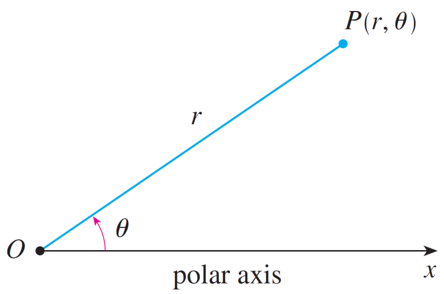
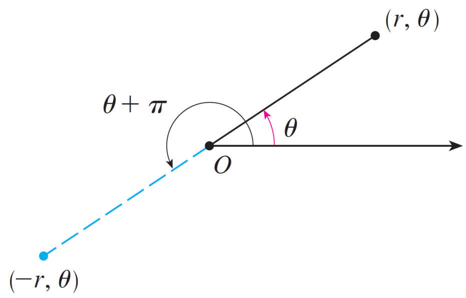
Polar Coordinates

DEFINITION: The **polar coordinate system** is a two-dimensional coordinate system in which each point P on a plane is determined by a distance r from a fixed point O that is called the **pole** (or origin) and an angle θ from a fixed direction. The point P is represented by the ordered pair (r, θ) and r, θ are called **polar coordinates.**

REMARK: We extend the meaning of polar coordinates (r, θ) to the case in which r is negative by agreeing that the points $(-r, \theta)$ and (r, θ) lie in the same line through O and at the same distance $|r|$ from O, but on opposite sides of O. If $r > 0$, the point (r, θ) lies in the same quadrant as θ; if $r < 0$, it lies in the quadrant on the opposite side of the pole.

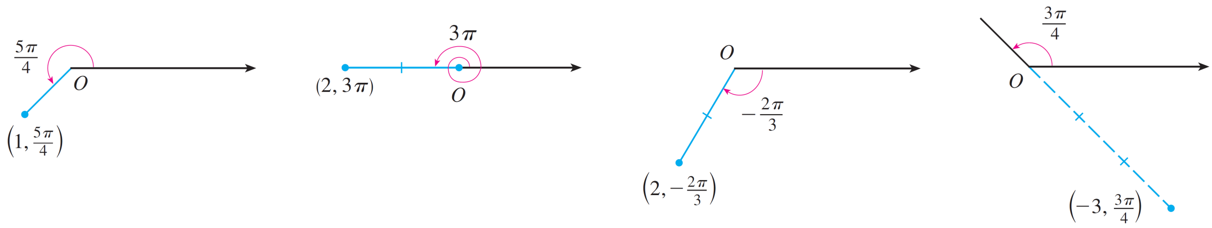




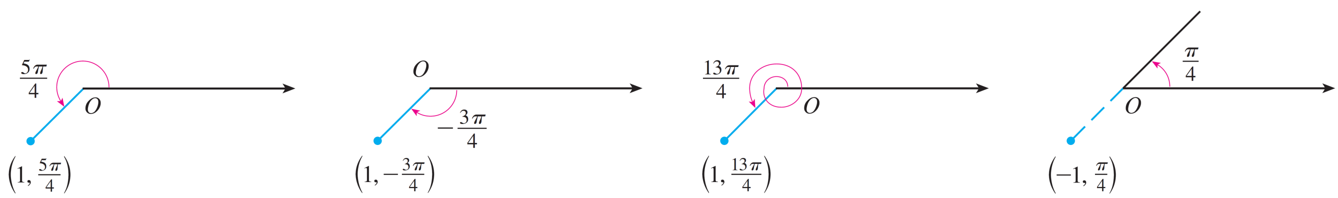
EXAMPLE: Plot the points whose polar coordinates are given:

(a) $(1, 5\pi/4)$ (b) $(2, 3\pi)$ (c) $(2, -2\pi/3)$ (d) $(-3, 3\pi/4)$

Solution:



REMARK: In the Cartesian coordinate system every point has only one representation, but in the polar coordinate system each point has many representations. For instance, the point $(1, 5\pi/4)$ in the Example above could be written as $(1, -3\pi/4)$ or $(1, 13\pi/4)$ or $(-1, \pi/4)$:



EXAMPLE: Find all the polar coordinates of the point $P(2, \pi/6)$.

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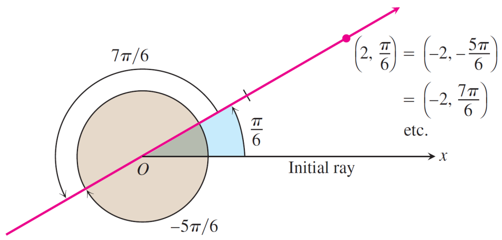
Solution: We sketch the initial ray of the coordinate system, draw the ray from the origin that makes an angle of $\pi/6$ radians with the initial ray, and mark the point $(2, \pi/6)$. We then find the angles for the other coordinate pairs of P in which $r = 2$ and $r = -2$.



For $r = 2$, the complete list of angles is

$$\frac{\pi}{6}, \quad \frac{\pi}{6} \pm 2\pi, \quad \frac{\pi}{6} \pm 4\pi, \quad \frac{\pi}{6} \pm 6\pi, \quad \ldots$$

For $r = -2$, the angles are

$$-\frac{5\pi}{6}, \quad -\frac{5\pi}{6} \pm 2\pi, \quad -\frac{5\pi}{6} \pm 4\pi, \quad -\frac{5\pi}{6} \pm 6\pi, \quad \ldots$$

The corresponding coordinate pairs of P are

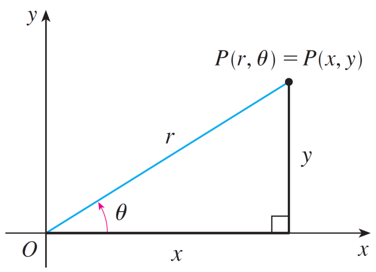
$$\left(2, \ \frac{\pi}{6} + 2n\pi\right), \quad n = 0, \ \pm 1, \ \pm 2, \ \ldots$$

and

$$\left(-2, \ -\frac{5\pi}{6} + 2n\pi\right), \quad n = 0, \ \pm 1, \ \pm 2, \ \ldots$$

When $n = 0$, the formulas give $(2, \pi/6)$ and $(-2, -5\pi/6)$. When $n = 1$, they give $(2, 13\pi/6)$ and $(-2, 7\pi/6)$, and so on.

The connection between polar and Cartesian coordinates can be seen from the figure below and described by the following formulas:



$$\boxed{x = r\cos\theta \qquad y = r\sin\theta}$$

$$\boxed{r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}}$$

EXAMPLE:

(a) Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates.

(b) Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

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(b) Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

Solution:

(a) We have:

$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2\cdot\frac{1}{2} = 1 \qquad y = r\sin\theta = 2\sin\frac{\pi}{3} = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$$

Therefore, the point is $(1, \sqrt{3})$ in Cartesian coordinates.

(b) If we choose r to be positive, then

$$r = \sqrt{x^2+y^2} = \sqrt{1^2+(-1)^2} = \sqrt{2} \qquad \tan\theta = \frac{y}{x} = -1$$

Since the point $(1, -1)$ lies in the fourth quadrant, we can choose $\theta = -\pi/4$ or $\theta = 7\pi/4$. Thus one possible answer is $(\sqrt{2}, -\pi/4)$; another is $(\sqrt{2}, 7\pi/4)$.

EXAMPLE: Express the equation $x = 1$ in polar coordinates.

Solution: We use the formula $x = r\cos\theta$.

$$x = 1$$

$$r\cos\theta = 1$$

$$r = \sec\theta$$

EXAMPLE: Express the equation $x^2 = 4y$ in polar coordinates.

Solution: We use the formulas $x = r\cos\theta$ and $y = r\sin\theta$.

$$x^2 = 4y$$

$$(r\cos\theta)^2 = 4r\sin\theta$$

$$r^2\cos^2\theta = 4r\sin\theta$$

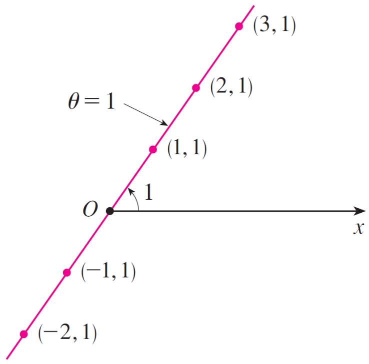
$$r = 4\frac{\sin\theta}{\cos^2\theta} = 4\sec\theta\tan\theta$$

Polar Curves

The **graph of a polar equation** $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

EXAMPLE: Sketch the polar curve $\theta = 1$.

Solution: This curve consists of all points (r, θ) such that the polar angle θ is 1 radian. It is the straight line that passes through O and makes an angle of 1 radian with the polar axis. Notice that the points $(r, 1)$ on the line with $r > 0$ are in the first quadrant, whereas those with $r < 0$ are in the third quadrant.



EXAMPLE: Sketch the following curves:

(a) $r = 2,\ 0 \le \theta \le 2\pi$.

(b) $r = 2\cos\theta,\ 0 \le \theta \le \pi$.

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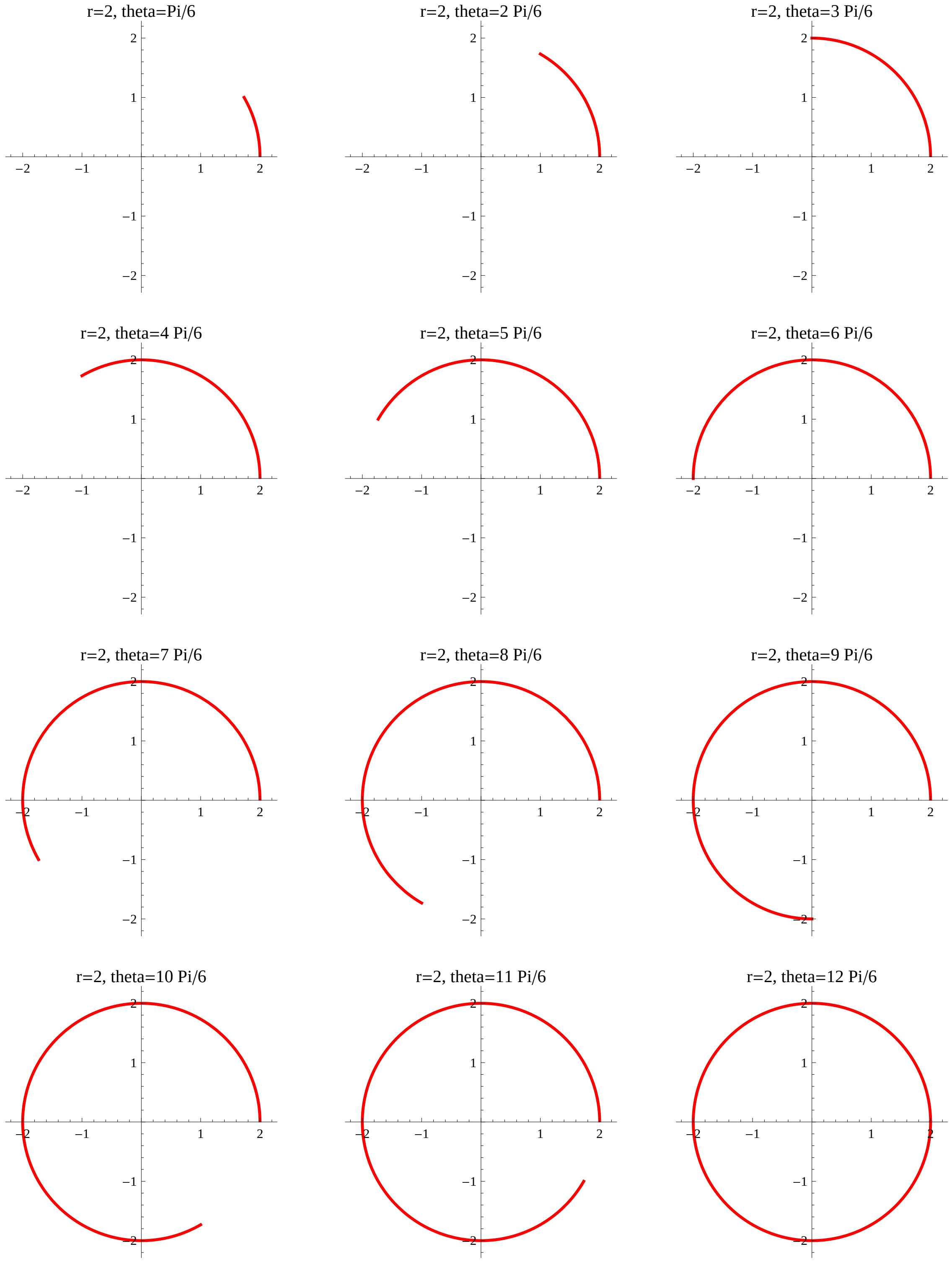
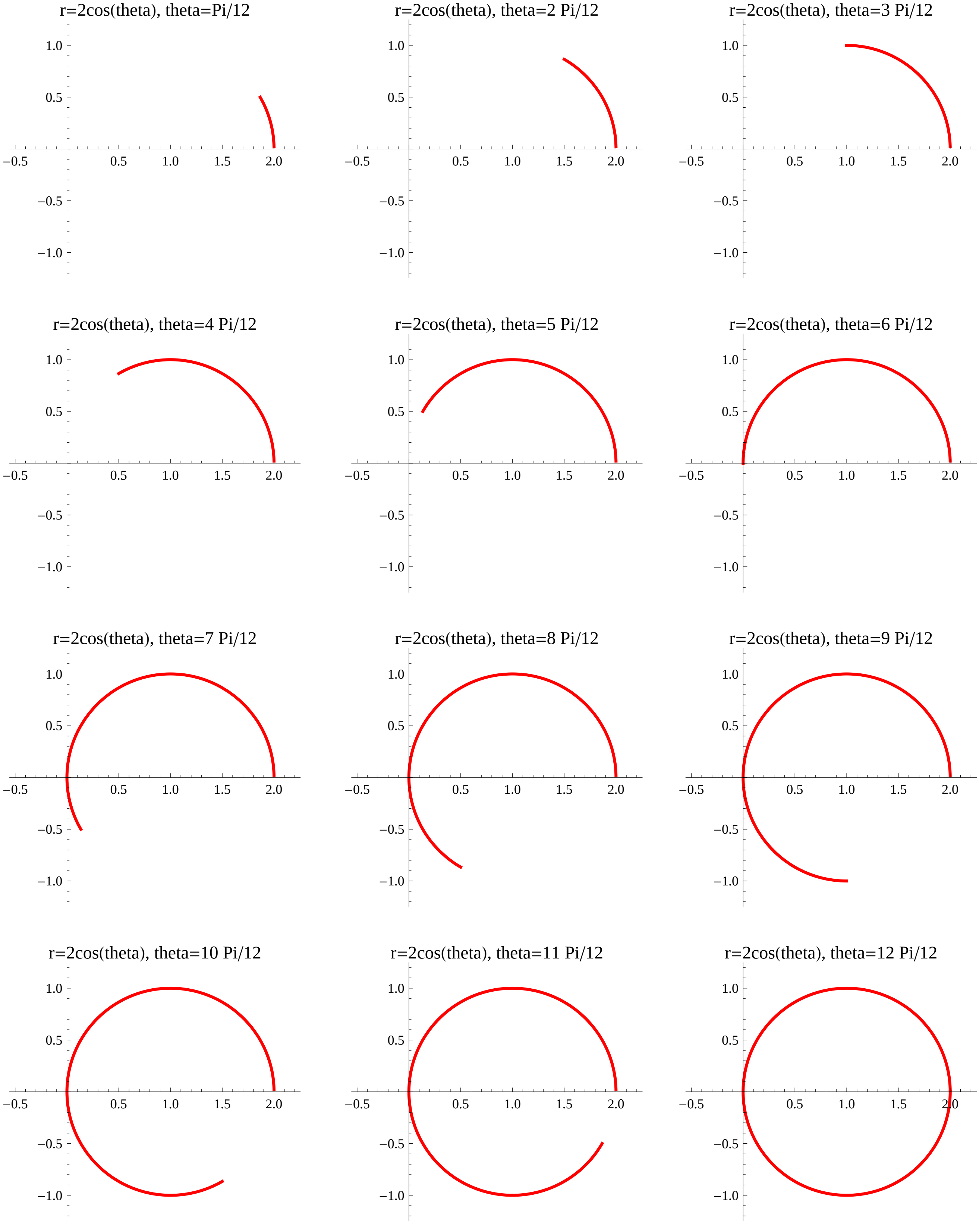
Solution: We have



EXAMPLE: Sketch the curve $r = 2\cos\theta,\ 0 \le \theta \le \pi$.

Solution: We have



EXAMPLE: Express the polar equation $r = 2\cos\theta$ in rectangular coordinates.

Solution: We use the formulas $r^2 = x^2 + y^2$ and $x = r\cos\theta$. We have

$$\begin{aligned} r &= 2\cos\theta \\ r^2 &= 2r\cos\theta \\ x^2 + y^2 &= 2x \\ x^2 - 2x + y^2 &= 0 \\ x^2 - 2x + 1 + y^2 &= 1 \\ (x-1)^2 + y^2 &= 1 \end{aligned}$$

EXAMPLE: Express the polar equation in rectangular coordinates. If possible, determine the graph of the equation from its rectangular form.

(a) $r = 5\sec\theta$ (b) $r = 2\sin\theta$ (c) $r = 2 + 2\cos\theta$

EXAMPLE: Express the polar equation in rectangular coordinates. If possible, determine the graph of the equation from its rectangular form.

(a) $r = 5\sec\theta$ (b) $r = 2\sin\theta$ (c) $r = 2 + 2\cos\theta$

Solution: We use the formulas $r^2 = x^2 + y^2$, $x = r\cos\theta$ and $y = r\sin\theta$.

(a) We have

$$r = 5\sec\theta$$

$$r\cos\theta = 5$$

$$x = 5$$

(b) We have

$$r = 2\sin\theta$$

$$r^2 = 2r\sin\theta$$

$$x^2 + y^2 = 2y$$

$$x^2 + y^2 - 2y = 0$$

$$x^2 + y^2 - 2y + 1 = 1$$

$$x^2 + (y-1)^2 = 1$$

(c) We have

$$r = 2 + 2\cos\theta$$

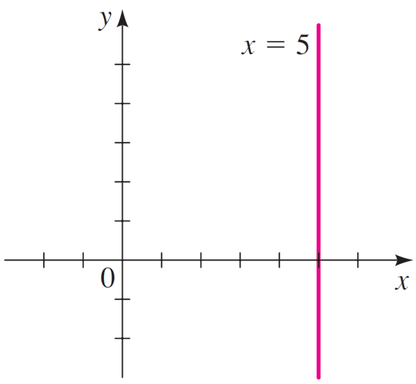
$$r^2 = 2r + 2r\cos\theta$$

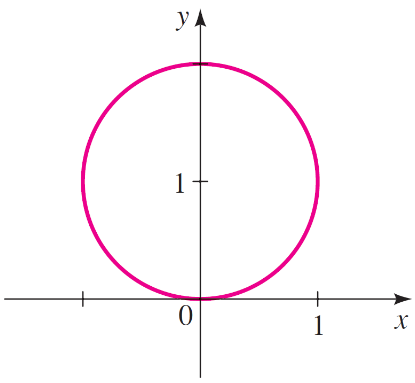
$$x^2 + y^2 = 2r + 2x$$

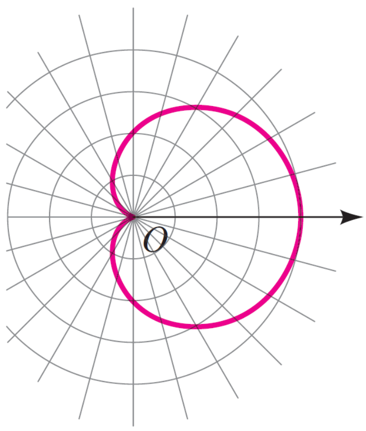
$$x^2 + y^2 - 2x = 2r$$

$$(x^2 + y^2 - 2x)^2 = 4r^2$$

$$(x^2 + y^2 - 2x)^2 = 4(x^2 + y^2)$$

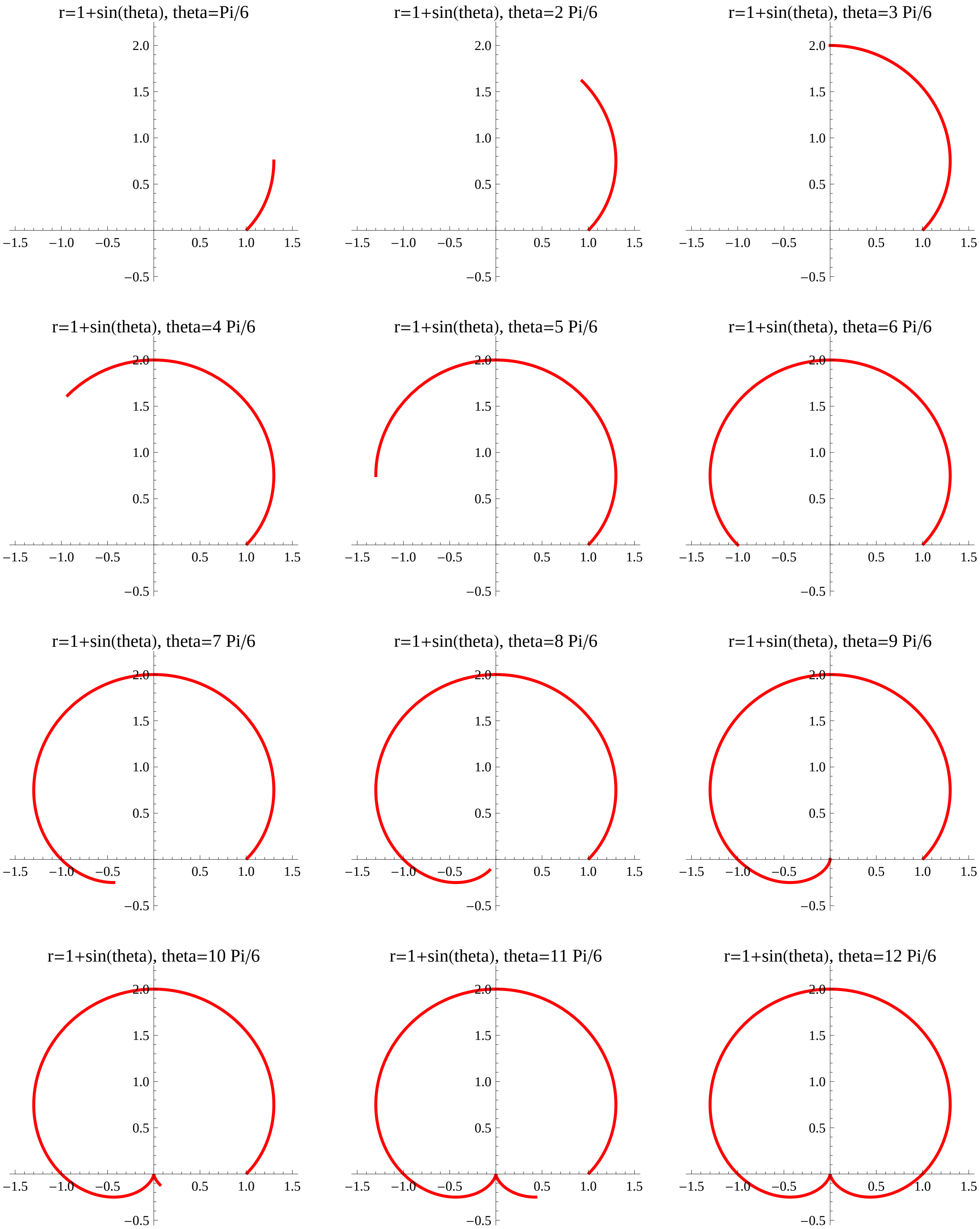






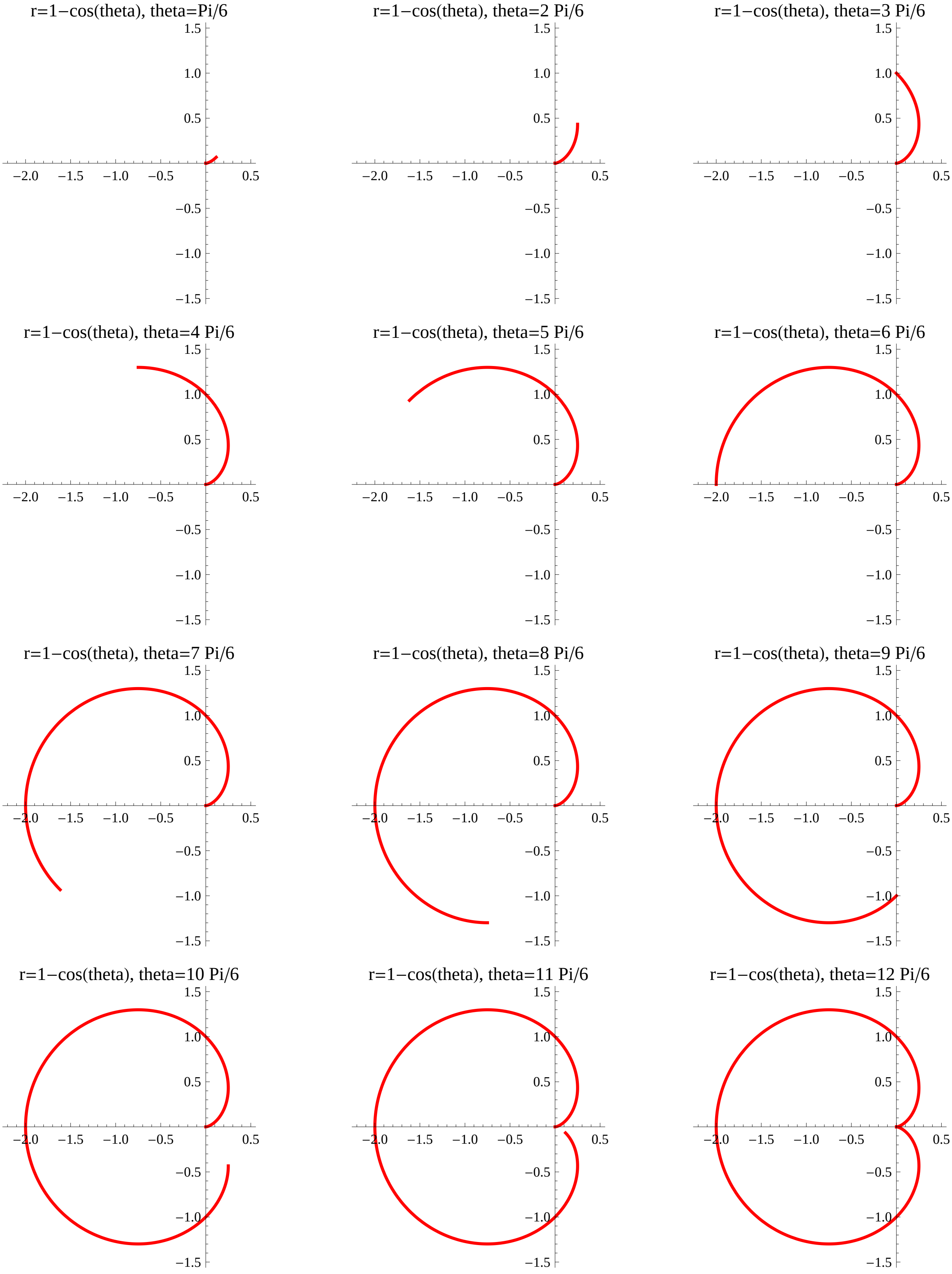
EXAMPLE: Sketch the curve $r = 1 + \sin\theta,\ 0 \leq \theta \leq 2\pi$ (cardioid).

Solution: We have



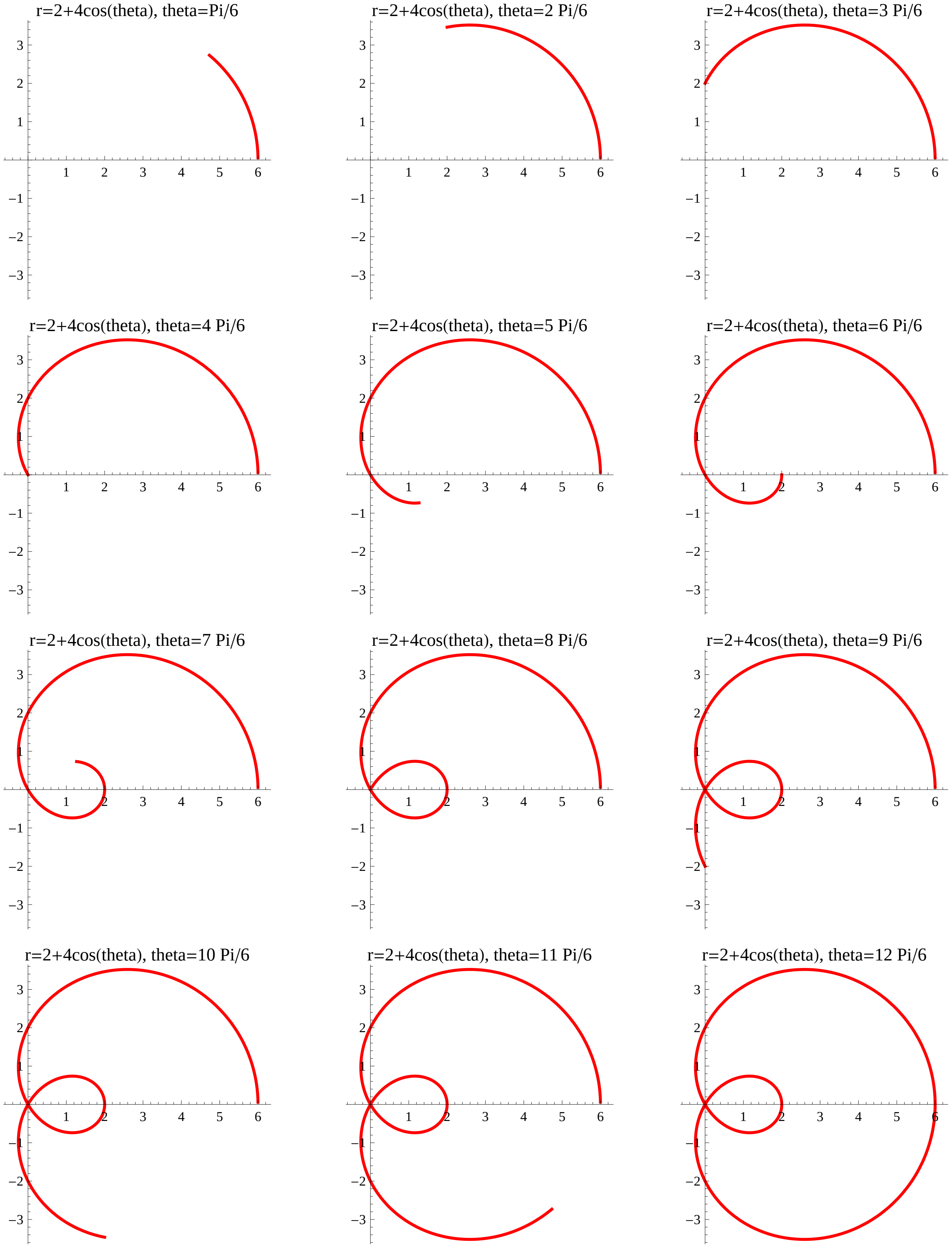
EXAMPLE: Sketch the curve $r = 1 - \cos\theta,\ 0 \le \theta \le 2\pi$ (cardioid).

Solution: We have



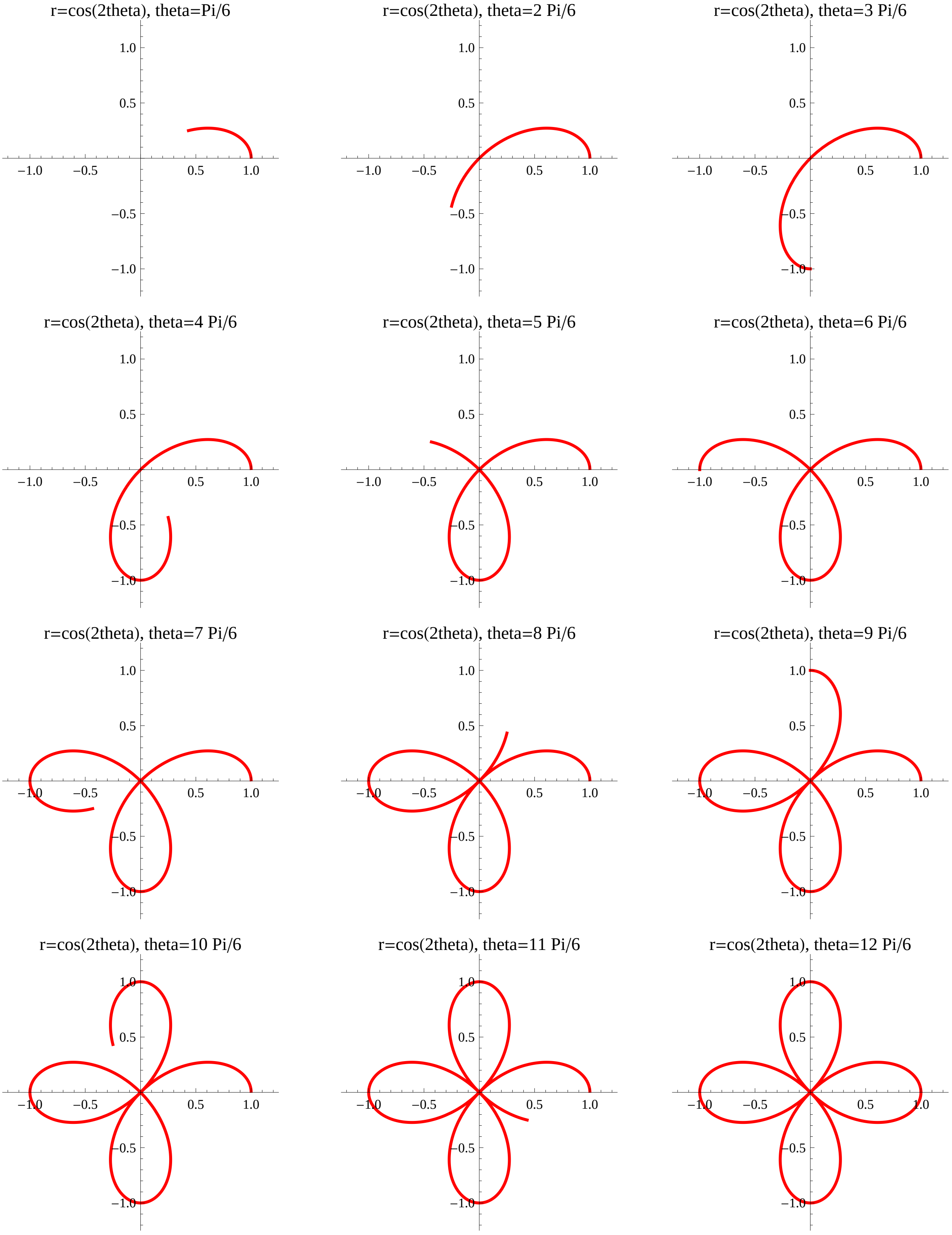
EXAMPLE: Sketch the curve $r = 2 + 4\cos\theta,\ 0 \leq \theta \leq 2\pi$.

Solution: We have



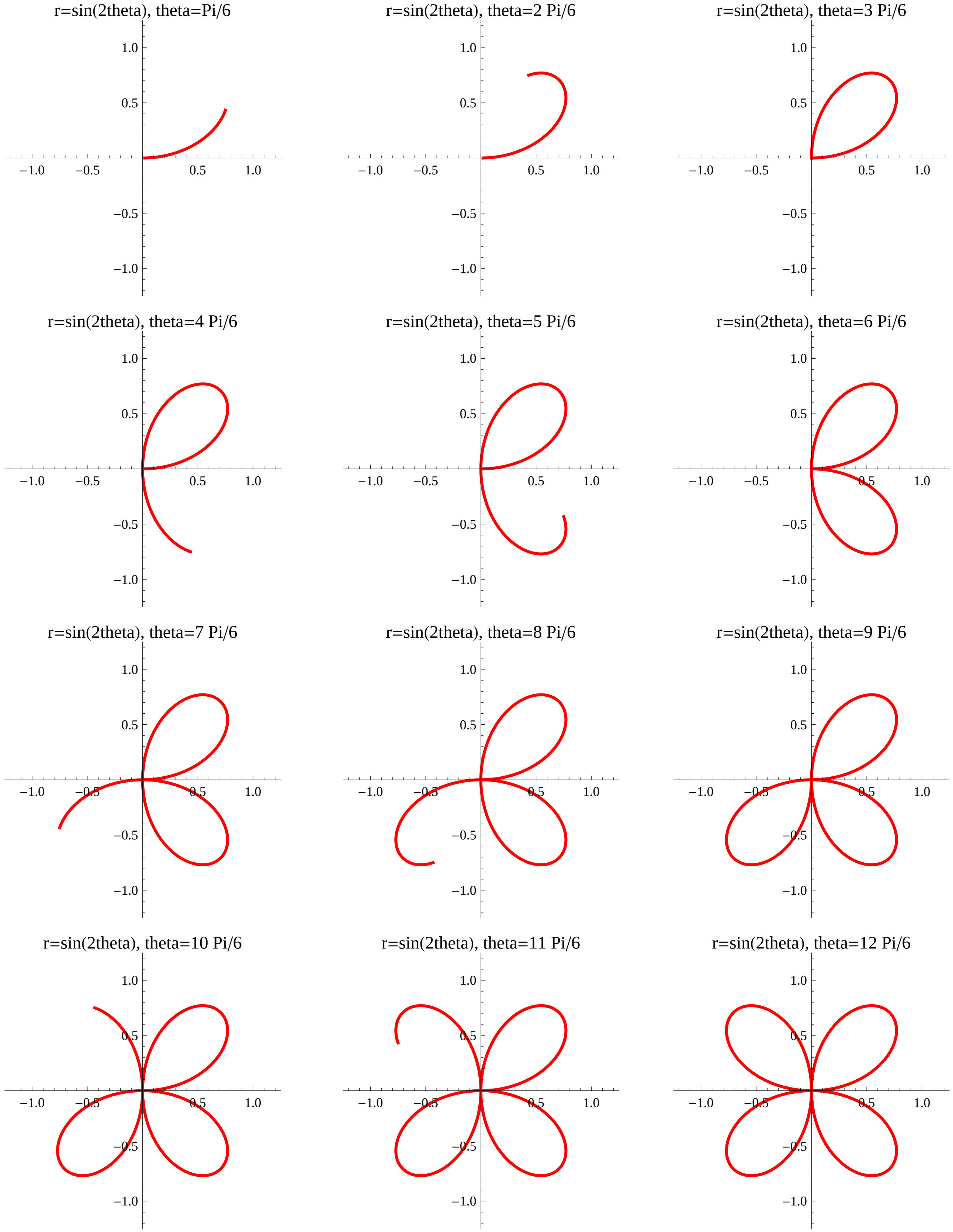
EXAMPLE: Sketch the curve $r = \cos(2\theta)$, $0 \le \theta \le 2\pi$ (four-leaved rose).

Solution: We have



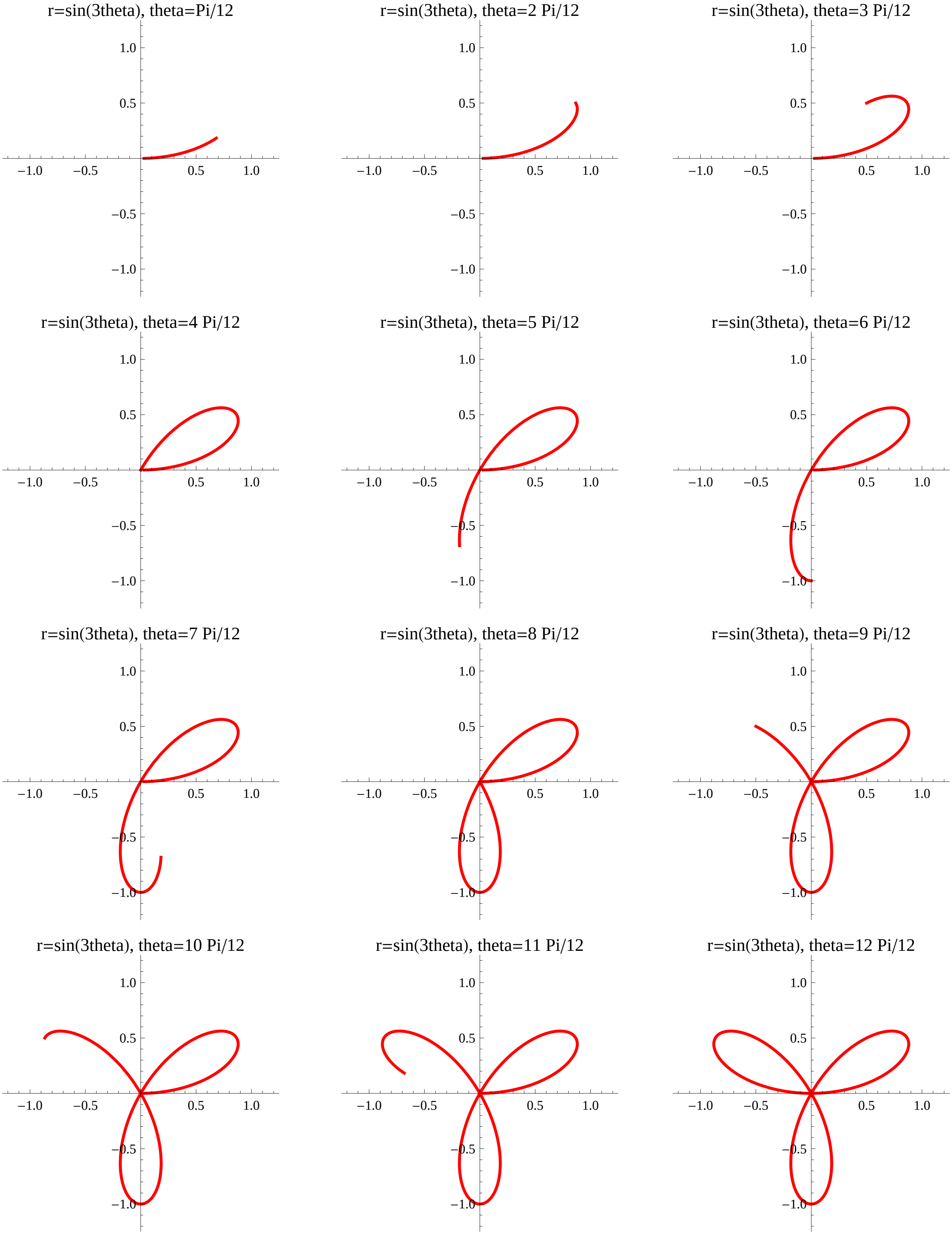
EXAMPLE: Sketch the curve $r = \sin(2\theta)$, $0 \le \theta \le 2\pi$ (four-leaved rose).

Solution: We have



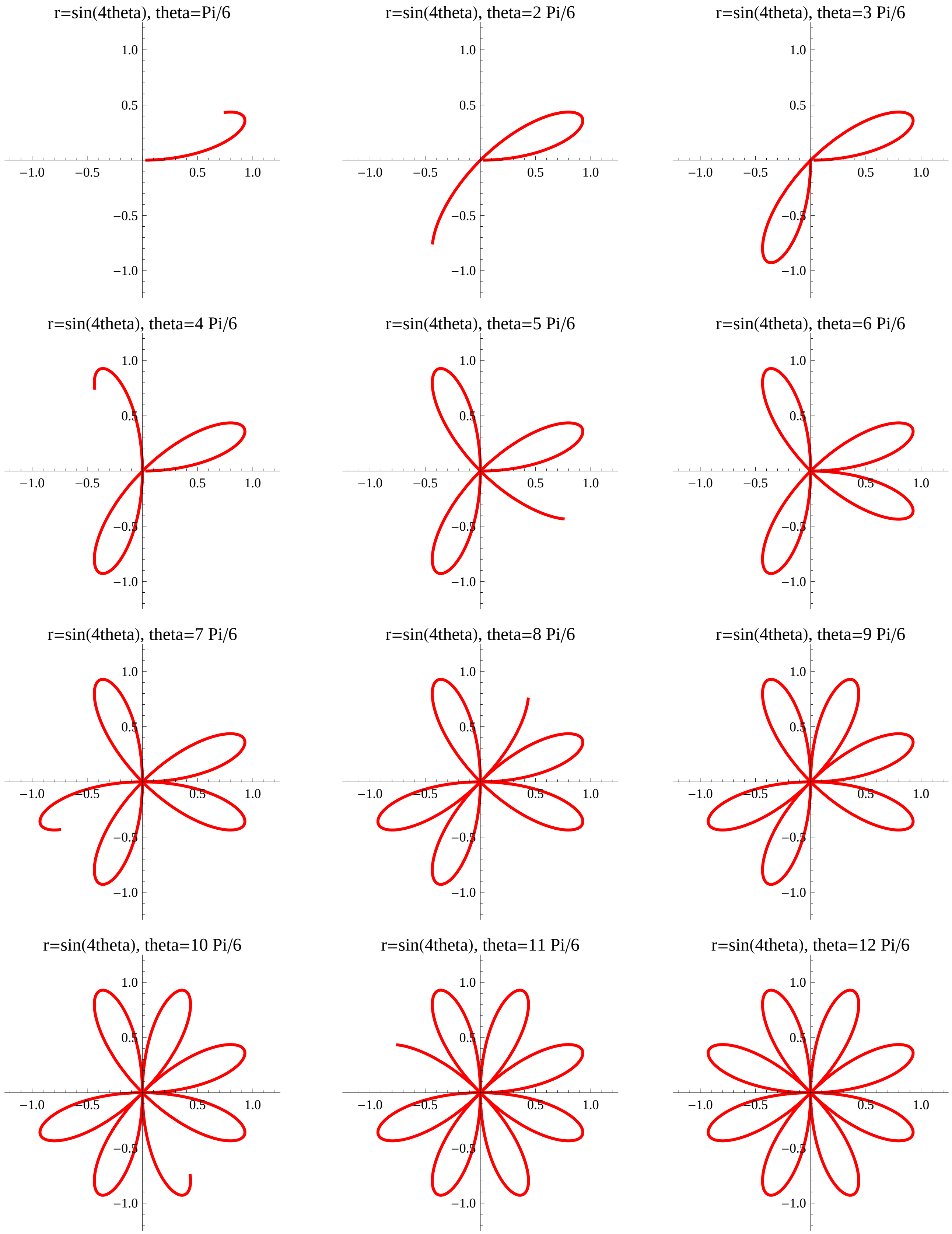
EXAMPLE: Sketch the curve $r = \sin(3\theta)$, $0 \le \theta \le \pi$ (three-leaved rose).

Solution: We have



EXAMPLE: Sketch the curve $r = \sin(4\theta)$, $0 \le \theta \le 2\pi$ (eight-leaved rose).

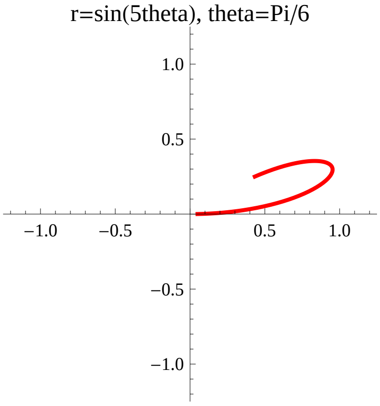
Solution: We have



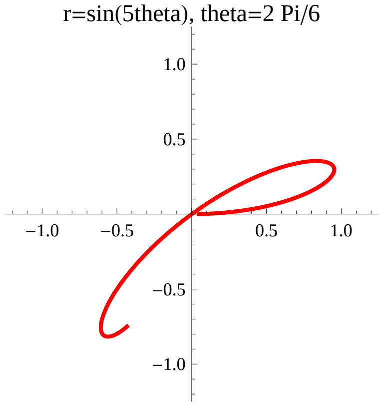
EXAMPLE: Sketch the curve $r = \sin(5\theta)$, $0 \le \theta \le 2\pi$ (five-leaved rose).

Solution: We have

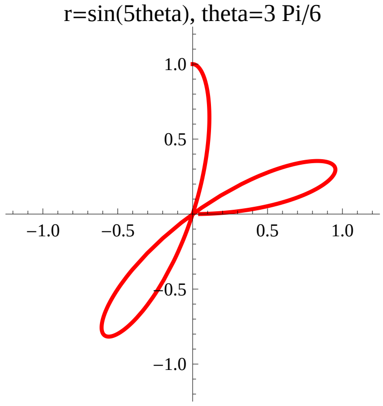
r=sin(5theta), theta=Pi/6



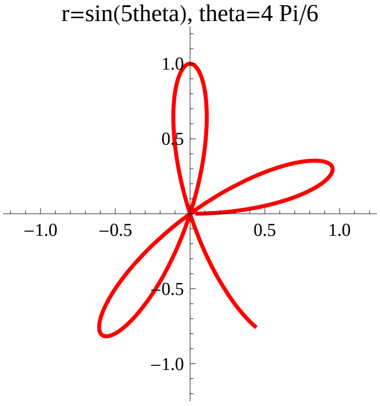
r=sin(5theta), theta=2 Pi/6



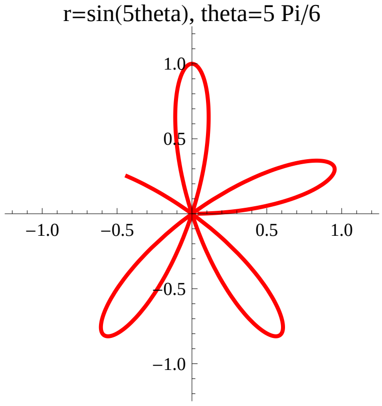
r=sin(5theta), theta=3 Pi/6



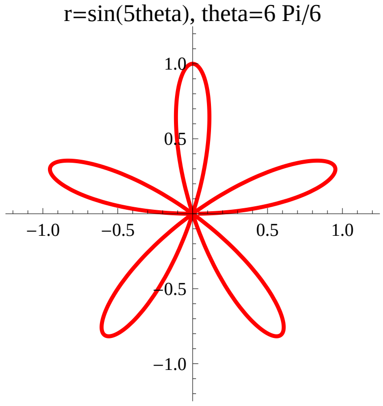
r=sin(5theta), theta=4 Pi/6



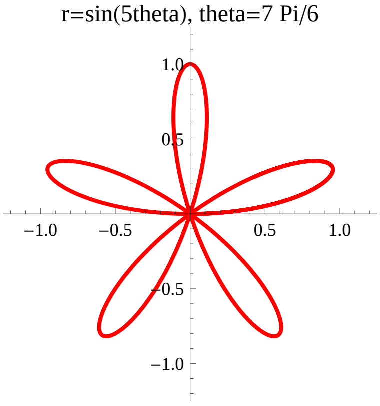
r=sin(5theta), theta=5 Pi/6



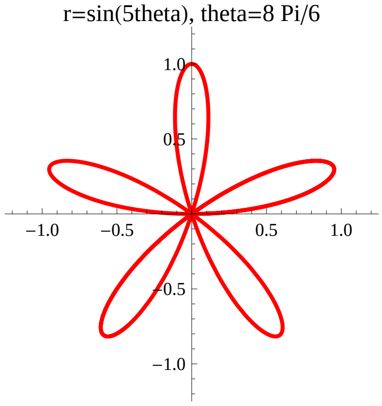
r=sin(5theta), theta=6 Pi/6



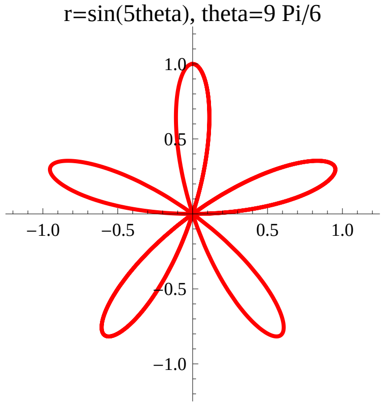
r=sin(5theta), theta=7 Pi/6



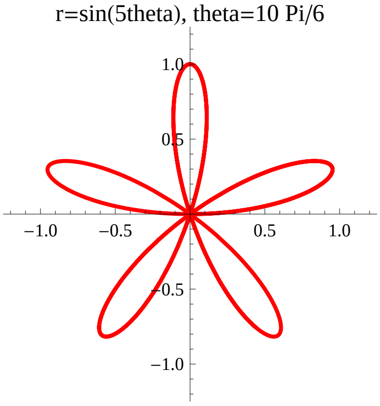
r=sin(5theta), theta=8 Pi/6



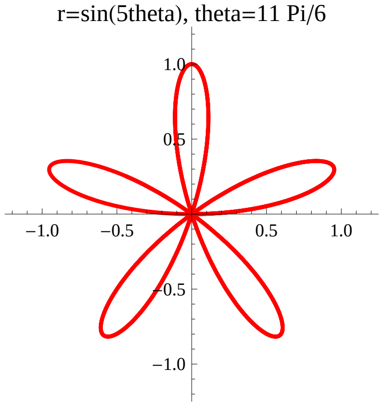
r=sin(5theta), theta=9 Pi/6



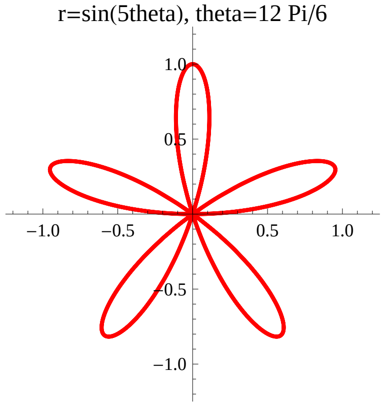
r=sin(5theta), theta=10 Pi/6



r=sin(5theta), theta=11 Pi/6

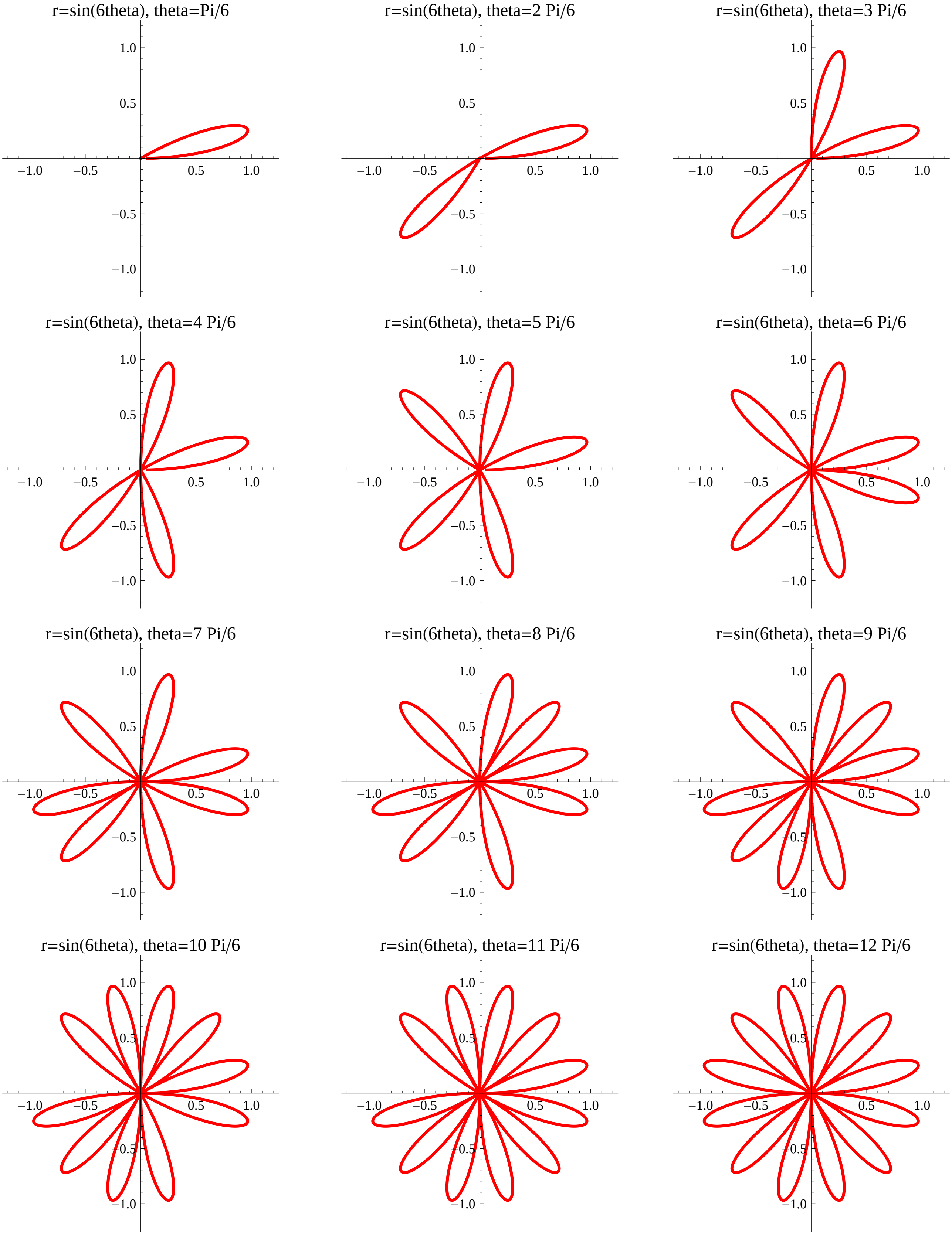


r=sin(5theta), theta=12 Pi/6



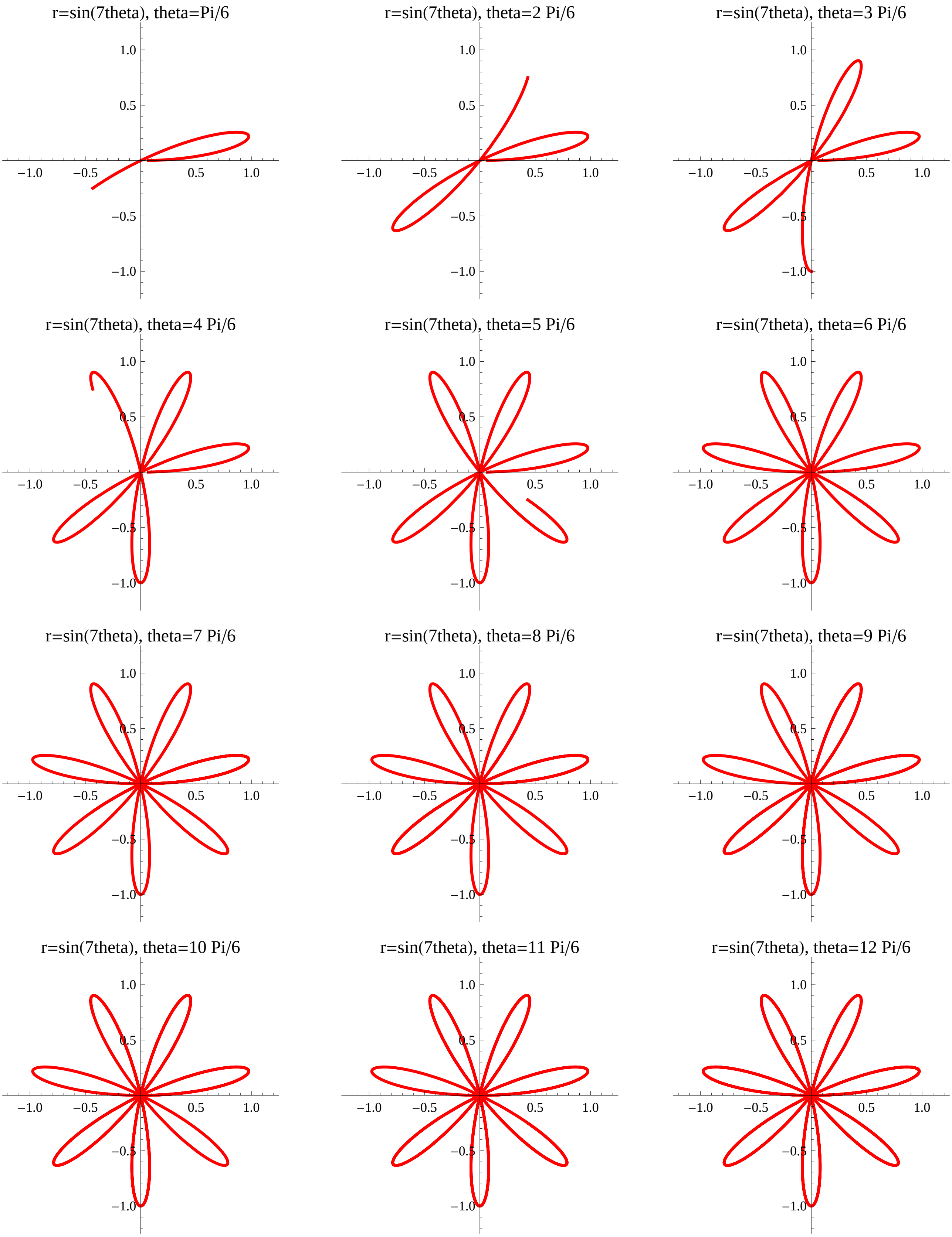
EXAMPLE: Sketch the curve $r = \sin(6\theta)$, $0 \leq \theta \leq 2\pi$ (twelve-leaved rose).

Solution: We have



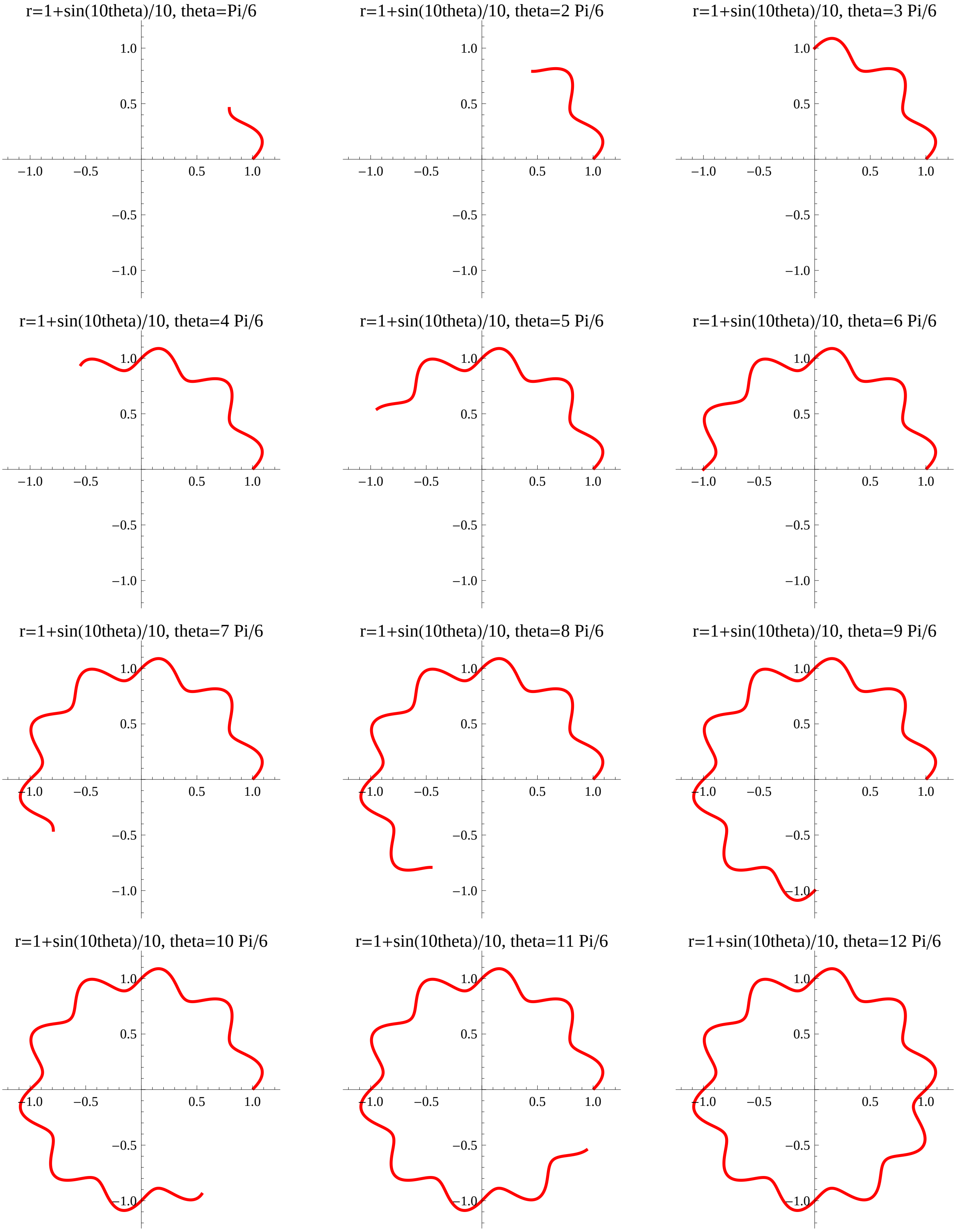
EXAMPLE: Sketch the curve $r = \sin(7\theta)$, $0 \le \theta \le 2\pi$ (seven-leaved rose).

Solution: We have



EXAMPLE: Sketch the curve $r = 1 + \dfrac{1}{10}\sin(10\theta),\ 0 \le \theta \le 2\pi$.

Solution: We have



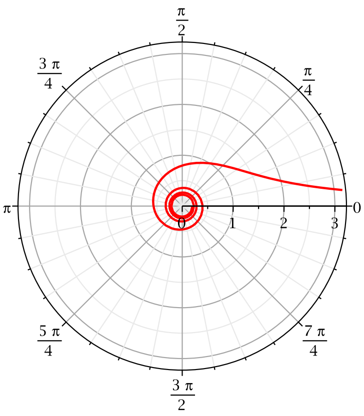
EXAMPLE: Match the polar equations with the graphs labeled I-VI:

(a) $r = \sin(\theta/2)$ (b) $r = \sin(\theta/4)$

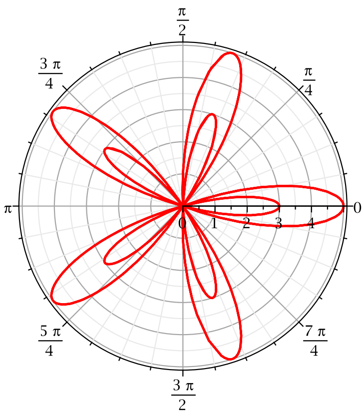
(c) $r = \sin\theta + \sin^3(5\theta/2)$ (d) $r = \theta\sin\theta$

(e) $r = 1 + 4\cos(5\theta)$ (f) $r = 1/\sqrt{\theta}$

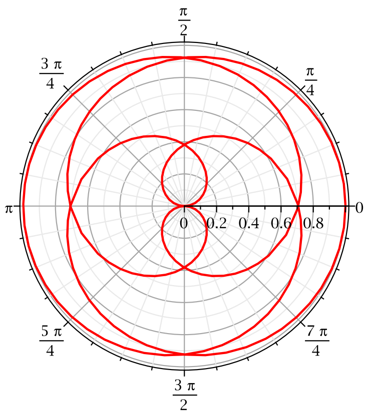
I



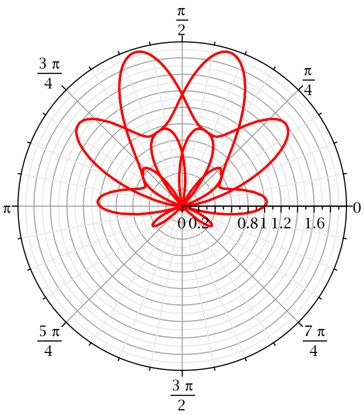
II



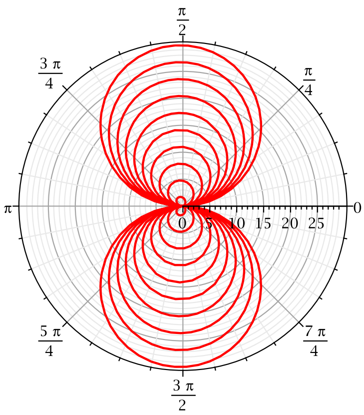
III



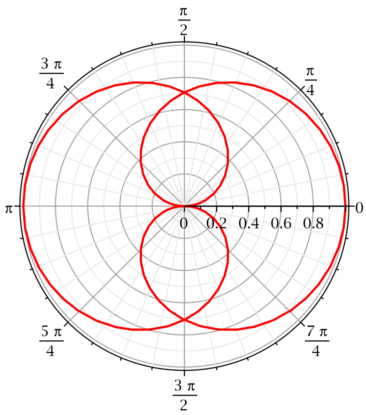
IV



V



VI



Tangents to Polar Curves

To find a tangent line to a polar curve $r = f(\theta)$ we regard θ as a parameter and write its parametric equations as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad\qquad y = r\sin\theta = f(\theta)\sin\theta$$

Then, using the method for finding slopes of parametric curves and the Product Rule, we have

$$\boxed{\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}} \tag{1}$$

EXAMPLE:

(a) For the cardioid $r = 1 + \sin\theta$, find the slope of the tangent line when $\theta = \pi/3$.

(b) Find the points on the cardioid where the tangent line is horizontal or vertical.

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(a) For the cardioid $r = 1 + \sin\theta$, find the slope of the tangent line when $\theta = \pi/3$.

(b) Find the points on the cardioid where the tangent line is horizontal or vertical.

Solution: Using Equation 1 with $r = 1 + \sin\theta$, we have

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta}$$

$$= \frac{\cos\theta(1+2\sin\theta)}{1 - 2\sin^2\theta - \sin\theta} = \frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}$$

(a) The slope of the tangent at the point where $\theta = \pi/3$ is

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/3} = \frac{\cos(\pi/3)(1+2\sin(\pi/3))}{(1+\sin(\pi/3))(1-2\sin(\pi/3))} = \frac{\frac{1}{2}(1+\sqrt{3})}{(1+\sqrt{3}/2)(1-\sqrt{3})}$$

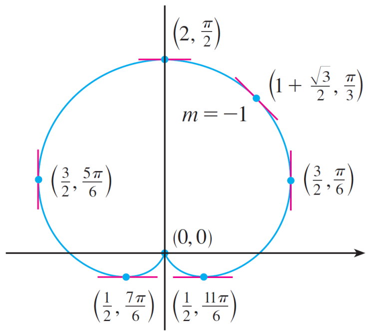
$$= \frac{1+\sqrt{3}}{(2+\sqrt{3})(1-\sqrt{3})} = \frac{1+\sqrt{3}}{-1-\sqrt{3}} = -1$$

(b) Observe that

$$\frac{dy}{d\theta} = \cos\theta(1+2\sin\theta) = 0 \quad \text{when} \quad \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\frac{dx}{d\theta} = (1+\sin\theta)(1-2\sin\theta) = 0 \quad \text{when} \quad \theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

Therefore there are horizontal tangents at the points $(2, \ \pi/2), (\frac{1}{2}, 7\pi/6), (\frac{1}{2}, 11\pi/6)$ and vertical tangents at $(\frac{3}{2}, \pi/6)$ and $(\frac{3}{2}, 5\pi/6)$. When $\theta = 3\pi/2$, both $dy/d\theta$ and $dx/d\theta$ are 0, so we must be careful. Using l'Hospital's Rule, we have



$$\lim_{\theta\to(3\pi/2)^-}\frac{dy}{dx} = \left(\lim_{\theta\to(3\pi/2)^-}\frac{1+2\sin\theta}{1-2\sin\theta}\right)\left(\lim_{\theta\to(3\pi/2)^-}\frac{\cos\theta}{1+\sin\theta}\right)$$

$$= -\frac{1}{3}\lim_{\theta\to(3\pi/2)^-}\frac{\cos\theta}{1+\sin\theta} = -\frac{1}{3}\lim_{\theta\to(3\pi/2)^-}\frac{-\sin\theta}{\cos\theta} = \infty$$

By symmetry,

$$\lim_{\theta\to(3\pi/2)^+}\frac{dy}{dx} = -\infty$$

Thus there is a vertical tangent line at the pole.

REMARK: Instead of having to remember Equation 1, we could employ the method used to derive it. For instance, in the above Example we could have written

$$x = r\cos\theta = (1+\sin\theta)\cos\theta = \cos\theta + \frac{1}{2}\sin 2\theta$$

$$y = r\sin\theta = (1+\sin\theta)\sin\theta = \sin\theta + \sin^2\theta$$

Then we would have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta + 2\sin\theta\cos\theta}{-\sin\theta + \cos 2\theta} = \frac{\cos\theta + \sin 2\theta}{-\sin\theta + \cos 2\theta}$$

which is equivalent to our previous expression.