2 + y^2 = 4$ in $\mathbb{R}^2$ and $\mathbb{R}^3$.

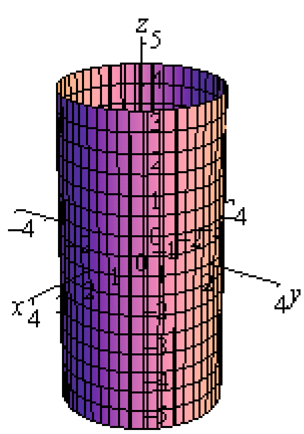
Solution
As with the previous example this won't have a 1-D graph since there are two variables.

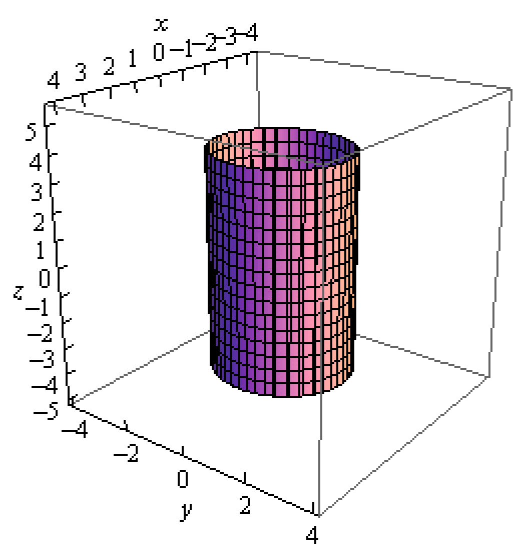
In $\mathbb{R}^2$ this is a circle centered at the origin with radius 2.

In $\mathbb{R}^3$ however, as with the previous example, this may or may not be a circle. Since we have not specified z in any way we must assume that z can take on any value. In other words, at any value of z this equation must be satisfied and so at any value z we have a circle of radius 2 centered on the z-axis. This means that we have a cylinder of radius 2 centered on the z-axis.

Here are the graphs for this example.







Notice that again, if we look to where the cylinder intersects the xy-plane we will again get the circle from $\mathbb{R}^2$.

We need to be careful with the last two examples. It would be tempting to take the results of these and say that we can't graph lines or circles in $\mathbb{R}^3$ and yet that doesn't really make sense. There is no reason for there to not be graphs of lines or circles in $\mathbb{R}^3$. Let's think about the example of the circle. To graph a circle in $\mathbb{R}^3$ we would need to do something like $x^2 + y^2 = 4$ at $z = 5$. This would be a circle of radius 2 centered on the *z*-axis at the level of $z = 5$. So, as long as we specify a *z* we will get a circle and not a cylinder. We will see an easier way to specify circles in a later section.

We could do the same thing with the line from the second example. However, we will be looking at lines in more generality in the next section and so we'll see a better way to deal with lines in $\mathbb{R}^3$ there.

The point of the examples in this section is to make sure that we are being careful with graphing equations and making sure that we always remember which coordinate system that we are in.

Another quick point to make here is that, as we've seen in the above examples, many graphs of equations in $\mathbb{R}^3$ are surfaces. That doesn't mean that we can't graph curves in $\mathbb{R}^3$. We can and will graph curves in $\mathbb{R}^3$ as well as we'll see later in this chapter.

Equations of Lines

In this section we need to take a look at the equation of a line in $\mathbb{R}^3$. As we saw in the previous section the equation $y = mx + b$ does not describe a line in $\mathbb{R}^3$, instead it describes a plane. This doesn't mean however that we can't write down an equation for a line in 3-D space. We're just going to need a new way of writing down the equation of a curve.

So, before we get into the equations of lines we first need to briefly look at vector functions. We're going to take a more in depth look at vector functions later. At this point all that we need to worry about is notational issues and how they can be used to give the equation of a curve.

The best way to get an idea of what a vector function is and what its graph looks like is to look at an example. So, consider the following vector function.

$$\vec{r}\left(t\right) = \left\langle t, 1\right\rangle$$

A vector function is a function that takes one or more variables, one in this case, and returns a vector. Note as well that a vector function can be a function of two or more variables. However, in those cases the graph may no longer be a curve in space.

The vector that the function gives can be a vector in whatever dimension we need it to be. In the example above it returns a vector in $\mathbb{R}^2$. When we get to the real subject of this section, equations of lines, we'll be using a vector function that returns a vector in $\mathbb{R}^3$

Now, we want to determine the graph of the vector function above. In order to find the graph of our function we'll think of the vector that the vector function returns as a position vector for points on the graph. Recall that a position vector, say $\vec{v} = \left\langle a, b\right\rangle$, is a vector that starts at the origin and ends at the point $\left(a, b\right)$.

So, to get the graph of a vector function all we need to do is plug in some values of the variable and then plot the point that corresponds to each position vector we get out of the function and play connect the dots. Here are some evaluations for our example.

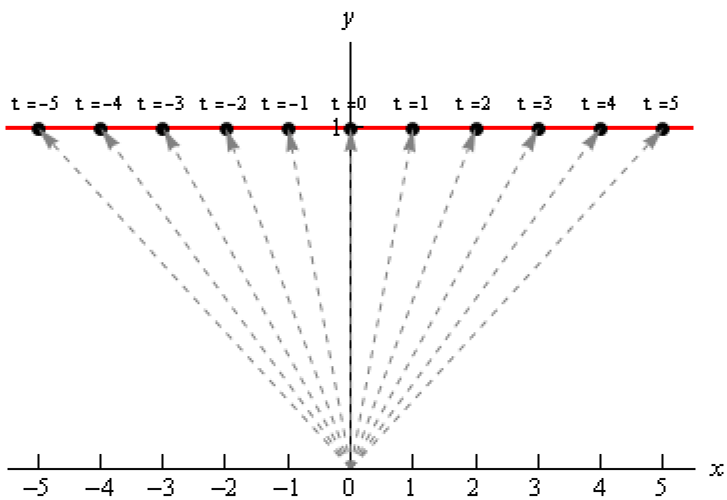
$$\vec{r}\left(-3\right) = \left\langle -3, 1\right\rangle \qquad \vec{r}\left(-1\right) = \left\langle -1, 1\right\rangle \qquad \vec{r}\left(2\right) = \left\langle 2, 1\right\rangle \qquad \vec{r}\left(5\right) = \left\langle 5, 1\right\rangle$$

So, each of these are position vectors representing points on the graph of our vector function. The points,

$$\left(-3, 1\right) \qquad \left(-1, 1\right) \qquad \left(2, 1\right) \qquad \left(5, 1\right)$$

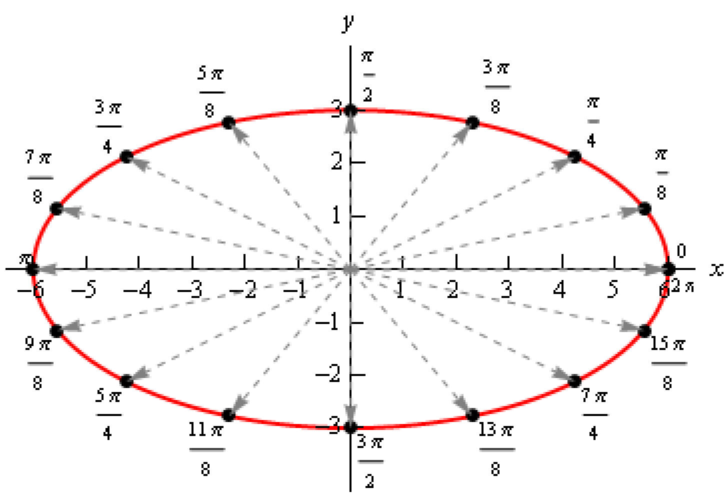
are all points that lie on the graph of our vector function.

If we do some more evaluations and plot all the points we get the following sketch.



In this sketch we've included the position vector (in gray and dashed) for several evaluations as well as the *t* (above each point) we used for each evaluation. It looks like, in this case the graph of the vector equation is in fact the line $y = 1$.

Here's another quick example. Here is the graph of $\vec{r}\left(t\right) = \left\langle 6\cos t, 3\sin t\right\rangle$.



In this case we get an ellipse. It is important to not come away from this section with the idea that vector functions only graph out lines. We'll be looking at lines in this section, but the graphs of vector functions do not have to be lines as the example above shows.

We'll leave this brief discussion of vector functions with another way to think of the graph of a vector function. Imagine that a pencil/pen is attached to the end of the position vector and as we increase the variable the resulting position vector moves and as it moves the pencil/pen on the end sketches out the curve for the vector function.

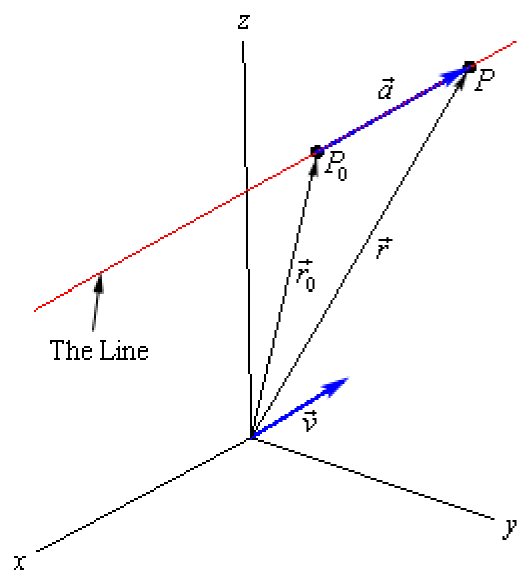
Okay, we now need to move into the actual topic of this section. We want to write down the equation of a line in $\mathbb{R}^3$ and as suggested by the work above we will need a vector function to do this. To see how we're going to do this let's think about what we need to write down the equation of a line in $\mathbb{R}^2$. In two dimensions we need the slope (m) and a point that was on the line in order to write down the equation.

In $\mathbb{R}^3$ that is still all that we need except in this case the "slope" won't be a simple number as it was in two dimensions. In this case we will need to acknowledge that a line can have a three dimensional slope. So, we need something that will allow us to describe a direction that is potentially in three dimensions. We already have a quantity that will do this for us. Vectors give directions and can be three dimensional objects.

So, let's start with the following information. Suppose that we know a point that is on the line, $P_0 = (x_0, y_0, z_0)$, and that $\vec{v} = \langle a, b, c \rangle$ is some vector that is parallel to the line. Note, in all likelihood, $\vec{v}$ will not be on the line itself. We only need $\vec{v}$ to be parallel to the line. Finally, let $P = (x, y, z)$ be any point on the line.

Now, since our "slope" is a vector let's also represent the two points on the line as vectors. We'll do this with position vectors. So, let $\vec{r_0}$ and $\vec{r}$ be the position vectors for P_0 and P respectively. Also, for no apparent reason, let's define $\vec{a}$ to be the vector with representation $\overrightarrow{P_0P}$.

We now have the following sketch with all these points and vectors on it.



Now, we've shown the parallel vector, $\vec{v}$, as a position vector but it doesn't need to be a position vector. It can be anywhere, a position vector, on the line or off the line, it just needs to be parallel to the line.

Next, notice that we can write $\vec{r}$ as follows,

$$\vec{r} = \vec{r_0} + \vec{a}$$

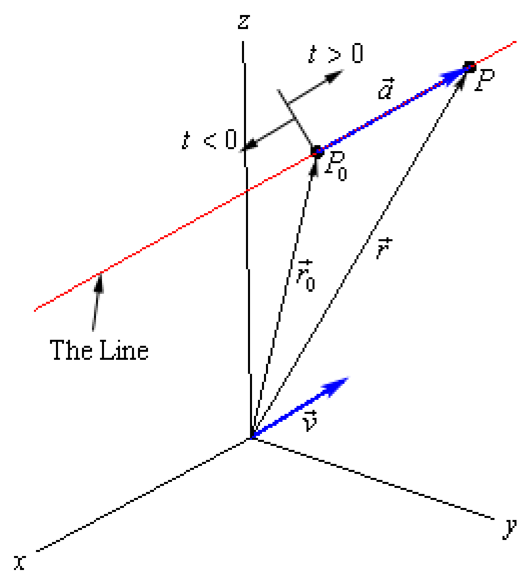
If you're not sure about this go back and check out the sketch for vector addition in the vector arithmetic section. Now, notice that the vectors $\vec{a}$ and $\vec{v}$ are parallel. Therefore there is a number, t, such that

$$\vec{a} = t\,\vec{v}$$

We now have,

$$\vec{r} = \overrightarrow{r_0} + t\,\vec{v} = \left\langle x_0, y_0, z_0 \right\rangle + t\left\langle a, b, c \right\rangle$$

This is called the **vector form of the equation of a line**. The only part of this equation that is not known is the *t*. Notice that $t\,\vec{v}$ will be a vector that lies along the line and it tells us how far from the original point that we should move. If *t* is positive we move away from the original point in the direction of $\vec{v}$ (right in our sketch) and if *t* is negative we move away from the original point in the opposite direction of $\vec{v}$ (left in our sketch). As *t* varies over all possible values we will completely cover the line. The following sketch shows this dependence on *t* of our sketch.



There are several other forms of the equation of a line. To get the first alternate form let's start with the vector form and do a slight rewrite.

$$\begin{aligned} \vec{r} &= \left\langle x_0, y_0, z_0 \right\rangle + t\left\langle a, b, c \right\rangle \\ \left\langle x, y, z \right\rangle &= \left\langle x_0 + ta, y_0 + tb, z_0 + tc \right\rangle \end{aligned}$$

The only way for two vectors to be equal is for the components to be equal. In other words,

$$\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned}$$

This set of equations is called the **parametric form of the equation of a line**. Notice as well that this is really nothing more than an extension of the parametric equations we've seen previously. The only difference is that we are now working in three dimensions instead of two dimensions.

To get a point on the line all we do is pick a t and plug into either form of the line. In the vector form of the line we get a position vector for the point and in the parametric form we get the actual coordinates of the point.

There is one more form of the line that we want to look at. If we assume that a, b, and c are all non-zero numbers we can solve each of the equations in the parametric form of the line for t. We can then set all of them equal to each other since t will be the same number in each. Doing this gives the following,

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

This is called the **symmetric equations of the line**.

If one of a, b, or c does happen to be zero we can still write down the symmetric equations. To see this let's suppose that $b=0$. In this case t will not exist in the parametric equation for y and so we will only solve the parametric equations for x and z for t. We then set those equal and acknowledge the parametric equation for y as follows,

$$\frac{x-x_0}{a}=\frac{z-z_0}{c} \qquad\qquad y=y_0$$

Let's take a look at an example.

Example 1 Write down the equation of the line that passes through the points $(2,-1,3)$ and $(1,4,-3)$. Write down all three forms of the equation of the line.

Solution
To do this we need the vector $\vec{v}$ that will be parallel to the line. This can be any vector as long as it's parallel to the line. In general, $\vec{v}$ won't lie on the line itself. However, in this case it will. All we need to do is let $\vec{v}$ be the vector that starts at the second point and ends at the first point. Since these two points are on the line the vector between them will also lie on the line and will hence be parallel to the line. So,

$$\vec{v}=\langle 1,-5,6\rangle$$

Note that the order of the points was chosen to reduce the number of minus signs in the vector. We could just have easily gone the other way.

Once we've got $\vec{v}$ there really isn't anything else to do. To use the vector form we'll need a point on the line. We've got two and so we can use either one. We'll use the first point. Here is the vector form of the line.

$$\vec{r}=\langle 2,-1,3\rangle+t\langle 1,-5,6\rangle=\langle 2+t,-1-5t,3+6t\rangle$$

Once we have this equation the other two forms follow. Here are the parametric equations of the line.

$$x = 2 + t$$
$$y = -1 - 5t$$
$$z = 3 + 6t$$

Here is the symmetric form.

$$\frac{x-2}{1} = \frac{y+1}{-5} = \frac{z-3}{6}$$

Example 2 Determine if the line that passes through the point $(0,-3,8)$ and is parallel to the line given by $x = 10 + 3t$, $y = 12t$ and $z = -3 - t$ passes through the *xz*-plane. If it does give the coordinates of that point.

Solution
To answer this we will first need to write down the equation of the line. We know a point on the line and just need a parallel vector. We know that the new line must be parallel to the line given by the parametric equations in the problem statement. That means that any vector that is parallel to the given line must also be parallel to the new line.

Now recall that in the parametric form of the line the numbers multiplied by *t* are the components of the vector that is parallel to the line. Therefore, the vector,

$$\vec{v} = \langle 3, 12, -1 \rangle$$

is parallel to the given line and so must also be parallel to the new line.

The equation of new line is then,

$$\vec{r} = \langle 0,-3,8 \rangle + t\langle 3,12,-1 \rangle = \langle 3t, -3+12t, 8-t \rangle$$

If this line passes through the *xz*-plane then we know that the *y*-coordinate of that point must be zero. So, let's set the *y* component of the equation equal to zero and see if we can solve for *t*. If we can, this will give the value of *t* for which the point will pass through the *xz*-plane.

$$-3 + 12t = 0 \qquad \Rightarrow \qquad t = \frac{1}{4}$$

So, the line does pass through the *xz*-plane. To get the complete coordinates of the point all we need to do is plug $t = \frac{1}{4}$ into any of the equations. We'll use the vector form.

$$\vec{r} = \left\langle 3\left(\frac{1}{4}\right), -3 + 12\left(\frac{1}{4}\right), 8 - \frac{1}{4} \right\rangle = \left\langle \frac{3}{4}, 0, \frac{31}{4} \right\rangle$$

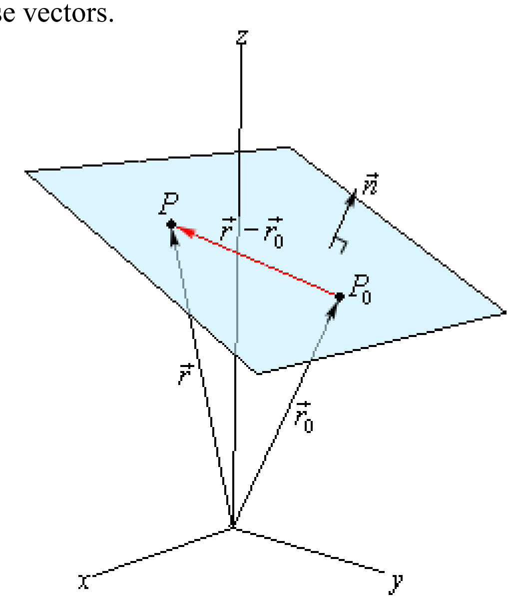
Recall that this vector is the position vector for the point on the line and so the coordinates of the point where the line will pass through the *xz*-plane are $\left(\frac{3}{4}, 0, \frac{31}{4}\right)$.

Equations of Planes

In the first section of this chapter we saw a couple of equations of planes. However, none of those equations had three variables in them and were really extensions of graphs that we could look at in two dimensions. We would like a more general equation for planes.

So, let's start by assuming that we know a point that is on the plane, $P_0 = (x_0, y_0, z_0)$. Let's also suppose that we have a vector that is orthogonal (perpendicular) to the plane, $\vec{n} = \langle a, b, c \rangle$. This vector is called the **normal vector**. Now, assume that $P = (x, y, z)$ is any point in the plane. Finally, since we are going to be working with vectors initially we'll let $\overrightarrow{r_0}$ and $\vec{r}$ be the position vectors for P_0 and P respectively.

Here is a sketch of all these vectors.



Notice that we added in the vector $\vec{r} - \overrightarrow{r_0}$ which will lie completely in the plane. Also notice that we put the normal vector on the plane, but there is actually no reason to expect this to be the case. We put it here to illustrate the point. It is completely possible that the normal vector does not touch the plane in any way.

Now, because $\vec{n}$ is orthogonal to the plane, it's also orthogonal to any vector that lies in the plane. In particular it's orthogonal to $\vec{r} - \overrightarrow{r_0}$. Recall from the Dot Product section that two orthogonal vectors will have a dot product of zero. In other words,

$$\vec{n} \bullet \left(\vec{r} - \overrightarrow{r_0} \right) = 0 \qquad \Rightarrow \qquad \vec{n} \bullet \vec{r} = \vec{n} \bullet \overrightarrow{r_0}$$

This is called the **vector equation of the plane**.

A slightly more useful form of the equations is as follows. Start with the first form of the vector equation and write down a vector for the difference.

$$\langle a,b,c\rangle \bullet \left(\langle x,y,z\rangle - \langle x_0,y_0,z_0\rangle\right) = 0$$
$$\langle a,b,c\rangle \bullet \langle x-x_0, y-y_0, z-z_0\rangle = 0$$

Now, actually compute the dot product to get,

$$a\left(x-x_0\right)+b\left(y-y_0\right)+c\left(z-z_0\right)=0$$

This is called the **scalar equation of plane**. Often this will be written as,

$$ax+by+cz=d$$

where $d = ax_0 + by_0 + cz_0$.

This second form is often how we are given equations of planes. Notice that if we are given the equation of a plane in this form we can quickly get a normal vector for the plane. A normal vector is,

$$\vec{n} = \langle a,b,c\rangle$$

Let's work a couple of examples.

Example 1 Determine the equation of the plane that contains the points $P=\left(1,-2,0\right)$, $Q=\left(3,1,4\right)$ and $R=\left(0,-1,2\right)$.

Solution

In order to write down the equation of plane we need a point (we've got three so we're cool there) and a normal vector. We need to find a normal vector. Recall however, that we saw how to do this in the Cross Product section.

We can form the following two vectors from the given points.

$$\overrightarrow{PQ} = \langle 2,3,4\rangle \qquad \overrightarrow{PR} = \langle -1,1,2\rangle$$

These two vectors will lie completely in the plane since we formed them from points that were in the plane. Notice as well that there are many possible vectors to use here, we just chose two of the possibilities.

Now, we know that the cross product of two vectors will be orthogonal to both of these vectors. Since both of these are in the plane any vector that is orthogonal to both of these will also be orthogonal to the plane. Therefore, we can use the cross product as the normal vector.

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 4 \\ -1 & 1 & 2 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ 2 & 3 \\ -1 & 1 \end{matrix} = 2\vec{i} - 8\vec{j} + 5\vec{k}$$

The equation of the plane is then,

$$2(x-1)-8(y+2)+5(z-0)=0$$
$$2x-8y+5z=18$$

We used *P* for the point, but could have used any of the three points.

Example 2 Determine if the plane given by $-x+2z=10$ and the line given by $\vec{r}=\langle 5,2-t,10+4t\rangle$ are orthogonal, parallel or neither.

Solution

This is not as difficult a problem as it may at first appear to be. We can pick off a vector that is normal to the plane. This is $\vec{n}=\langle -1,0,2\rangle$. We can also get a vector that is parallel to the line. This is $v=\langle 0,-1,4\rangle$.

Now, if these two vectors are parallel then the line and the plane will be orthogonal. If you think about it this makes some sense. If $\vec{n}$ and $\vec{v}$ are parallel, then $\vec{v}$ is orthogonal to the plane, but $\vec{v}$ is also parallel to the line. So, if the two vectors are parallel the line and plane will be orthogonal.

Let's check this.

$$\vec{n}\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -1 & 0 & 2\\ 0 & -1 & 4\end{vmatrix}\begin{matrix}\vec{i} & \vec{j}\\ -1 & 0\\ 0 & -1\end{matrix}=2\vec{i}+4\vec{j}+\vec{k}\neq\vec{0}$$

So, the vectors aren't parallel and so the plane and the line are not orthogonal.

Now, let's check to see if the plane and line are parallel. If the line is parallel to the plane then any vector parallel to the line will be orthogonal to the normal vector of the plane. In other words, if $\vec{n}$ and $\vec{v}$ are orthogonal then the line and the plane will be parallel.

Let's check this.

$$\vec{n}\bullet\vec{v}=0+0+8=8\neq 0$$

The two vectors aren't orthogonal and so the line and plane aren't parallel.

So, the line and the plane are neither orthogonal nor parallel.

Quadric Surfaces

In the previous two sections we've looked at lines and planes in three dimensions (or $\mathbb{R}^3$) and while these are used quite heavily at times in a Calculus class there are many other surfaces that are also used fairly regularly and so we need to take a look at those.

In this section we are going to be looking at quadric surfaces. Quadric surfaces are the graphs of any equation that can be put into the general form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

where A, … , J are constants.

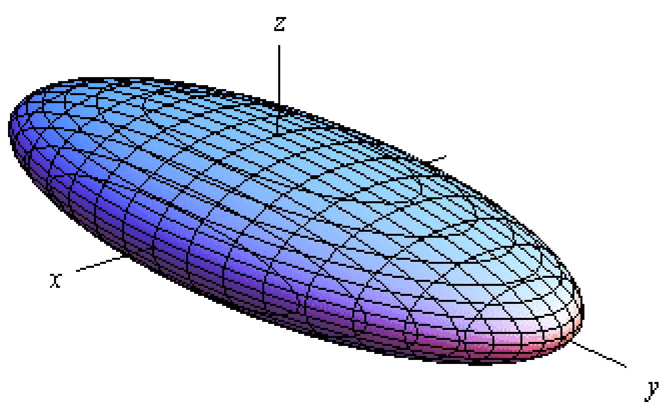
There is no way that we can possibly list all of them, but there are some standard equations so here is a list of some of the more common quadric surfaces.

Ellipsoid

Here is the general equation of an ellipsoid.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Here is a sketch of a typical ellipsoid.



If $a = b = c$ then we will have a sphere.

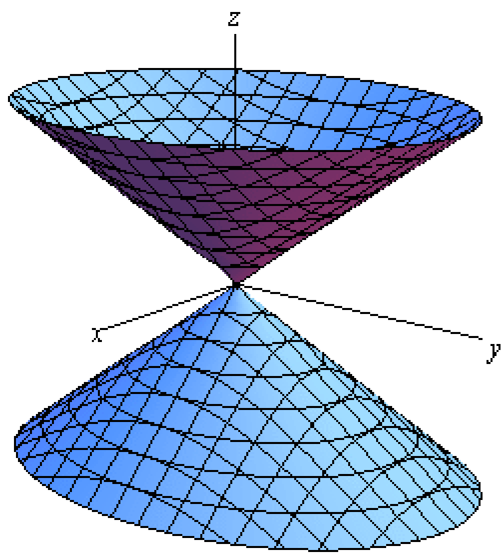
Notice that we only gave the equation for the ellipsoid that has been centered on the origin. Clearly ellipsoids don't have to be centered on the origin. However, in order to make the discussion in this section a little easier we have chosen to concentrate on surfaces that are "centered" on the origin in one way or another.

Cone

Here is the general equation of a cone.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Here is a sketch of a typical cone.



Note that this is the equation of a cone that will open along the z-axis. To get the equation of a cone that opens along one of the other axes all we need to do is make a slight modification of the equation. This will be the case for the rest of the surfaces that we'll be looking at in this section as well.

In the case of a cone the variable that sits by itself on one side of the equal sign will determine the axis that the cone opens up along. For instance, a cone that opens up along the x-axis will have the equation,

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{x^2}{a^2}$$

For most of the following surfaces we will not give the other possible formulas. We will however acknowledge how each formula needs to be changed to get a change of orientation for the surface.

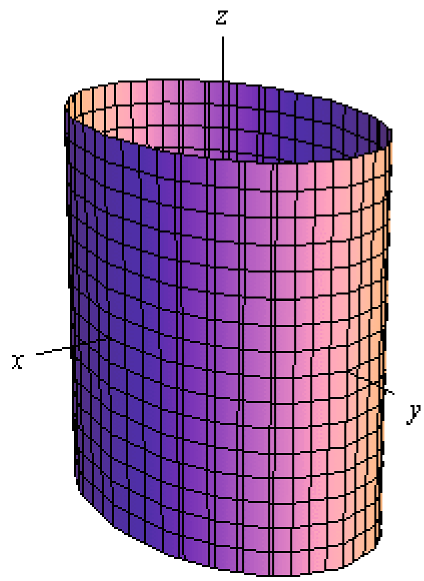
Cylinder
Here is the general equation of a cylinder.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This is a cylinder whose cross section is an ellipse. If $a = b$ we have a cylinder whose cross section is a circle. We'll be dealing with those kinds of cylinders more than the general form so the equation of a cylinder with a circular cross section is,

$$x^2 + y^2 = r^2$$

Here is a sketch of typical cylinder with an ellipse cross section.



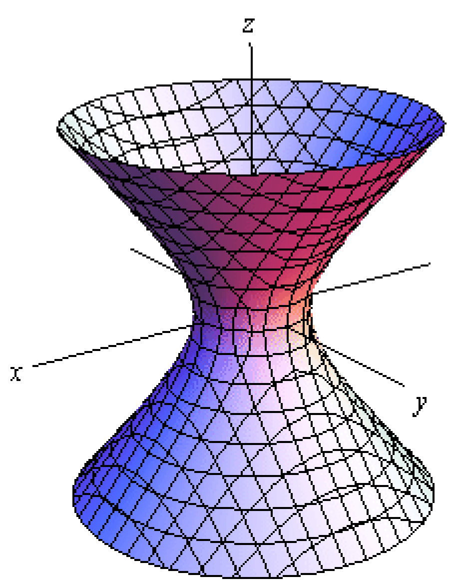
The cylinder will be centered on the axis corresponding to the variable that does not appear in the equation.

Be careful to not confuse this with a circle. In two dimensions it is a circle, but in three dimensions it is a cylinder.

Hyperboloid of One Sheet
Here is the equation of a hyperboloid of one sheet.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

Here is a sketch of a typical hyperboloid of one sheet.

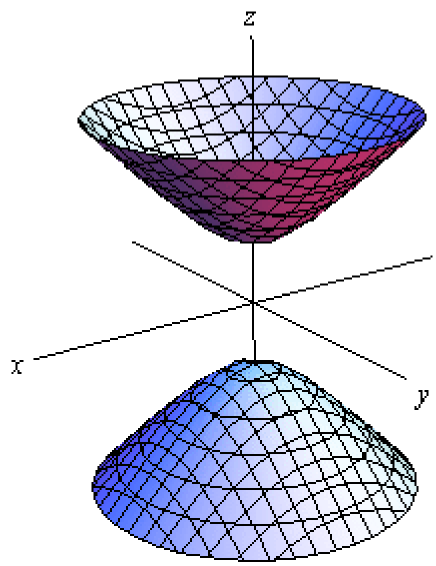


The variable with the negative in front of it will give the axis along which the graph is centered.

Hyperboloid of Two Sheets
Here is the equation of a hyperboloid of two sheets.

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Here is a sketch of a typical hyperboloid of two sheets.



The variable with the positive in front of it will give the axis along which the graph is centered.

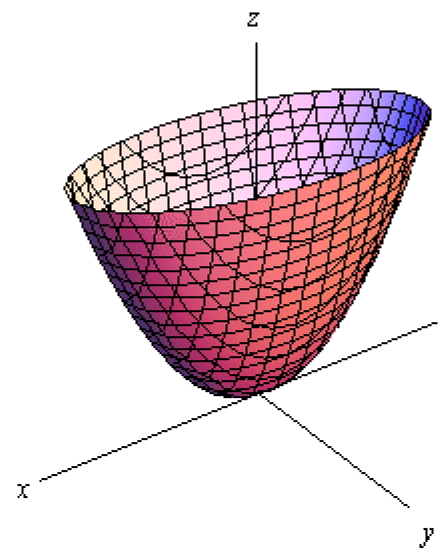
Notice that the only difference between the hyperboloid of one sheet and the hyperboloid of two sheets is the signs in front of the variables. They are exactly the opposite signs.

Elliptic Paraboloid
Here is the equation of an elliptic paraboloid.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$$

As with cylinders this has a cross section of an ellipse and if $a=b$ it will have a cross section of a circle. When we deal with these we'll generally be dealing with the kind that have a circle for a cross section.

Here is a sketch of a typical elliptic paraboloid.

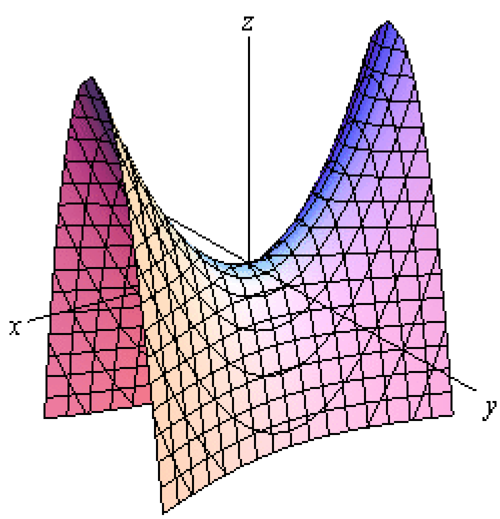


In this case the variable that isn't squared determines the axis upon which the paraboloid opens up. Also, the sign of c will determine the direction that the paraboloid opens. If c is positive then it opens up and if c is negative then it opens down.

Hyperbolic Paraboloid
Here is the equation of a hyperbolic paraboloid.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$$

Here is a sketch of a typical hyperbolic paraboloid.



These graphs are vaguely saddle shaped and as with the elliptic paraoloid the sign of c will determine the direction in which the surface "opens up". The graph above is shown for c positive.

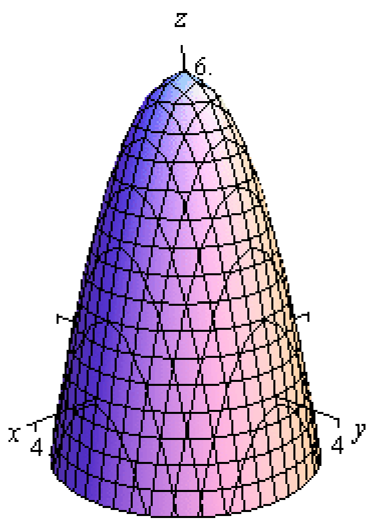
With the both of the types of paraboloids discussed above the surface can be easily moved up or down by adding/subtracting a constant from the left side.

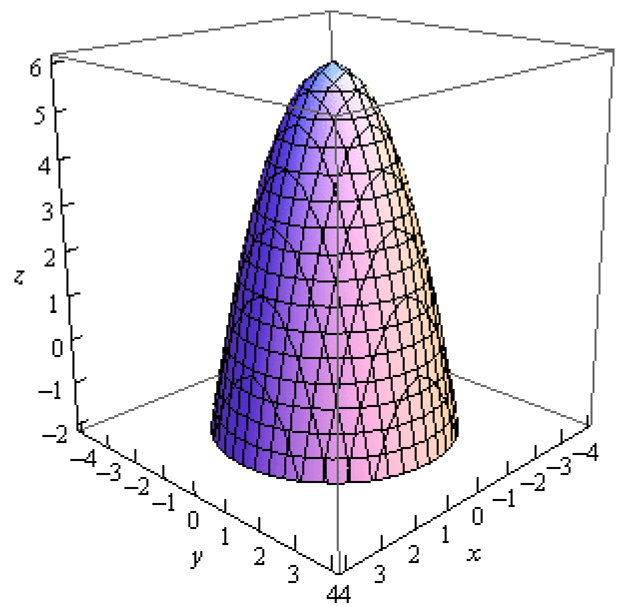
For instance

$$z = -x^2 - y^2 + 6$$

is an elliptic paraboloid that opens downward (be careful, the "-" is on the *x* and *y* instead of the *z*) and starts at $z = 6$ instead of $z = 0$.

Here are a couple of quick sketches of this surface.



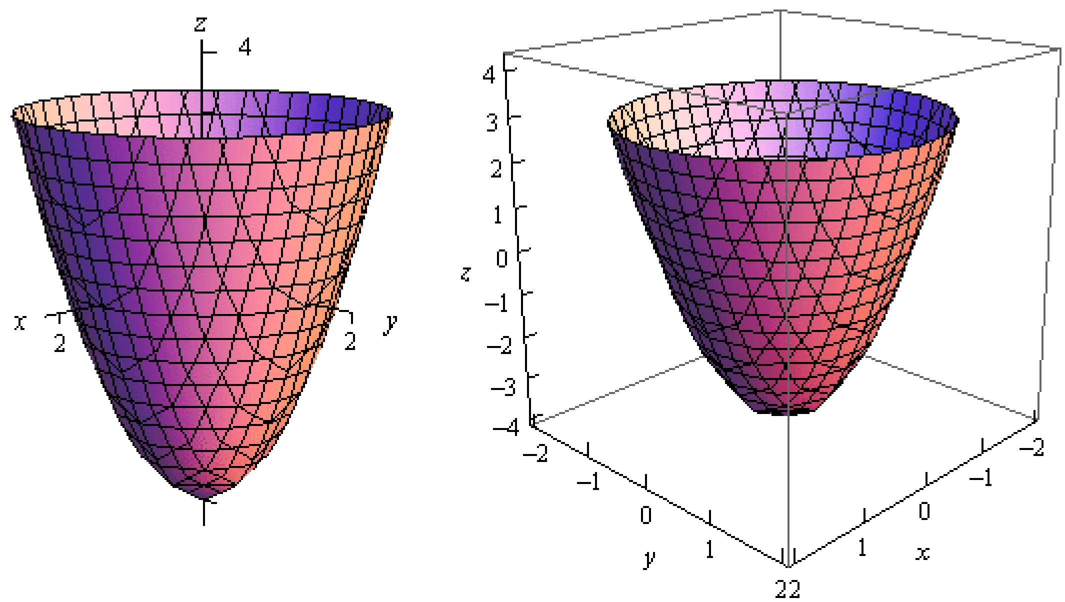


Note that we've given two forms of the sketch here. The sketch on the right has the standard set of axes but it is difficult to see the numbers on the axis. The sketch on the left has been "boxed" and this makes it easier to see the numbers to give a sense of perspective to the sketch. In most sketches that actually involve numbers on the axis system we will give both sketches to help get a feel for what the sketch looks like.

Functions of Several Variables

In this section we want to go over some of the basic ideas about functions of more than one variable.

First, remember that graphs of functions of two variables, $z = f(x, y)$ are surfaces in three dimensional space. For example here is the graph of $z = 2x^2 + 2y^2 - 4$.



This is an elliptic parabaloid and is an example of a quadric surface. We saw several of these in the previous section. We will be seeing quadric surfaces fairly regularly later on in Calculus III.

Another common graph that we'll be seeing quite a bit in this course is the graph of a plane. We have a convention for graphing planes that will make them a little easier to graph and hopefully visualize.

Recall that the equation of a plane is given by

$$ax + by + cz = d$$

or if we solve this for z we can write it in terms of function notation. This gives,

$$f(x, y) = Ax + By + D$$

To graph a plane we will generally find the intersection points with the three axes and then graph the triangle that connects those three points. This triangle will be a portion of the plane and it will give us a fairly decent idea on what the plane itself should look like. For example let's graph the plane given by,

$$f(x, y) = 12 - 3x - 4y$$

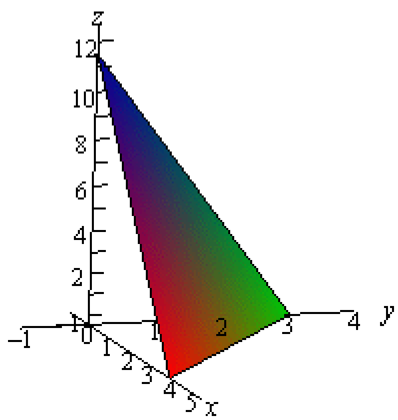
For purposes of graphing this it would probably be easier to write this as,

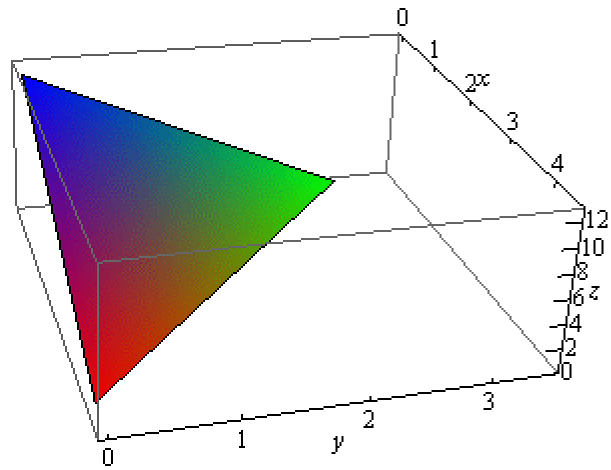
$$z = 12 - 3x - 4y \qquad \Rightarrow \qquad 3x + 4y + z = 12$$

Now, each of the intersection points with the three main coordinate axes is defined by the fact that two of the coordinates are zero. For instance, the intersection with the *z*-axis is defined by $x = y = 0$. So, the three intersection points are,

$$\begin{aligned} x-\text{axis} &: \left(4,0,0\right) \\ y-\text{axis} &: \left(0,3,0\right) \\ z-\text{axis} &: \left(0,0,12\right) \end{aligned}$$

Here is the graph of the plane.





Now, to extend this out, graphs of functions of the form $w = f\left(x, y, z\right)$ would be four dimensional surfaces. Of course we can't graph them, but it doesn't hurt to point this out.

We next want to talk about the domains of functions of more than one variable. Recall that domains of functions of a single variable, $y = f\left(x\right)$, consisted of all the values of *x* that we could plug into the function and get back a real number. Now, if we think about it, this means that the domain of a function of a single variable is an interval (or intervals) of values from the number line, or one dimensional space.

The domain of functions of two variables, $z = f\left(x, y\right)$, are regions from two dimensional space and consist of all the coordinate pairs, $\left(x, y\right)$, that we could plug into the function and get back a real number.

Example 1 Determine the domain of each of the following.

(a) $f(x,y)=\sqrt{x+y}$ [Solution]

(b) $f(x,y)=\sqrt{x}+\sqrt{y}$ [Solution]

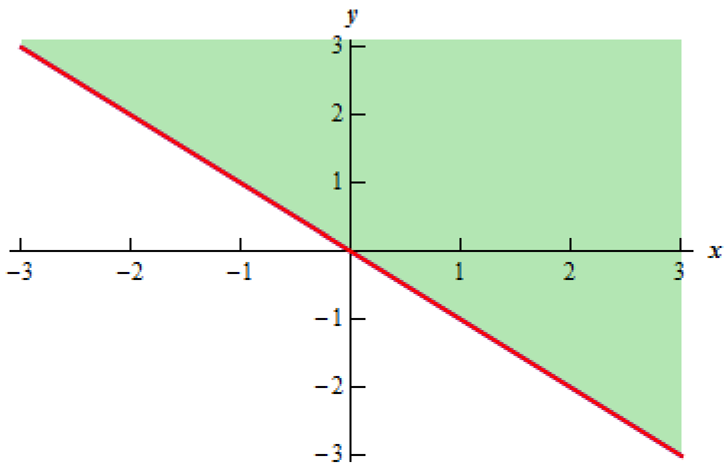
(c) $f(x,y)=\ln\left(9-x^2-9y^2\right)$ [Solution]

Solution

(a) In this case we know that we can't take the square root of a negative number so this means that we must require,

$$x+y\geq 0$$

Here is a sketch of the graph of this region.

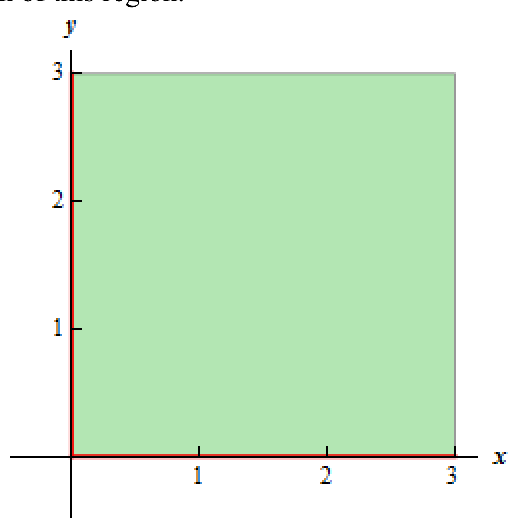


[Return to Problems]

(b) This function is different from the function in the previous part. Here we must require that,

$$x\geq 0 \quad \text{and} \quad y\geq 0$$

and they really do need to be separate inequalities. There is one for each square root in the function. Here is the sketch of this region.

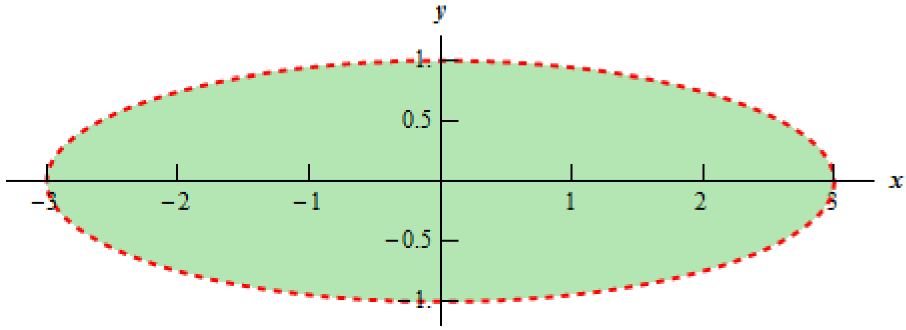


[Return to Problems]

(c) In this final part we know that we can't take the logarithm of a negative number or zero. Therefore we need to require that,

$$9 - x^2 - 9y^2 > 0 \qquad \Rightarrow \qquad \frac{x^2}{9} + y^2 < 1$$

and upon rearranging we see that we need to stay interior to an ellipse for this function. Here is a sketch of this region.



[Return to Problems]

Note that domains of functions of three variables, $w = f(x, y, z)$, will be regions in three dimensional space.

Example 2 Determine the domain of the following function,

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2 - 16}}$$

Solution

In this case we have to deal with the square root and division by zero issues. These will require,

$$x^2 + y^2 + z^2 - 16 > 0 \qquad \Rightarrow \qquad x^2 + y^2 + z^2 > 16$$

So, the domain for this function is the set of points that lies completely outside a sphere of radius 4 centered at the origin.

The next topic that we should look at is that of **level curves** or **contour curves**. The level curves of the function $z = f(x, y)$ are two dimensional curves we get by setting $z = k$, where k is any number. So the equations of the level curves are $f(x, y) = k$. Note that sometimes the equation will be in the form $f(x, y, z) = 0$ and in these cases the equations of the level curves are $f(x, y, k) = 0$.

You've probably seen level curves (or contour curves, whatever you want to call them) before. If you've ever seen the elevation map for a piece of land, this is nothing more than the contour curves for the function that gives the elevation of the land in that area. Of course, we probably don't have the function that gives the elevation, but we can at least graph the contour curves.

Let's do a quick example of this.

Example 3 Identify the level curves of $f(x,y)=\sqrt{x^2+y^2}$. Sketch a few of them.

Solution

First, for the sake of practice, let's identify what this surface given by $f(x,y)$ is. To do this let's rewrite it as,

$$z=\sqrt{x^2+y^2}$$

Now, this equation is not listed in the Quadric Surfaces section, but if we square both sides we get,

$$z^2=x^2+y^2$$

and this is listed in that section. So, we have a cone, or at least a portion of a cone. Since we know that square roots will only return positive numbers, it looks like we've only got the upper half of a cone.

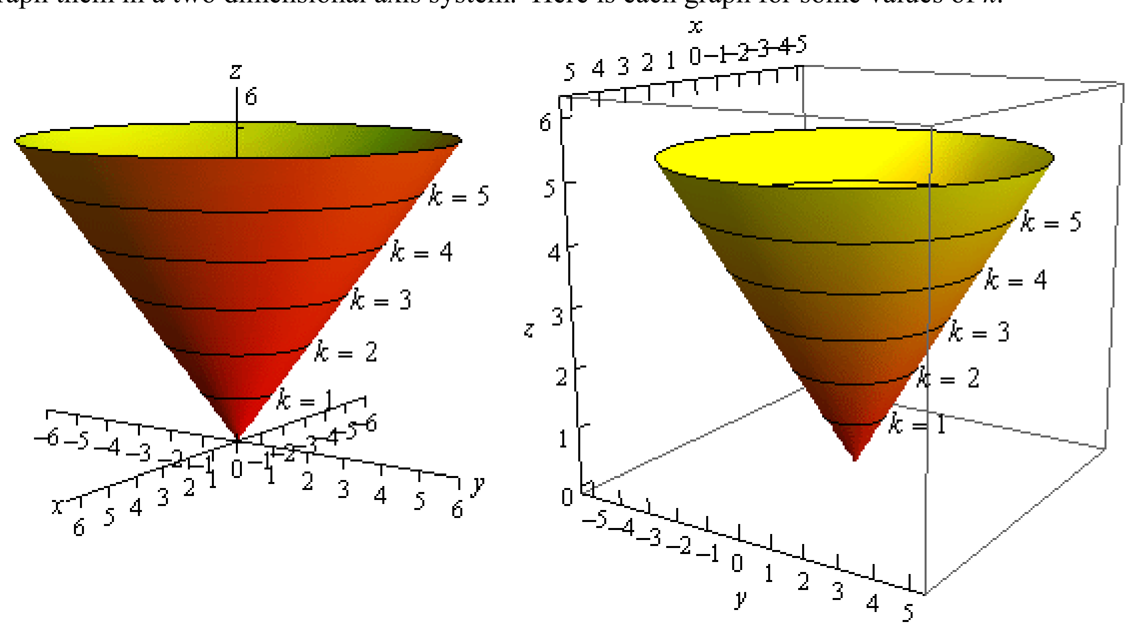
Note that this was not required for this problem. It was done for the practice of identifying the surface and this may come in handy down the road.

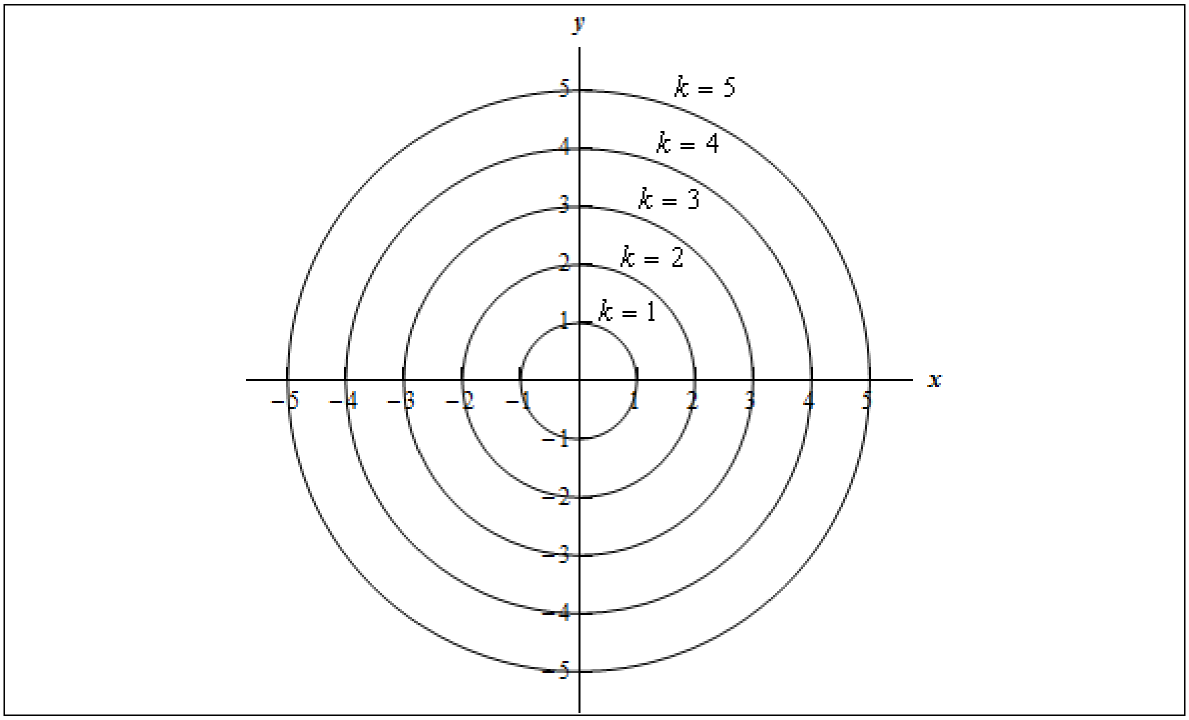
Now on to the real problem. The level curves (or contour curves) for this surface are given by the equation are found by substituting $z=k$. In the case of our example this is,

$$k=\sqrt{x^2+y^2} \qquad \Rightarrow \qquad x^2+y^2=k^2$$

where k is any number. So, in this case, the level curves are circles of radius k with center at the origin.

We can graph these in one of two ways. We can either graph them on the surface itself or we can graph them in a two dimensional axis system. Here is each graph for some values of k.





Note that we can think of contours in terms of the intersection of the surface that is given by $z = f(x, y)$ and the plane $z = k$. The contour will represent the intersection of the surface and the plane.

For functions of the form $f(x, y, z)$ we will occasionally look at **level surfaces**. The equations of level surfaces are given by $f(x, y, z) = k$ where k is any number.

The final topic in this section is that of **traces**. In some ways these are similar to contours. As noted above we can think of contours as the intersection of the surface given by $z = f(x, y)$ and the plane $z = k$. Traces of surfaces are curves that represent the intersection of the surface and the plane given by $x = a$ or $y = b$.

Let's take a quick look at an example of traces.

Example 4 Sketch the traces of $f(x, y) = 10 - 4x^2 - y^2$ for the plane $x = 1$ and $y = 2$.

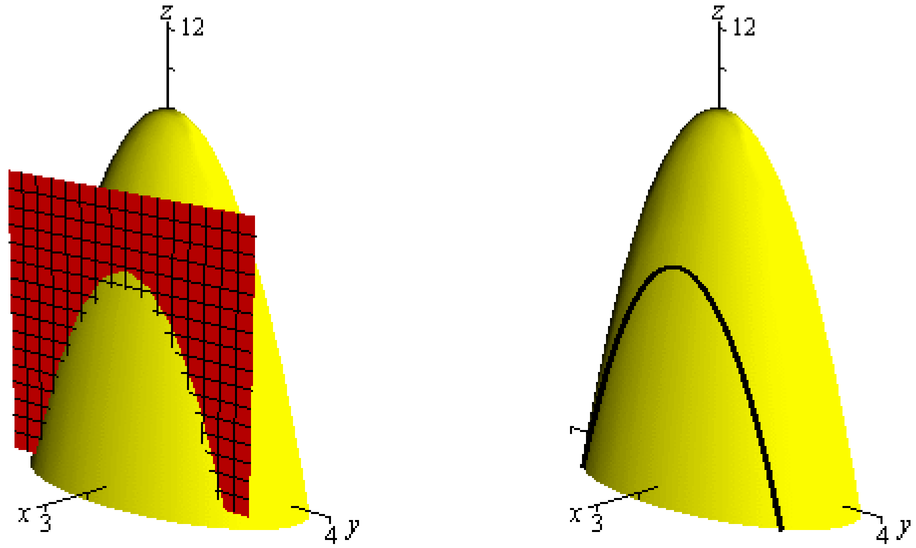
Solution

We'll start with $x = 1$. We can get an equation for the trace by plugging $x = 1$ into the equation. Doing this gives,

$$z = f(1, y) = 10 - 4(1)^2 - y^2 \qquad \Rightarrow \qquad z = 6 - y^2$$

and this will be graphed in the plane given by $x = 1$.

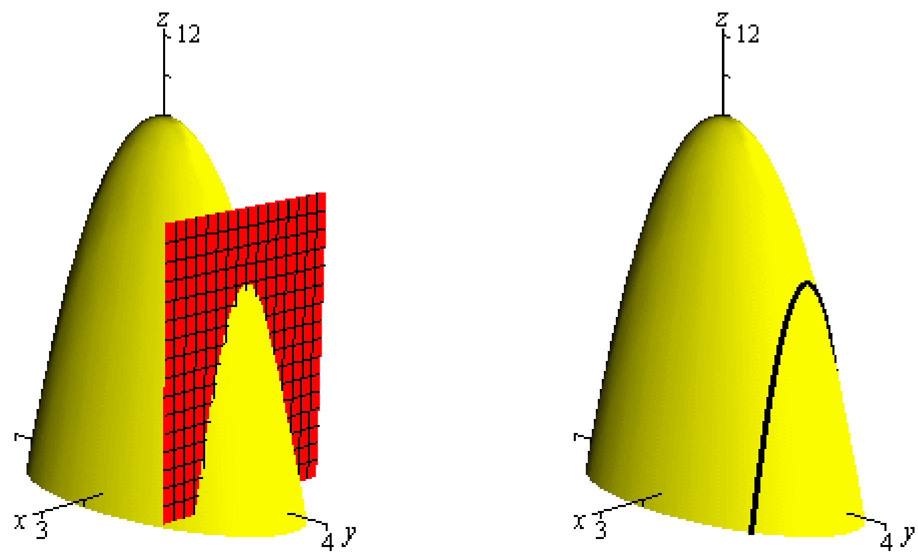
Below are two graphs. The graph on the left is a graph showing the intersection of the surface and the plane given by $x = 1$. On the right is a graph of the surface and the trace that we are after in this part.



For $y = 2$ we will do pretty much the same thing that we did with the first part. Here is the equation of the trace,

$$z = f(x, 2) = 10 - 4x^2 - (2)^2 \qquad \Rightarrow \qquad z = 6 - 4x^2$$

and here are the sketches for this case.



Vector Functions

We first saw vector functions back when we were looking at the Equation of Lines. In that section we talked about them because we wrote down the equation of a line in $\mathbb{R}^3$ in terms of a **vector function** (sometimes called a **vector-valued function**). In this section we want to look a little closer at them and we also want to look at some vector functions in $\mathbb{R}^3$ other than lines.

A vector function is a function that takes one or more variables and returns a vector. We'll spend most of this section looking at vector functions of a single variable as most of the places where vector functions show up here will be vector functions of single variables. We will however briefly look at vector functions of two variables at the end of this section.

A vector functions of a single variable in $\mathbb{R}^2$ and $\mathbb{R}^3$ have the form,

$$\vec{r}(t) = \langle f(t), g(t) \rangle \qquad\qquad \vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

respectively, where $f(t)$, $g(t)$ and $h(t)$ are called the **component functions**.

The main idea that we want to discuss in this section is that of graphing and identifying the graph given by a vector function. Before we do that however, we should talk briefly about the domain of a vector function. The **domain** of a vector function is the set of all *t*'s for which all the component functions are defined.

Example 1 Determine the domain of the following function.

$$\vec{r}(t) = \left\langle \cos t, \ln(4-t), \sqrt{t+1} \right\rangle$$

Solution

The first component is defined for all *t*'s. The second component is only defined for $t < 4$. The third component is only defined for $t \geq -1$. Putting all of these together gives the following domain.

$$[-1, 4)$$

This is the largest possible interval for which all three components are defined.

Let's now move into looking at the graph of vector functions. In order to graph a vector function all we do is think of the vector returned by the vector function as a position vector for points on the graph. Recall that a position vector, say $\vec{v} = \langle a, b, c \rangle$, is a vector that starts at the origin and ends at the point (a, b, c).

So, in order to sketch the graph of a vector function all we need to do is plug in some values of *t* and then plot points that correspond to the resulting position vector we get out of the vector function.

Because it is a little easier to visualize things we'll start off by looking at graphs of vector functions in $\mathbb{R}^2$.

Example 2 Sketch the graph of each of the following vector functions.

(a) $\vec{r}(t) = \langle t, 1 \rangle$ [Solution]

(b) $\vec{r}(t) = \langle t, t^3 - 10t + 7 \rangle$ [Solution]

Solution

(a) $\vec{r}(t) = \langle t, 1 \rangle$

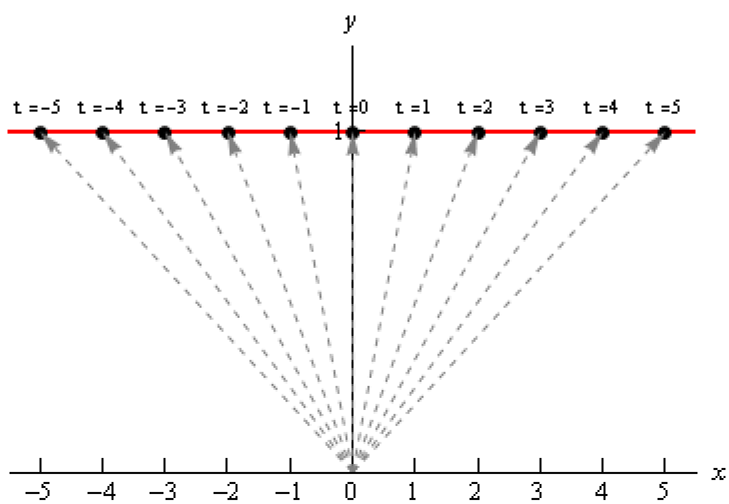
Okay, the first thing that we need to do is plug in a few values of *t* and get some position vectors. Here are a few,

$$\vec{r}(-3) = \langle -3, 1 \rangle \qquad \vec{r}(-1) = \langle -1, 1 \rangle \qquad \vec{r}(2) = \langle 2, 1 \rangle \qquad \vec{r}(5) = \langle 5, 1 \rangle$$

So, what this tells us is that the following points are all on the graph of this vector function.

$$(-3,1) \qquad (-1,1) \qquad (2,1) \qquad (5,1)$$

Here is a sketch of this vector function.



In this sketch we've included many more evaluations that just those above. Also note that we've put in the position vectors (in gray and dashed) so you can see how all this is working. Note however, that in practice the position vectors are generally not included in the sketch.

In this case it looks like we've got the graph of the line $y = 1$.

[Return to Problems]

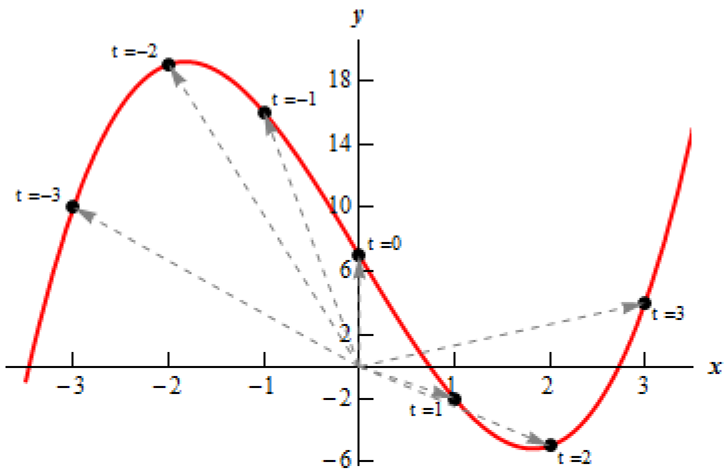
(b) $\vec{r}(t) = \langle t, t^3 - 10t + 7 \rangle$

Here are a couple of evaluations for this vector function.

$$\vec{r}(-3) = \langle -3, 10 \rangle \qquad \vec{r}(-1) = \langle -1, 16 \rangle \qquad \vec{r}(1) = \langle 1, -2 \rangle \qquad \vec{r}(3) = \langle 3, 4 \rangle$$

So, we've got a few points on the graph of this function. However, unlike the first part this isn't really going to be enough points to get a good idea of this graph. In general, it can take quite a few function evaluations to get an idea of what the graph is and it's usually easier to use a computer to do the graphing.

Here is a sketch of this graph. We've put in a few vectors/evaluations to illustrate them, but the reality is that we did have to use a computer to get a good sketch here.



[Return to Problems]

Both of the vector functions in the above example were in the form,

$$\vec{r}\left(t\right)=\left\langle t,g\left(t\right)\right\rangle$$

and what we were really sketching is the graph of $y=g\left(x\right)$ as you probably caught onto. Let's graph a couple of other vector functions that do not fall into this pattern.

Example 3 Sketch the graph of each of the following vector functions.

(a) $\vec{r}\left(t\right)=\left\langle 6\cos t,3\sin t\right\rangle$ [Solution]

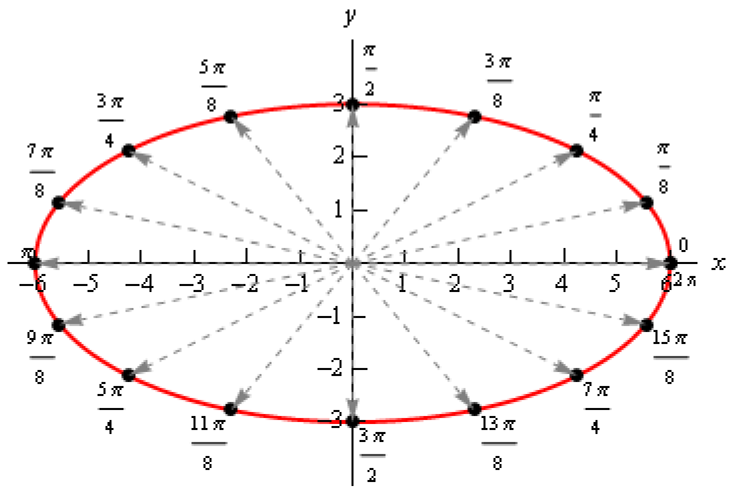
(b) $\vec{r}\left(t\right)=\left\langle t-2\sin t,t^2\right\rangle$ [Solution]

Solution

As we saw in the last part of the previous example it can really take quite a few function evaluations to really be able to sketch the graph of a vector function. Because of that we'll be skipping all the function evaluations here and just giving the graph. The main point behind this set of examples is to not get you too locked into the form we were looking at above. The first part will also lead to an important idea that we'll discuss after this example.

So, with that said here are the sketches of each of these.

(a) $\vec{r}\left(t\right)=\left\langle 6\cos t, 3\sin t\right\rangle$

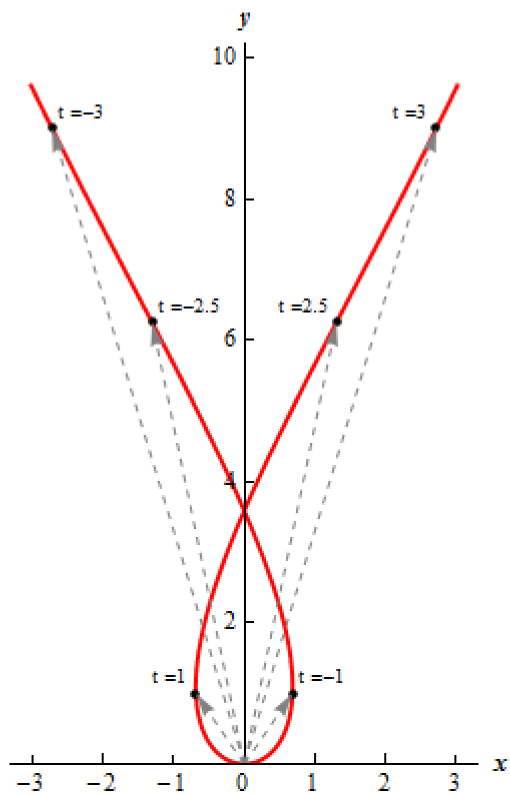


So, in this case it looks like we've got an ellipse.

[Return to Problems]

(b) $\vec{r}\left(t\right)=\left\langle t-2\sin t, t^2\right\rangle$

Here's the sketch for this vector function.



[Return to Problems]

Before we move on to vector functions in $\mathbb{R}^3$ let's go back and take a quick look at the first vector function we sketched in the previous example, $\vec{r}\left(t\right)=\left\langle 6\cos t, 3\sin t\right\rangle$. The fact that we got an ellipse here should not come as a surprise to you. We know that the first component function gives the x coordinate and the second component function gives the y coordinates of the point that we graph. If we strip these out to make this clear we get,

$$x = 6\cos t \qquad\qquad y = 3\sin t$$

This should look familiar to you. Back when we were looking at Parametric Equations we saw that this was nothing more than one of the sets of parametric equations that gave an ellipse.

This is an important idea in the study of vector functions. Any vector function can be broken down into a set of parametric equations that represent the same graph. In general, the two dimensional vector function, $\vec{r}\left(t\right) = \left\langle f\left(t\right), g\left(t\right)\right\rangle$, can be broken down into the parametric equations,

$$x = f\left(t\right) \qquad\qquad y = g\left(t\right)$$

Likewise, a three dimensional vector function, $\vec{r}\left(t\right) = \left\langle f\left(t\right), g\left(t\right), h\left(t\right)\right\rangle$, can be broken down into the parametric equations,

$$x = f\left(t\right) \qquad y = g\left(t\right) \qquad z = h\left(t\right)$$

Do not get too excited about the fact that we're now looking at parametric equations in $\mathbb{R}^3$. They work in exactly the same manner as parametric equations in $\mathbb{R}^2$ which we're used to dealing with already. The only difference is that we now have a third component.

Let's take a look at a couple of graphs of vector functions.

Example 4 Sketch the graph of the following vector function.

$$\vec{r}\left(t\right) = \left\langle 2-4t, -1+5t, 3+t\right\rangle$$

Solution

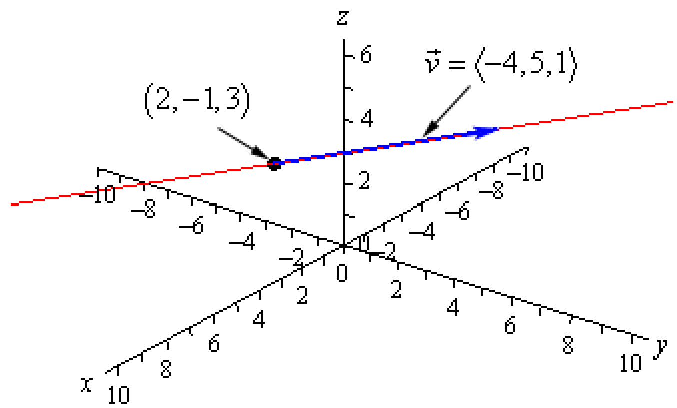
Notice that this is nothing more than a line. It might help if we rewrite it a little.

$$\vec{r}\left(t\right) = \left\langle 2,-1,3\right\rangle + t\left\langle -4,5,1\right\rangle$$

In this form we can see that this is the equation of a line that goes through the point $\left(2,-1,3\right)$ and is parallel to the vector $\vec{v} = \left\langle -4,5,1\right\rangle$.

To graph this line all that we need to do is plot the point and then sketch in the parallel vector. In order to get the sketch will assume that the vector is on the line and will start at the point in the line. To sketch in the line all we do this is extend the parallel vector into a line.

Here is a sketch.



Example 5 Sketch the graph of the following vector function.

$$\vec{r}(t) = \langle 2\cos t, 2\sin t, 3 \rangle$$

Solution

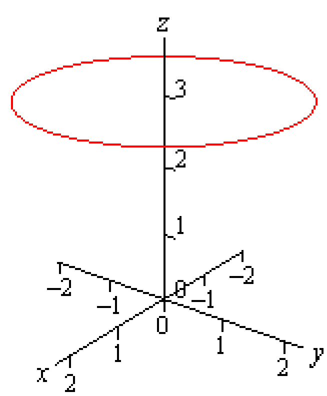
In this case to see what we've got for a graph let's get the parametric equations for the curve.

$$x = 2\cos t \qquad\qquad y = 2\sin t \qquad\qquad z = 3$$

If we ignore the *z* equation for a bit we'll recall (hopefully) that the parametric equations for *x* and *y* give a circle of radius 2 centered on the origin (or about the *z*-axis since we are in $\mathbb{R}^3$).

Now, all the parametric equations here tell us is that no matter what is going on in the graph all the *z* coordinates must be 3. So, we get a circle of radius 2 centered on the *z*-axis and at the level of $z = 3$.

Here is a sketch.



Note that it is very easy to modify the above vector function to get a circle centered on the *x* or *y*-axis as well. For instance,

$$\vec{r}(t) = \langle 10\sin t, -3, 10\cos t \rangle$$

will be a circle of radius 10 centered on the *y*-axis and at $y = -3$. In other words, as long as two of the terms are a sine and a cosine (with the same coefficient) and the other is a fixed number then we will have a circle that is centered on the axis that is given by the fixed number.

Let's take a look at a modification of this.

Example 6 Sketch the graph of the following vector function.

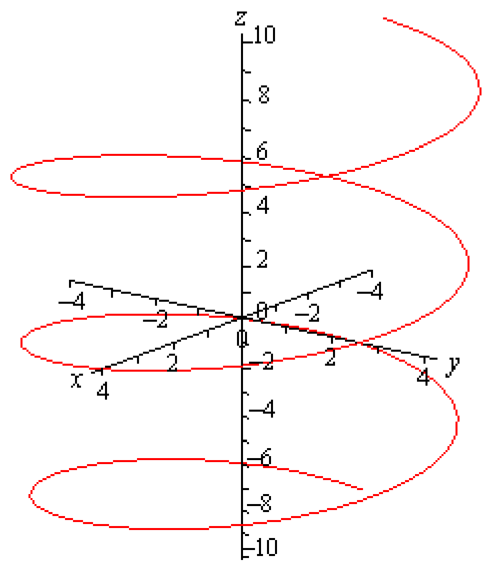
$$\vec{r}(t) = \langle 4\cos t, 4\sin t, t \rangle$$

Solution

If this one had a constant in the *z* component we would have another circle. However, in this case we don't have a constant. Instead we've got a *t* and that will change the curve. However, because the *x* and *y* component functions are still a circle in parametric equations our curve should have a circular nature to it in some way.

In fact, the only change is in the *z* component and as *t* increases the *z* coordinate will increase. Also, as *t* increases the *x* and *y* coordinates will continue to form a circle centered on the *z*-axis. Putting these two ideas together tells us that at we increase *t* the circle that is being traced out in the *x* and *y* directions should be also be rising.

Here is a sketch of this curve.



So, we've got a helix (or spiral, depending on what you want to call it) here.

As with circles the component that has the *t* will determine the axis that the helix rotates about. For instance,

$$\vec{r}(t) = \langle t, 6\cos t, 6\sin t \rangle$$

is a helix that rotates around the *x*-axis.

Also note that if we allow the coefficients on the sine and cosine for both the circle and helix to be different we will get ellipses.

For example,

$$\vec{r}(t) = \langle 9\cos t, t, 2\sin t \rangle$$

will be a helix that rotates about the y-axis and is in the shape of an ellipse.

There is a nice formula that we should derive before moving onto vector functions of two variables.

Example 7 Determine the vector equation for the line segment starting at the point $P = (x_1, y_1, z_1)$ and ending at the point $Q = (x_2, y_2, z_2)$.

Solution
It is important to note here that we only want the equation of the line segment that starts at P and ends at Q. We don't want any other portion of the line and we do want the direction of the line segment preserved as we increase t. With all that said, let's not worry about that and just find the vector equation of the line that passes through the two points. Once we have this we will be able to get what we're after.

So, we need a point on the line. We've got two and we will use P. We need a vector that is parallel to the line and since we've got two points we can find the vector between them. This vector will lie on the line and hence be parallel to the line. Also, let's remember that we want to preserve the starting and ending point of the line segment so let's construct the vector using the same "orientation".

$$\vec{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Using this vector and the point P we get the following vector equation of the line.

$$\vec{r}(t) = \langle x_1, y_1, z_1 \rangle + t \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

While this is the vector equation of the line, let's rewrite the equation slightly.

$$\begin{aligned} \vec{r}(t) &= \langle x_1, y_1, z_1 \rangle + t \langle x_2, y_2, z_2 \rangle - t \langle x_1, y_1, z_1 \rangle \\ &= (1-t) \langle x_1, y_1, z_1 \rangle + t \langle x_2, y_2, z_2 \rangle \end{aligned}$$

This is the equation of the line that contains the points P and Q. We of course just want the line segment that starts at P and ends at Q. We can get this by simply restricting the values of t.

Notice that

$$\vec{r}(0) = \langle x_1, y_1, z_1 \rangle \qquad\qquad \vec{r}(1) = \langle x_2, y_2, z_2 \rangle$$

So, if we restrict t to be between zero and one we will cover the line segment and we will start and end at the correct point.

So the vector equation of the line segment that starts at $P = (x_1, y_1, z_1)$ and ends at $Q = (x_2, y_2, z_2)$ is,

$$\vec{r}(t) = (1-t) \langle x_1, y_1, z_1 \rangle + t \langle x_2, y_2, z_2 \rangle \qquad\qquad 0 \le t \le 1$$

As noted briefly at the beginning of this section we can also have vector functions of two variables. In these case the graphs of vector function of two variables are surfaces. So, to make sure that we don't forget that let's work an example with that as well.

Example 8 Identify the surface that is described by $\vec{r}\left(x,y\right)=x\,\vec{i}+y\,\vec{j}+\left(x^2+y^2\right)\vec{k}$.

Solution
First, notice that in this case the vector function will in fact be a function of two variables. This will always be the case when we are using vector functions to represent surfaces.

To identify the surface let's go back to parametric equations.

$$x=x \qquad\qquad y=y \qquad\qquad z=x^2+y^2$$

The first two are really only acknowledging that we are picking x and y for free and then determining z from our choices of these two. The last equation is the one that we want. We should recognize that function from the section on quadric surfaces. The third equation is the equation of an elliptic paraboloid and so the vector function represents an elliptic paraboloid.

As a final topic for this section let's generalize the idea from the previous example and note that given any function of one variable ($y=f\left(x\right)$ or $x=h\left(y\right)$) or any function of two variables ($z=g\left(x,y\right), x=g\left(y,z\right)$, or $y=g\left(x,z\right)$) we can always write down a vector form of the equation.

For a function of one variable this will be,

$$\vec{r}\left(x\right)=x\,\vec{i}+f\left(x\right)\vec{j} \qquad\qquad \vec{r}\left(y\right)=h\left(y\right)\vec{i}+y\,\vec{j}$$

and for a function of two variables the vector form will be,

$$\vec{r}\left(x,y\right)=x\,\vec{i}+y\,\vec{j}+g\left(x,y\right)\vec{k} \qquad\qquad \vec{r}\left(y,z\right)=g\left(y,z\right)\vec{i}+y\,\vec{j}+z\,\vec{k}$$

$$\vec{r}\left(x,z\right)=x\,\vec{i}+g\left(x,z\right)\vec{j}+z\,\vec{k}$$

depending upon the original form of the function.

For example the hyperbolic paraboloid $y=2x^2-5z^2$ can be written as the following vector function.

$$\vec{r}\left(x,z\right)=x\,\vec{i}+\left(2x^2-5z^2\right)\vec{j}+z\,\vec{k}$$

This is a fairly important idea and we will be doing quite a bit of this kind of thing in Calculus III.

Calculus with Vector Functions

In this section we need to talk briefly about limits, derivatives and integrals of vector functions. As you will see, these behave in a fairly predictable manner. We will be doing all of the work in $\mathbb{R}^3$ but we can naturally extend the formulas/work in this section to $\mathbb{R}^n$ (*i.e.* *n*-dimensional space).

Let's start with limits. Here is the limit of a vector function.

$$\begin{aligned}\lim_{t\to a}\vec{r}\left(t\right)&=\lim_{t\to a}\left\langle f\left(t\right),g\left(t\right),h\left(t\right)\right\rangle\\&=\left\langle\lim_{t\to a}f\left(t\right),\lim_{t\to a}g\left(t\right),\lim_{t\to a}h\left(t\right)\right\rangle\\&=\lim_{t\to a}f\left(t\right)\vec{i}+\lim_{t\to a}g\left(t\right)\vec{j}+\lim_{t\to a}h\left(t\right)\vec{k}\end{aligned}$$

So, all that we do is take the limit of each of the component's functions and leave it as a vector.

Example 1 Compute $\lim_{t\to 1}\vec{r}\left(t\right)$ where $\vec{r}\left(t\right)=\left\langle t^3,\frac{\sin\left(3t-3\right)}{t-1},\mathbf{e}^{2t}\right\rangle$.

Solution
There really isn't all that much to do here.

$$\begin{aligned}\lim_{t\to 1}\vec{r}\left(t\right)&=\left\langle\lim_{t\to 1}t^3,\lim_{t\to 1}\frac{\sin\left(3t-3\right)}{t-1},\lim_{t\to 1}\mathbf{e}^{2t}\right\rangle\\&=\left\langle\lim_{t\to 1}t^3,\lim_{t\to 1}\frac{3\cos\left(3t-3\right)}{1},\lim_{t\to 1}\mathbf{e}^{2t}\right\rangle\\&=\left\langle 1,3,\mathbf{e}^2\right\rangle\end{aligned}$$

Notice that we had to use L'Hospital's Rule on the y component.

Now let's take care of derivatives and after seeing how limits work it shouldn't be too surprising that we have the following for derivatives.

$$\vec{r}'\left(t\right)=\left\langle f'\left(t\right),g'\left(t\right),h'\left(t\right)\right\rangle=f'\left(t\right)\vec{i}+g'\left(t\right)\vec{j}+h'\left(t\right)\vec{k}$$

Example 2 Compute $\vec{r}'\left(t\right)$ for $\vec{r}\left(t\right)=t^6\,\vec{i}+\sin\left(2t\right)\vec{j}-\ln\left(t+1\right)\vec{k}$.

Solution
There really isn't too much to this problem other than taking the derivatives.

$$\vec{r}'\left(t\right)=6t^5\,\vec{i}+2\cos\left(2t\right)\vec{j}-\frac{1}{t+1}\vec{k}$$

Most of the basic facts that we know about derivatives still hold however, just to make it clear here are some facts about derivatives of vector functions.

Facts

$$\frac{d}{dt}\left(\vec{u}+\vec{v}\right)=\vec{u}'+\vec{v}'$$
$$\left(c\vec{u}\right)'=c\,\vec{u}'$$
$$\frac{d}{dt}\left(f\left(t\right)\vec{u}\left(t\right)\right)=f'\left(t\right)\vec{u}\left(t\right)+f\left(t\right)\vec{u}'$$
$$\frac{d}{dt}\left(\vec{u}\bullet\vec{v}\right)=\vec{u}'\bullet\vec{v}+\vec{u}\bullet\vec{v}'$$
$$\frac{d}{dt}\left(\vec{u}\times\vec{v}\right)=\vec{u}'\times\vec{v}+\vec{u}\times\vec{v}'$$
$$\frac{d}{dt}\left(\vec{u}\left(f\left(t\right)\right)\right)=f'\left(t\right)\vec{u}'\left(f\left(t\right)\right)$$

There is also one quick definition that we should get out of the way so that we can use it when we need to.

A **smooth curve** is any curve for which $\vec{r}'\left(t\right)$ is continuous and $\vec{r}'\left(t\right)\neq 0$ for any *t* except possibly at the endpoints. A helix is a smooth curve, for example.

Finally, we need to discuss integrals of vector functions. Using both limits and derivatives as a guide it shouldn't be too surprising that we also have the following for integration for indefinite integrals

$$\int\vec{r}\left(t\right)=\left\langle\int f\left(t\right)dt,\int g\left(t\right)dt,\int h\left(t\right)dt\right\rangle+\vec{c}$$
$$\int\vec{r}\left(t\right)=\int f\left(t\right)dt\;\vec{i}+\int g\left(t\right)dt\;\vec{j}+\int h\left(t\right)dt\;\vec{k}+\vec{c}$$

and the following for definite integrals.

$$\int_a^b\vec{r}\left(t\right)dt=\left\langle\int_a^b f\left(t\right)dt,\int_a^b g\left(t\right)dt,\int_a^b h\left(t\right)dt\right\rangle$$
$$\int_a^b\vec{r}\left(t\right)dt=\int_a^b f\left(t\right)dt\;\vec{i}+\int_a^b g\left(t\right)dt\;\vec{j}+\int_a^b h\left(t\right)dt\;\vec{k}$$

With the indefinite integrals we put in a constant of integration to make sure that it was clear that the constant in this case needs to be a vector instead of a regular constant.

Also, for the definite integrals we will sometimes write it as follows,

$$\int_a^b\vec{r}\left(t\right)dt=\left(\left\langle\int f\left(t\right)dt,\int g\left(t\right)dt,\int h\left(t\right)dt\right\rangle\right)\Big|_a^b$$
$$\int_a^b\vec{r}\left(t\right)dt=\left(\int f\left(t\right)dt\;\vec{i}+\int g\left(t\right)dt\;\vec{j}+\int h\left(t\right)dt\;\vec{k}\right)\Big|_a^b$$

In other words, we will do the indefinite integral and then do the evaluation of the vector as a whole instead of on a component by component basis.

Example 3 Compute $\int \vec{r}\left(t\right)dt$ for $\vec{r}\left(t\right) = \left\langle \sin\left(t\right), 6, 4t \right\rangle$.

Solution

All we need to do is integrate each of the components and be done with it.

$$\int \vec{r}\left(t\right)dt = \left\langle -\cos\left(t\right), 6t, 2t^2 \right\rangle + \vec{c}$$

Example 4 Compute $\int_0^1 \vec{r}\left(t\right)dt$ for $\vec{r}\left(t\right) = \left\langle \sin\left(t\right), 6, 4t \right\rangle$.

Solution

In this case all that we need to do is reuse the result from the previous example and then do the evaluation.

$$\begin{aligned} \int_0^1 \vec{r}\left(t\right)dt &= \left(\left\langle -\cos\left(t\right), 6t, 2t^2 \right\rangle\right)\Big|_0^1 \\ &= \left\langle -\cos\left(1\right), 6, 2 \right\rangle - \left\langle -1, 0, 0 \right\rangle \\ &= \left\langle 1 - \cos\left(1\right), 6, 2 \right\rangle \end{aligned}$$

Tangent, Normal and Binormal Vectors

In this section we want to look at an application of derivatives for vector functions. Actually, there are a couple of applications, but they all come back to needing the first one.

In the past we've used the fact that the derivative of a function was the slope of the tangent line. With vector functions we get exactly the same result, with one exception.

Given the vector function, $\vec{r}\left(t\right)$, we call $\vec{r}'\left(t\right)$ the **tangent vector** provided it exists and provided $\vec{r}'\left(t\right) \neq \vec{0}$. The tangent line to $\vec{r}\left(t\right)$ at *P* is then the line that passes through the point *P* and is parallel to the tangent vector, $\vec{r}'\left(t\right)$. Note that we really do need to require $\vec{r}'\left(t\right) \neq \vec{0}$ in order to have a tangent vector. If we had $\vec{r}'\left(t\right) = \vec{0}$ we would have a vector that had no magnitude and so couldn't give us the direction of the tangent.

Also, provided $\vec{r}'\left(t\right) \neq \vec{0}$, the **unit tangent vector** to the curve is given by,

$$\vec{T}\left(t\right) = \frac{\vec{r}'\left(t\right)}{\left\|\vec{r}'\left(t\right)\right\|}$$

While, the components of the unit tangent vector can be somewhat messy on occasion there are times when we will need to use the unit tangent vector instead of the tangent vector.

Example 1 Find the general formula for the tangent vector and unit tangent vector to the curve given by $\vec{r}\left(t\right) = t^2\,\vec{i} + 2\sin t\,\vec{j} + 2\cos t\,\vec{k}$.

Solution
First, by general formula we mean that we won't be plugging in a specific *t* and so we will be finding a formula that we can use at a later date if we'd like to find the tangent at any point on the curve. With that said there really isn't all that much to do at this point other than to do the work.

Here is the tangent vector to the curve.

$$\vec{r}'\left(t\right) = 2t\,\vec{i} + 2\cos t\,\vec{j} - 2\sin t\,\vec{k}$$

To get the unit tangent vector we need the length of the tangent vector.

$$\begin{aligned}\left\|\vec{r}'\left(t\right)\right\| &= \sqrt{4t^2 + 4\cos^2 t + 4\sin^2 t}\\ &= \sqrt{4t^2+4}\end{aligned}$$

The unit tangent vector is then,

$$\begin{aligned}\vec{T}\left(t\right) &= \frac{1}{\sqrt{4t^2+4}}\left(2t\,\vec{i} + 2\cos t\,\vec{j} - 2\sin t\,\vec{k}\right)\\ &= \frac{2t}{\sqrt{4t^2+4}}\vec{i} + \frac{2\cos t}{\sqrt{4t^2+4}}\vec{j} - \frac{2\sin t}{\sqrt{4t^2+4}}\vec{k}\end{aligned}$$

Example 2 Find the vector equation of the tangent line to the curve given by $\vec{r}(t) = t^2\,\vec{i} + 2\sin t\,\vec{j} + 2\cos t\,\vec{k}$ at $t = \frac{\pi}{3}$.

Solution
First we need the tangent vector and since this is the function we were working with in the previous example we can just reuse the tangent vector from that example and plug in $t = \frac{\pi}{3}$.

$$\vec{r}'\left(\frac{\pi}{3}\right) = \frac{2\pi}{3}\vec{i} + 2\cos\left(\frac{\pi}{3}\right)\vec{j} - 2\sin\left(\frac{\pi}{3}\right)\vec{k} = \frac{2\pi}{3}\vec{i} + \vec{j} - \sqrt{3}\,\vec{k}$$

We'll also need the point on the line at $t = \frac{\pi}{3}$ so,

$$\vec{r}\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}\vec{i} + \sqrt{3}\,\vec{j} + \vec{k}$$

The vector equation of the line is then,

$$\vec{r}(t) = \left\langle \frac{\pi^2}{9}, \sqrt{3}, 1 \right\rangle + t\left\langle \frac{2\pi}{3}, 1, -\sqrt{3} \right\rangle$$

Before moving on let's note a couple of things about the previous example. First, we could have used the unit tangent vector had we wanted to for the parallel vector. However, that would have made for a more complicated equation for the tangent line.

Second, notice that we used $\vec{r}(t)$ to represent the tangent line despite the fact that we used that as well for the function. Do not get excited about that. The $\vec{r}(t)$ here is much like y is with normal functions. With normal functions, y is the generic letter that we used to represent functions and $\vec{r}(t)$ tends to be used in the same way with vector functions.

Next we need to talk about the **unit normal** and the **binormal** vectors.

The unit normal vector is defined to be,

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\left\|\vec{T}'(t)\right\|}$$

The unit normal is orthogonal (or normal, or perpendicular) to the unit tangent vector and hence to the curve as well. We've already seen normal vectors when we were dealing with Equations of Planes. They will show up with some regularity in several Calculus III topics.

The definition of the unit normal vector always seems a little mysterious when you first see it. It follows directly from the following fact.

Fact

Suppose that $\vec{r}(t)$ is a vector such that $\left\|\vec{r}(t)\right\| = c$ for all t. Then $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$.

To prove this fact is pretty simple. From the fact statement and the relationship between the magnitude of a vector and the dot product we have the following.

$$\vec{r}\left(t\right)\bullet\vec{r}\left(t\right)=\left\|\vec{r}\left(t\right)\right\|^{2}=c^{2}\qquad\text{for all } t$$

Now, because this is true for all t we can see that,

$$\frac{d}{dt}\left(\vec{r}\left(t\right)\bullet\vec{r}\left(t\right)\right)=\frac{d}{dt}\left(c^{2}\right)=0$$

Also, recalling the fact from the previous section about differentiating a dot product we see that,

$$\frac{d}{dt}\left(\vec{r}\left(t\right)\bullet\vec{r}\left(t\right)\right)=\vec{r}'\left(t\right)\bullet\vec{r}\left(t\right)+\vec{r}\left(t\right)\bullet\vec{r}'\left(t\right)=2\vec{r}'\left(t\right)\bullet\vec{r}\left(t\right)$$

Or, upon putting all this together we get,

$$2\vec{r}'\left(t\right)\bullet\vec{r}\left(t\right)=0\qquad\Rightarrow\qquad\vec{r}'\left(t\right)\bullet\vec{r}\left(t\right)=0$$

Therefore $\vec{r}'\left(t\right)$ is orthogonal to $\vec{r}\left(t\right)$.

The definition of the unit normal then falls directly from this. Because $\vec{T}\left(t\right)$ is a unit vector we know that $\left\|\vec{T}\left(t\right)\right\|=1$ for all t and hence by the Fact $\vec{T}'\left(t\right)$ is orthogonal to $\vec{T}\left(t\right)$. However, because $\vec{T}\left(t\right)$ is tangent to the curve, $\vec{T}'\left(t\right)$ must be orthogonal, or normal, to the curve as well and so be a normal vector for the curve. All we need to do then is divide by $\left\|\vec{T}'\left(t\right)\right\|$ to arrive at a unit normal vector.

Next, is the binormal vector. The binormal vector is defined to be,

$$\vec{B}\left(t\right)=\vec{T}\left(t\right)\times\vec{N}\left(t\right)$$

Because the binormal vector is defined to be the cross product of the unit tangent and unit normal vector we then know that the binormal vector is orthogonal to both the tangent vector and the normal vector.

Example 3 Find the normal and binormal vectors for $\vec{r}\left(t\right)=\left\langle t,3\sin t,3\cos t\right\rangle$.

Solution

We first need the unit tangent vector so first get the tangent vector and its magnitude.

$$\vec{r}'\left(t\right)=\left\langle 1,3\cos t,-3\sin t\right\rangle$$

$$\left\|\vec{r}'\left(t\right)\right\|=\sqrt{1+9\cos^{2}t+9\sin^{2}t}=\sqrt{10}$$

The unit tangent vector is then,

$$\vec{T}\left(t\right)=\left\langle\frac{1}{\sqrt{10}},\frac{3}{\sqrt{10}}\cos t,-\frac{3}{\sqrt{10}}\sin t\right\rangle$$

The unit normal vector will now require the derivative of the unit tangent and its magnitude.

$$\vec{T}'(t) = \left\langle 0, -\frac{3}{\sqrt{10}}\sin t, -\frac{3}{\sqrt{10}}\cos t \right\rangle$$

$$\left\| \vec{T}'(t) \right\| = \sqrt{\frac{9}{10}\sin^2 t + \frac{9}{10}\cos^2 t} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$$

The unit normal vector is then,

$$\vec{N}(t) = \frac{\sqrt{10}}{3}\left\langle 0, -\frac{3}{\sqrt{10}}\sin t, -\frac{3}{\sqrt{10}}\cos t \right\rangle = \left\langle 0, -\sin t, -\cos t \right\rangle$$

Finally, the binormal vector is,

$$\begin{aligned} \vec{B}(t) &= \vec{T}(t) \times \vec{N}(t) \\ &= \left| \begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\cos t & -\frac{3}{\sqrt{10}}\sin t & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\cos t \\ 0 & -\sin t & -\cos t & 0 & -\sin t \end{array} \right. \\ &= -\frac{3}{\sqrt{10}}\cos^2 t\,\vec{i} - \frac{1}{\sqrt{10}}\sin t\,\vec{k} + \frac{1}{\sqrt{10}}\cos t\;\vec{j} - \frac{3}{\sqrt{10}}\sin^2 t\,\vec{i} \\ &= -\frac{3}{\sqrt{10}}\vec{i} + \frac{1}{\sqrt{10}}\cos t\;\vec{j} - \frac{1}{\sqrt{10}}\sin t\;\vec{k} \end{aligned}$$

Arc Length with Vector Functions

In this section we'll recast an old formula into terms of vector functions. We want to determine the length of a vector function,

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

on the interval $a \le t \le b$.

We actually already know how to do this. Recall that we can write the vector function into the parametric form,

$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

Also, recall that with two dimensional parametric curves the arc length is given by,

$$L = \int_a^b \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2}\, dt$$

There is a natural extension of this to three dimensions. So, the length of the curve $\vec{r}(t)$ on the interval $a \le t \le b$ is,

$$L = \int_a^b \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2 + \left[h'(t)\right]^2}\, dt$$

There is a nice simplification that we can make for this. Notice that the integrand (the function we're integrating) is nothing more than the magnitude of the tangent vector,

$$\left\|\vec{r}'(t)\right\| = \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2 + \left[h'(t)\right]^2}$$

Therefore, the arc length can be written as,

$$L = \int_a^b \left\|\vec{r}'(t)\right\| dt$$

Let's work a quick example of this.

Example 1 Determine the length of the curve $\vec{r}(t) = \langle 2t, 3\sin(2t), 3\cos(2t) \rangle$ on the interval $0 \le t \le 2\pi$.

Solution
We will first need the tangent vector and its magnitude.

$$\vec{r}'(t) = \langle 2, 6\cos(2t), -6\sin(2t) \rangle$$

$$\left\|\vec{r}'(t)\right\| = \sqrt{4 + 36\cos^2(2t) + 36\sin^2(2t)} = \sqrt{4+36} = 2\sqrt{10}$$

The length is then,

$$\begin{aligned} L &= \int_a^b \left\| \vec{r}\,'(t) \right\| dt \\ &= \int_0^{2\pi} 2\sqrt{10}\, dt \\ &= 4\pi\sqrt{10} \end{aligned}$$

We need to take a quick look at another concept here. We define the **arc length function** as,

$$s(t) = \int_0^t \left\| \vec{r}\,'(u) \right\| du$$

Before we look at why this might be important let's work a quick example.

Example 2 Determine the arc length function for $\vec{r}(t) = \langle 2t, 3\sin(2t), 3\cos(2t) \rangle$.

Solution
From the previous example we know that,

$$\left\| \vec{r}\,'(t) \right\| = 2\sqrt{10}$$

The arc length function is then,

$$s(t) = \int_0^t 2\sqrt{10}\, du = \left(2\sqrt{10}\, u \right)\Big|_0^t = 2\sqrt{10}\, t$$

Okay, just why would we want to do this? Well let's take the result of the example above and solve it for *t*.

$$t = \frac{s}{2\sqrt{10}}$$

Now, taking this and plugging it into the original vector function and we can **reparameterize** the function into the form, $\vec{r}(t(s))$. For our function this is,

$$\vec{r}(t(s)) = \left\langle \frac{s}{\sqrt{10}}, 3\sin\left(\frac{s}{\sqrt{10}} \right), 3\cos\left(\frac{s}{\sqrt{10}} \right) \right\rangle$$

So, why would we want to do this? Well with the reparameterization we can now tell where we are on the curve after we've traveled a distance of *s* along the curve. Note as well that we will start the measurement of distance from where we are at $t = 0$.

Example 3 Where on the curve $\vec{r}(t) = \langle 2t, 3\sin(2t), 3\cos(2t) \rangle$ are we after traveling for a distance of $\frac{\pi\sqrt{10}}{3}$?

Solution
To determine this we need the reparameterization, which we have from above.

$$\vec{r}\left(t\left(s\right)\right) = \left\langle \frac{s}{\sqrt{10}}, 3\sin\left(\frac{s}{\sqrt{10}}\right), 3\cos\left(\frac{s}{\sqrt{10}}\right) \right\rangle$$

Then, to determine where we are all that we need to do is plug in $s = \frac{\pi\sqrt{10}}{3}$ into this and we'll get our location.

$$\vec{r}\left(t\left(\frac{\pi\sqrt{10}}{3}\right)\right) = \left\langle \frac{\pi}{3}, 3\sin\left(\frac{\pi}{3}\right), 3\cos\left(\frac{\pi}{3}\right) \right\rangle = \left\langle \frac{\pi}{3}, \frac{3\sqrt{3}}{2}, \frac{3}{2} \right\rangle$$

So, after traveling a distance of $\frac{\pi\sqrt{10}}{3}$ along the curve we are at the point $\left(\frac{\pi}{3}, \frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$.

Curvature

In this section we want to briefly discuss the **curvature** of a smooth curve (recall that for a smooth curve we require $\vec{r}'(t)$ is continuous and $\vec{r}'(t) \neq 0$). The curvature measures how fast a curve is changing direction at a given point.

There are several formulas for determining the curvature for a curve. The formal definition of curvature is,

$$\kappa = \left| \frac{d\vec{T}}{ds} \right|$$

where $\vec{T}$ is the unit tangent and s is the arc length. Recall that we saw in a previous section how to reparameterize a curve to get it into terms of the arc length.

In general the formal definition of the curvature is not easy to use so there are two alternate formulas that we can use. Here they are.

$$\kappa = \frac{\left\| \vec{T}'(t) \right\|}{\left\| \vec{r}'(t) \right\|} \qquad\qquad \kappa = \frac{\left\| \vec{r}'(t) \times \vec{r}''(t) \right\|}{\left\| \vec{r}'(t) \right\|^3}$$

These may not be particularly easy to deal with either, but at least we don't need to reparameterize the unit tangent.

Example 1 Determine the curvature for $\vec{r}(t) = \langle t, 3\sin t, 3\cos t \rangle$.

Solution
Back in the section when we introduced the tangent vector we computed the tangent and unit tangent vectors for this function. These were,

$$\vec{r}'(t) = \langle 1, 3\cos t, -3\sin t \rangle$$

$$\vec{T}(t) = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\cos t, -\frac{3}{\sqrt{10}}\sin t \right\rangle$$

The derivative of the unit tangent is,

$$\vec{T}'(t) = \left\langle 0, -\frac{3}{\sqrt{10}}\sin t, -\frac{3}{\sqrt{10}}\cos t \right\rangle$$

The magnitudes of the two vectors are,

$$\left\| \vec{r}'(t) \right\| = \sqrt{1 + 9\cos^2 t + 9\sin^2 t} = \sqrt{10}$$

$$\left\| \vec{T}'(t) \right\| = \sqrt{0 + \frac{9}{10}\sin^2 t + \frac{9}{10}\cos^2 t} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$$

The curvature is then,

$$\kappa = \frac{\left\|\vec{T}'(t)\right\|}{\left\|\vec{r}'(t)\right\|} = \frac{3/\sqrt{10}}{\sqrt{10}} = \frac{3}{10}$$

In this case the curvature is constant. This means that the curve is changing direction at the same rate at every point along it. Recalling that this curve is a helix this result makes sense.

Example 2 Determine the curvature of $\vec{r}(t) = t^2\,\vec{i} + t\,\vec{k}$.

Solution

In this case the second form of the curvature would probably be easiest. Here are the first couple of derivatives.

$$\vec{r}'(t) = 2t\,\vec{i} + \vec{k} \qquad\qquad \vec{r}''(t) = 2\,\vec{i}$$

Next, we need the cross product.

$$\begin{aligned}\vec{r}'(t)\times\vec{r}''(t) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & 0 & 1 \\ 2 & 0 & 0 \end{vmatrix}\begin{matrix} \vec{i} & \vec{j} \\ 2t & 0 \\ 2 & 0 \end{matrix} \\ &= 2\,\vec{j}\end{aligned}$$

The magnitudes are,

$$\left\|\vec{r}'(t)\times\vec{r}''(t)\right\| = 2 \qquad\qquad \left\|\vec{r}'(t)\right\| = \sqrt{4t^2+1}$$

The curvature at any value of t is then,

$$\kappa = \frac{2}{\left(4t^2+1\right)^{\frac{3}{2}}}$$

There is a special case that we can look at here as well. Suppose that we have a curve given by $y = f(x)$ and we want to find its curvature.

As we saw when we first looked at vector functions we can write this as follows,

$$\vec{r}(x) = x\,\vec{i} + f(x)\,\vec{j}$$

If we then use the second formula for the curvature we will arrive at the following formula for the curvature.

$$\kappa = \frac{\left|f''(x)\right|}{\left(1+\left[f'(x)\right]^2\right)^{\frac{3}{2}}}$$

Velocity and Acceleration

In this section we need to take a look at the velocity and acceleration of a moving object.

From Calculus I we know that given the position function of an object that the velocity of the object is the first derivative of the position function and the acceleration of the object is the second derivative of the position function.

So, given this it shouldn't be too surprising that if the position function of an object is given by the vector function $\vec{r}\left(t\right)$ then the velocity and acceleration of the object is given by,

$$\vec{v}\left(t\right)=\vec{r}'\left(t\right) \qquad\qquad \vec{a}\left(t\right)=\vec{r}''\left(t\right)$$

Notice that the velocity and acceleration are also going to be vectors as well.

In the study of the motion of objects the acceleration is often broken up into a **tangential component**, a_T, and a **normal component**, a_N. The tangential component is the part of the acceleration that is tangential to the curve and the normal component is the part of the acceleration that is normal (or orthogonal) to the curve. If we do this we can write the acceleration as,

$$\vec{a}=a_T\vec{T}+a_N\vec{N}$$

where $\vec{T}$ and $\vec{N}$ are the unit tangent and unit normal for the position function.

If we define $v=\left\|\vec{v}\left(t\right)\right\|$ then the tangential and normal components of the acceleration are given by,

$$a_T=v'=\frac{\vec{r}'\left(t\right)\bullet\vec{r}''\left(t\right)}{\left\|r'\left(t\right)\right\|} \qquad\qquad a_N=\kappa v^2=\frac{\left\|\vec{r}'\left(t\right)\times\vec{r}''\left(t\right)\right\|}{\left\|r'\left(t\right)\right\|}$$

where κ is the curvature for the position function.

There are two formulas to use here for each component of the acceleration and while the second formula may seem overly complicated it is often the easier of the two. In the tangential component, v, may be messy and computing the derivative may be unpleasant. In the normal component we will already be computing both of these quantities in order to get the curvature and so the second formula in this case is definitely the easier of the two.

Let's take a quick look at a couple of examples.

Example 1 If the acceleration of an object is given by $\vec{a}=\vec{i}+2\vec{j}+6t\vec{k}$ find the object's velocity and position functions given that the initial velocity is $\vec{v}\left(0\right)=\vec{j}-\vec{k}$ and the initial position is $\vec{r}\left(0\right)=\vec{i}-2\vec{j}+3\vec{k}$.

Solution

We'll first get the velocity. To do this all (well almost all) we need to do is integrate the acceleration.

$$\begin{aligned}\vec{v}(t) &= \int \vec{a}(t)\,dt \\ &= \int \vec{i} + 2\vec{j} + 6t\vec{k}\,dt \\ &= t\,\vec{i} + 2t\,\vec{j} + 3t^2\,\vec{k} + \vec{c}\end{aligned}$$

To completely get the velocity we will need to determine the "constant" of integration. We can use the initial velocity to get this.

$$\vec{j} - \vec{k} = \vec{v}(0) = \vec{c}$$

The velocity of the object is then,

$$\begin{aligned}\vec{v}(t) &= t\,\vec{i} + 2t\,\vec{j} + 3t^2\,\vec{k} + \vec{j} - \vec{k} \\ &= t\,\vec{i} + (2t+1)\,\vec{j} + \left(3t^2 - 1\right)\vec{k}\end{aligned}$$

We will find the position function by integrating the velocity function.

$$\begin{aligned}\vec{r}(t) &= \int \vec{v}(t)\,dt \\ &= \int t\,\vec{i} + (2t+1)\,\vec{j} + \left(3t^2-1\right)\vec{k}\,dt \\ &= \frac{1}{2}t^2\,\vec{i} + \left(t^2+t\right)\vec{j} + \left(t^3-t\right)\vec{k} + \vec{c}\end{aligned}$$

Using the initial position gives us,

$$\vec{i} - 2\vec{j} + 3\vec{k} = \vec{r}(0) = \vec{c}$$

So, the position function is,

$$\vec{r}(t) = \left(\frac{1}{2}t^2 + 1\right)\vec{i} + \left(t^2 + t - 2\right)\vec{j} + \left(t^3 - t + 3\right)\vec{k}$$

Example 2 For the object in the previous example determine the tangential and normal components of the acceleration.

Solution

There really isn't much to do here other than plug into the formulas. To do this we'll need to notice that,

$$\begin{aligned}\vec{r}\,'(t) &= t\,\vec{i} + (2t+1)\,\vec{j} + \left(3t^2-1\right)\vec{k} \\ \vec{r}\,''(t) &= \vec{i} + 2\vec{j} + 6t\vec{k}\end{aligned}$$

Let's first compute the dot product and cross product that we'll need for the formulas.

$$\vec{r}\,'(t)\bullet\vec{r}\,''(t) = t + 2(2t+1) + 6t\left(3t^2-1\right) = 18t^3 - t + 2$$

$$\vec{r}'(t)\times\vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t & 2t+1 & 3t^2-1 \\ 1 & 2 & 6t \end{vmatrix}\begin{matrix} \vec{i} & \vec{j} \\ t & 2t+1 \\ 1 & 2 \end{matrix}$$

$$= (6t)(2t+1)\vec{i} + (3t^2-1)\vec{j} + 2t\vec{k} - 6t^2\vec{j} - 2(3t^2-1)\vec{i} - (2t+1)\vec{k}$$

$$= (6t^2+6t+2)\vec{i} - (3t^2+1)\vec{j} - \vec{k}$$

Next, we also need a couple of magnitudes.

$$\left\|\vec{r}'(t)\right\| = \sqrt{t^2 + (2t+1)^2 + (3t^2-1)^2} = \sqrt{9t^4 - t^2 + 4t + 2}$$

$$\left\|\vec{r}'(t)\times\vec{r}''(t)\right\| = \sqrt{(6t^2+6t+2)^2 + (3t^2+1)^2 + 1} = \sqrt{45t^4 + 72t^3 + 66t^2 + 24t + 6}$$

The tangential component of the acceleration is then,

$$a_T = \frac{18t^3 - t + 2}{\sqrt{9t^4 - t^2 + 4t + 2}}$$

The normal component of the acceleration is,

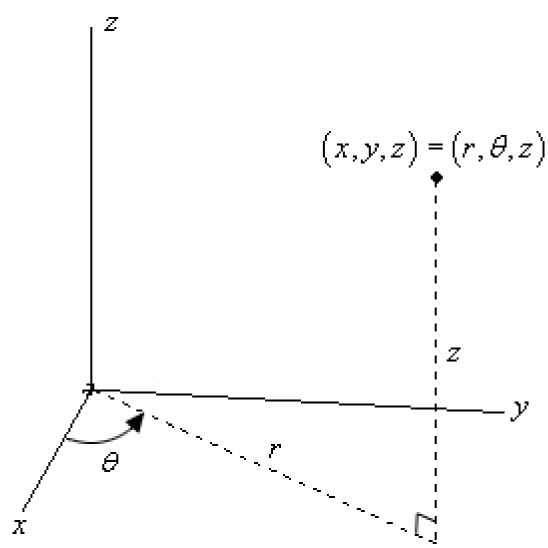
$$a_N = \frac{\sqrt{45t^4 + 72t^3 + 66t^2 + 24t + 6}}{\sqrt{9t^4 - t^2 + 4t + 2}} = \sqrt{\frac{45t^4 + 72t^3 + 66t^2 + 24t + 6}{9t^4 - t^2 + 4t + 2}}$$

Cylindrical Coordinates

As with two dimensional space the standard (x, y, z) coordinate system is called the Cartesian coordinate system. In the last two sections of this chapter we'll be looking at some alternate coordinate systems for three dimensional space.

We'll start off with the cylindrical coordinate system. This one is fairly simple as it is nothing more than an extension of polar coordinates into three dimensions. Not only is it an extension of polar coordinates, but we extend it into the third dimension just as we extend Cartesian coordinates into the third dimension. All that we do is add a z on as the third coordinate. The r and θ are the same as with polar coordinates.

Here is a sketch of a point in $\mathbb{R}^3$.



The conversions for x and y are the same conversions that we used back when we were looking at polar coordinates. So, if we have a point in cylindrical coordinates the Cartesian coordinates can be found by using the following conversions.

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

The third equation is just an acknowledgement that the z-coordinate of a point in Cartesian and polar coordinates is the same.

Likewise, if we have a point in Cartesian coordinates the cylindrical coordinates can be found by using the following conversions.

$$r = \sqrt{x^2 + y^2} \qquad \text{OR} \qquad r^2 = x^2 + y^2$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$z = z$$

Let's take a quick look at some surfaces in cylindrical coordinates.

Example 1 Identify the surface for each of the following equations.

(a) $r = 5$

(b) $r^2 + z^2 = 100$

(c) $z = r$

Solution

(a) In two dimensions we know that this is a circle of radius 5. Since we are now in three dimensions and there is no z in equation this means it is allowed to vary freely. So, for any given z we will have a circle of radius 5 centered on the z-axis.

In other words, we will have a cylinder of radius 5 centered on the z-axis.

(b) This equation will be easy to identify once we convert back to Cartesian coordinates.

$$r^2 + z^2 = 100$$
$$x^2 + y^2 + z^2 = 100$$

So, this is a sphere centered at the origin with radius 10.

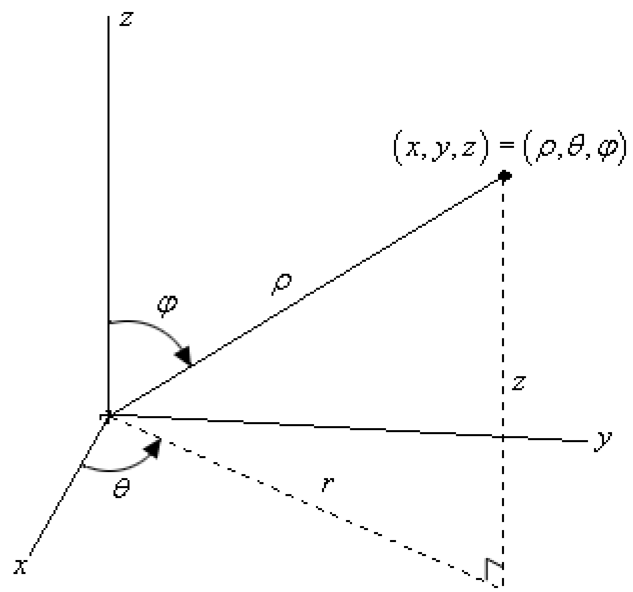
(c) Again, this one won't be too bad if we convert back to Cartesian. For reasons that will be apparent eventually, we'll first square both sides, then convert.

$$z^2 = r^2$$
$$z^2 = x^2 + y^2$$

From the section on quadric surfaces we know that this is the equation of a cone.

Spherical Coordinates

In this section we will introduce spherical coordinates. Spherical coordinates can take a little getting used to. It's probably easiest to start things off with a sketch.



Spherical coordinates consist of the following three quantities.

First there is ρ. This is the distance from the origin to the point and we will require $\rho \geq 0$.

Next there is θ. This is the same angle that we saw in polar/cylindrical coordinates. It is the angle between the positive *x*-axis and the line above denoted by *r* (which is also the same *r* as in polar/cylindrical coordinates). There are no restrictions on θ.

Finally there is φ. This is the angle between the positive *z*-axis and the line from the origin to the point. We will require $0 \leq \varphi \leq \pi$.

In summary, ρ is the distance from the origin to the point, φ is the angle that we need to rotate down from the positive *z*-axis to get to the point and θ is how much we need to rotate around the *z*-axis to get to the point.

We should first derive some conversion formulas. Let's first start with a point in spherical coordinates and ask what the cylindrical coordinates of the point are. So, we know (ρ, θ, φ) and want to find (r, θ, z). Of course we really only need to find *r* and *z* since θ is the same in both coordinate systems.

We will be able to do all of our work by looking at the right triangle shown above in our sketch. With a little geometry we see that the angle between *z* and ρ is φ and so we can see that,

$$z = \rho\cos\varphi$$
$$r = \rho\sin\varphi$$

and these are exactly the formulas that we were looking for. So, given a point in spherical coordinates the cylindrical coordinates of the point will be,

$$r = \rho\sin\varphi$$
$$\theta = \theta$$
$$z = \rho\cos\varphi$$

Note as well that,

$$r^2 + z^2 = \rho^2\cos^2\varphi + \rho^2\sin^2\varphi = \rho^2\left(\cos^2\varphi + \sin^2\varphi\right) = \rho^2$$

Or,

$$\rho^2 = r^2 + z^2$$

Next, let's find the Cartesian coordinates of the same point. To do this we'll start with the cylindrical conversion formulas from the previous section.

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Now all that we need to do is use the formulas from above for r and z to get,

$$x = \rho\sin\varphi\cos\theta$$
$$y = \rho\sin\varphi\sin\theta$$
$$z = \rho\cos\varphi$$

Also note that since we know that $r^2 = x^2 + y^2$ we get,

$$\rho^2 = x^2 + y^2 + z^2$$

Converting points from Cartesian or cylindrical coordinates into spherical coordinates is usually done with the same conversion formulas. To see how this is done let's work an example of each.

Example 1 Perform each of the following conversions.

(a) Convert the point $\left(\sqrt{6}, \frac{\pi}{4}, \sqrt{2}\right)$ from cylindrical to spherical coordinates.

[Solution]

(b) Convert the point $\left(-1, 1, -\sqrt{2}\right)$ from Cartesian to spherical coordinates.

[Solution]

Solution

(a) Convert the point $\left(\sqrt{6},\frac{\pi}{4},\sqrt{2}\right)$ from cylindrical to spherical coordinates.

We'll start by acknowledging that θ is the same in both coordinate systems and so we don't need to do anything with that.

Next, let's find ρ.

$$\rho=\sqrt{r^2+z^2}=\sqrt{6+2}=\sqrt{8}=2\sqrt{2}$$

Finally, let's get φ. To do this we can use either the conversion for *r* or *z*. We'll use the conversion for *z*.

$$z=\rho\cos\varphi \qquad \Rightarrow \qquad \cos\varphi=\frac{z}{\rho}=\frac{\sqrt{2}}{2\sqrt{2}} \qquad \Rightarrow \qquad \varphi=\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$$

Notice that there are many possible values of φ that will give $\cos\varphi=\frac{1}{2}$, however, we have restricted φ to the range $0\le\varphi\le\pi$ and so this is the only possible value in that range.

So, the spherical coordinates of this point will are $\left(2\sqrt{2},\frac{\pi}{4},\frac{\pi}{3}\right)$.

[Return to Problems]

(b) Convert the point $\left(-1,1,-\sqrt{2}\right)$ from Cartesian to spherical coordinates.

The first thing that we'll do here is find ρ.

$$\rho=\sqrt{x^2+y^2+z^2}=\sqrt{1+1+2}=2$$

Now we'll need to find φ. We can do this using the conversion for *z*.

$$z=\rho\cos\varphi \qquad \Rightarrow \qquad \cos\varphi=\frac{z}{\rho}=\frac{-\sqrt{2}}{2} \qquad \Rightarrow \qquad \varphi=\cos^{-1}\left(\frac{-\sqrt{2}}{2}\right)=\frac{3\pi}{4}$$

As with the last parts this will be the only possible φ in the range allowed.

Finally, let's find θ. To do this we can use the conversion for *x* or *y*. We will use the conversion for *y* in this case.

$$\sin\theta=\frac{y}{\rho\sin\varphi}=\frac{1}{2\left(\frac{\sqrt{2}}{2}\right)}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2} \qquad \Rightarrow \qquad \theta=\frac{\pi}{4}\text{ or }\theta=\frac{3\pi}{4}$$

Now, we actually have more possible choices for θ but all of them will reduce down to one of the two angles above since they will just be one of these two angles with one or more complete rotations around the unit circle added on.

We will however, need to decide which one is the correct angle since only one will be. To do this let's notice that, in two dimensions, the point with coordinates $x = -1$ and $y = 1$ lies in the second quadrant. This means that θ must be angle that will put the point into the second quadrant. Therefore, the second angle, $\theta = \frac{3\pi}{4}$, must be the correct one.

The spherical coordinates of this point are then $\left(2, \frac{3\pi}{4}, \frac{3\pi}{4}\right)$.

[Return to Problems]

Now, let's take a look at some equations and identify the surfaces that they represent.

Example 2 Identify the surface for each of the following equations.

(a) $\rho = 5$ [Solution]

(b) $\varphi = \dfrac{\pi}{3}$ [Solution]

(c) $\theta = \dfrac{2\pi}{3}$ [Solution]

(d) $\rho \sin\varphi = 2$ [Solution]

Solution

(a) $\rho = 5$

There are a couple of ways to think about this one.

First, think about what this equation is saying. This equation says that, no matter what θ and φ are, the distance from the origin must be 5. So, we can rotate as much as we want away from the *z*-axis and around the *z*-axis, but we must always remain at a fixed distance from the origin. This is exactly what a sphere is. So, this is a sphere of radius 5 centered at the origin.

The other way to think about it is to just convert to Cartesian coordinates.

$$\begin{aligned} \rho &= 5 \\ \rho^2 &= 25 \\ x^2 + y^2 + z^2 &= 25 \end{aligned}$$

Sure enough a sphere of radius 5 centered at the origin.

[Return to Problems]

(b) $\varphi = \dfrac{\pi}{3}$

In this case there isn't an easy way to convert to Cartesian coordinates so we'll just need to think about this one a little. This equation says that no matter how far away from the origin that we move and no matter how much we rotate around the *z*-axis the point must always be at an angle of $\frac{\pi}{3}$ from the *z*-axis.

This is exactly what happens in a cone. All of the points on a cone are a fixed angle from the *z*-

axis. So, we have a cone whose points are all at an angle of $\frac{\pi}{3}$ from the z-axis.

[Return to Problems]

(c) $\theta = \dfrac{2\pi}{3}$

As with the last part we won't be able to easily convert to Cartesian coordinates here. In this case no matter how far from the origin we get or how much we rotate down from the positive z-axis the points must always form an angle of $\frac{2\pi}{3}$ with the x-axis.

Points in a vertical plane will do this. So, we have a vertical plane that forms an angle of $\frac{2\pi}{3}$ with the positive x-axis.

[Return to Problems]

(d) $\rho \sin\varphi = 2$

In this case we can convert to Cartesian coordinates so let's do that. There are actually two ways to do this conversion. We will look at both since both will be used on occasion.

Solution 1
In this solution method we will convert directly to Cartesian coordinates. To do this we will first need to square both sides of the equation.

$$\rho^2 \sin^2 \varphi = 4$$

Now, for no apparent reason add $\rho^2 \cos^2 \varphi$ to both sides.

$$\begin{aligned} \rho^2 \sin^2 \varphi + \rho^2 \cos^2 \varphi &= 4 + \rho^2 \cos^2 \varphi \\ \rho^2 \left(\sin^2 \varphi + \cos^2 \varphi \right) &= 4 + \rho^2 \cos^2 \varphi \\ \rho^2 &= 4 + \left(\rho \cos\varphi \right)^2 \end{aligned}$$

Now we can convert to Cartesian coordinates.

$$\begin{aligned} x^2 + y^2 + z^2 &= 4 + z^2 \\ x^2 + y^2 &= 4 \end{aligned}$$

So, we have a cylinder of radius 2 centered on the z-axis.

This solution method wasn't too bad, but it did require some not so obvious steps to complete.

Solution 2
This method is much shorter, but also involves something that you may not see the first time around. In this case instead of going straight to Cartesian coordinates we'll first convert to cylindrical coordinates.

This won't always work, but in this case all we need to do is recognize that $r = \rho \sin\varphi$ and we will get something we can recognize. Using this we get,

$$\rho \sin\varphi = 2$$
$$r = 2$$

At this point we know this is a cylinder (remember that we're in three dimensions and so this isn't a circle!). However, let's go ahead and finish the conversion process out.

$$r^2 = 4$$
$$x^2 + y^2 = 4$$

[Return to Problems]

So, as we saw in the last part of the previous example it will *sometimes* be easier to convert equations in spherical coordinates into cylindrical coordinates before converting into Cartesian coordinates. This won't always be easier, but it can make some of the conversions quicker and easier.

The last thing that we want to do in this section is generalize the first three parts of the previous example.

$\rho = a$	sphere of radius a centered at the origin
$\varphi = \alpha$	cone that makes an angle of α with the positive $z-$axis
$\theta = \beta$	vertical plane that makes an angle of β with the positive $x-$axis