B U Department of Mathematics

Math 102 Calculus II

Fall 2000 Second Midterm

1. Consider the limit:

$$\lim_{(x,y)\to(0,1)} \frac{x+y-1}{\sqrt{x^2+(y-1)^2}}.$$

(a) Write three particular paths to approach the point (0,1) while taking the limit. What is the equation of the general family of linear straight paths for such approaches? Determine constants in this equation as much as you can.
(b) Verify that the limit does NOT exist.

Solution:

(a) Three particular paths might be: (i) $x=0$, (ii) $y=1$ and (iii) $y=(x+1)^2$. The equation of the family of linear paths is: $y=mx+n$, where m and n are constants. Since each of these paths passes through the point $(0,1)$, it follows that $1=m.0+n$, therefore $n=1$. Hence, the equation of the family of linear paths through the point $(0,1)$ is $y=mx+1$.
(b) Let us evaluate the limit along $y=1$:

$$\lim_{\substack{(x,y)\to(0,1)\\ y=1}} \frac{x+y-1}{x^2+(y-1)^2} = \lim_{x\to 0}\frac{x}{\sqrt{x^2}} = \begin{cases} 1, & x\to 0^+ \\ -1, & x\to 0^- \end{cases}.$$

Therefore, NO LIMIT at $(0,1)$.

2. Find the maximum and minimum values of the function $f(x,y,z) = x+y+z$ subject to $x^2+y^2+z^2=\frac{3}{8}$.

Solution:

One uses the Method of Langrange Multipliers: set $g(x,y,z)=x^2+y^2+z^2-\frac{3}{8}$ so that the constraint for $f(x,y,z)$ is $g=0$. Then, requiring $\vec{\nabla} f(x,y,z)=\lambda\vec{\nabla} g(x,y,z)$, we get:

$$1=\lambda(2x), \quad 1=\lambda(2y), \quad 1=\lambda(2z)$$

which yields $x=y=z=1/2\lambda$. Replacing into the constraint equation:

$$\left(\frac{1}{2\lambda}\right)^2+\left(\frac{1}{2\lambda}\right)^2+\left(\frac{1}{2\lambda}\right)^2=\frac{3}{8}\Rightarrow\frac{3}{4\lambda^2}=\frac{3}{8}\Rightarrow\lambda^2=2\Rightarrow\lambda=\pm\sqrt{2}.$$

So the extremum points subject to $g(x,y,z)=0$ are $x=y=z=\frac{1}{2\lambda}=\mp\frac{1}{2\sqrt{2}}$.The values of f at these points are

$$f(\frac{1}{2\sqrt{2}},\frac{1}{2\sqrt{2}},\frac{1}{2\sqrt{2}})=\frac{3}{4}\sqrt{2}$$

and

$$f(-\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}) = -\frac{3}{4}\sqrt{2}.$$

The first one is the absolute maximum and the second one is the absolute minimum for f subject to the constraint.

3. Consider the function $H(x, y) = -x^3 - y^3 + 3xy + 10$.
(a) Find all critical points for $H(x, y)$.
(b) Is there an <u>absolute</u> minimum or maximum for $H(x, y)$? (No calculations, but justify your answer.)
(c) Suppose that H represents the height locally. Starting at $(-1, -1)$, which direction one should follow with a constant speed to go to a higher place as quick as possible? What is the direction for a lower place? (Give the directions as unit vectors).
(d) For a person at $(-1, -1)$, what is the rate of change of the height towards the origin?
(e) Write the equations for the line normal to the graph of $H(x, y)$ and for the tangent plane at $(-1, -1)$.

Solution:

(a) At critical points, $\frac{\partial H}{\partial x} = 0$ and $\frac{\partial H}{\partial y} = 0$, or, one or both of them are undefined. $\frac{\partial H}{\partial x} = -3x^2 + 3y$, $\frac{\partial H}{\partial y} = -3y^2 + 3x$ are defined everywhere. We equate them to zero and get $y = x^2$ from the former and $x = y^2$ from the latter. Hence $x = x^4 \Rightarrow x(1 - x^3) = 0$. So, the critical points are $(0, 0)$ and $(1, 1)$.

(b) NO. First note that since $H(x, y)$ is defined everywhere, it does not necessarily have absolute extrema. Now, as both x and $y \to +\infty$, $H(x, y) \to -\infty$. This is not easy to observe but it is true. Also, as both x and $y \to -\infty$, $H(x, y) \to +\infty$ so, there is no absolute minimum nor maximum.

(c) For path of steepest ascent, we need $D_{\hat{u}}H(x, y) = \vec{\nabla}H(x, y) \cdot \hat{u}$ highest so that we get

$$\vec{\nabla}H(x, y) // \hat{u} \Rightarrow \hat{u} = \frac{\vec{\nabla}H(-1, -1)}{\left|\left|\vec{\nabla}H(-1, -1)\right|\right|}.$$

Since $\vec{\nabla}H(x, y) = (-3x^2 + 3y)\vec{\imath} + (-3y^2 + 3x)\vec{\jmath}$ and $\vec{\nabla}H(-1, -1) = -6\vec{\imath} - 6\vec{\jmath}$:

$$\hat{u} = \frac{\vec{\nabla}H(-1, -1)}{\left|\left|\vec{\nabla}H(-1, -1)\right|\right|} = \frac{-6\vec{\imath} - 6\vec{\jmath}}{\sqrt{(-6)^2 + (-6)^2}} = -\frac{1}{\sqrt{2}}(\vec{\imath} + \vec{\jmath})$$

is for a higher place the quickest direction.
For low places, any unit direction $\hat{v}$ such that $\hat{v} \cdot \hat{u} < 0$ will do. In particular, the quickest descent is along $-\hat{u}$.

(d) Towards the origin, the direction is given by the vector

$$\vec{u} = (0 - (-1))\vec{\imath} + (0 - (-1))\vec{\jmath} = \vec{\imath} + \vec{\jmath}$$

The rate of change of $H(x, y)$ towards the origin at $(-1, -1)$ is given by:

$$D_{\hat{u}}H(-1, -1) = (-6\vec{\imath} - 6\vec{\jmath}) \cdot (\frac{\vec{\imath} + \vec{\jmath}}{\sqrt{2}}) = -\frac{12}{\sqrt{2}} = -6\sqrt{2}$$

(e) At $(-1,-1)$, $z = H(-1,-1) = 15$. The upward normal vector $\vec{N}$ to the graph of $z = H(x,y)$ at $(-1,-1,15)$, say P, is given by

$$\vec{N} = \vec{\nabla}(z + x^3 + y^3 - 3xy - 10) = 6\vec{\imath} + 6\vec{\jmath} + \vec{k}.$$

So the normal line is parametrized as

$$l : \vec{r}(t) = (-\vec{\imath} - \vec{\jmath} + 15\vec{k}) + t(6\vec{\imath} + 6\vec{\jmath} + \vec{k}).$$

If $Q(x,y,z)$ is an arbitrary point on the tangent plane σ at P, the equation of the tangent plane is given by:

$$\begin{aligned}
&\sigma : \vec{N} \cdot \vec{PQ} = 0 \\
&(6\vec{\imath} + 6\vec{\jmath} + \vec{k}) \cdot \left[(x+1)\vec{\imath} + (y+1)\vec{\jmath} + (z-15)\vec{k}\right] = 0 \\
&\sigma : 6x + 6y + z = 3.
\end{aligned}$$

4. Consider a function $f = f(\rho)$ where $\rho = \sqrt{x^2 + y^2 + z^2}$. If f satisfies the Laplacian:

$$\vec{\nabla}^2 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = a\frac{\partial^2 f}{\partial \rho^2} + b\frac{\partial f}{\partial \rho},$$

find a and b.

Solution:

One uses chain rule succesively:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}\frac{\partial f}{\partial \rho} = \frac{x}{\rho}\frac{\partial f}{\partial \rho}.$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x}{\rho}\right)\frac{\partial f}{\partial \rho} + \frac{x}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \rho}\right) &= \left(\frac{1}{\rho} - \frac{x^2}{\rho^3}\right)\frac{\partial f}{\partial \rho} + \frac{x}{\rho}\left[\frac{\partial}{\partial \rho}\left(\frac{\partial f}{\partial \rho}\right)\frac{\partial \rho}{\partial x}\right] \\
&= \left(\frac{1}{\rho} - \frac{x^2}{\rho^3}\right)\frac{\partial f}{\partial \rho} + \frac{x^2}{\rho^2}\frac{\partial^2 f}{\partial \rho^2}.
\end{aligned}$$

Noticing the x, y, z symmetry of the Laplacian and of $\rho(x,y,z)$, one obtains (without calculations)

$$\frac{\partial^2 f}{\partial y^2} = \left(\frac{1}{\rho} - \frac{y^2}{\rho^3}\right)\frac{\partial f}{\partial \rho} + \frac{y^2}{\rho^2}\frac{\partial^2 f}{\partial \rho^2}$$

and

$$\frac{\partial^2 f}{\partial z^2} = \left(\frac{1}{\rho} - \frac{z^2}{\rho^3}\right)\frac{\partial f}{\partial \rho} + \frac{z^2}{\rho^2}\frac{\partial^2 f}{\partial \rho^2}$$

So adding them up:

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} &= \frac{x^2 + y^2 + z^2}{\rho^2}\frac{\partial^2 f}{\partial \rho^2} + \left(\frac{3}{\rho} - \frac{x^2 + y^2 + z^2}{\rho^3}\right)\frac{\partial f}{\partial \rho} \\
&= \frac{\partial^2 f}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial f}{\partial \rho} \\
&\Rightarrow a = 1 \text{ and } b = \frac{2}{\rho}.
\end{aligned}$$

B U Department of Mathematics

Math 102 Calculus II

Fall 2001 Second Midterm

1. **a)** Let $f(x,y) = \begin{cases} \dfrac{4xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. Is $f(x,y)$ continuous at $(0,0)$? Justify your answer!

Solution:

First notice that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ has a $\frac{0}{0}$ type indeterminacy. Use polar coordinates: $x = r\cos\theta$ and $y = r\sin\theta$ so that $r^2 = x^2+y^2$, $r \geq 0$. We now have:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} \frac{4r^2\sin\theta\cos\theta}{\sqrt{r^2}} = \lim_{r\to 0} 4r\sin\theta\cos\theta = 0$$

Hence, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$. Thus, $f(x,y)$ is continuous at $(0,0)$.

b) Prove or disprove that the directional derivative of $f(x,y)$ in the direction of $\boldsymbol{u}+\boldsymbol{v}$ at the point $P(x_0,y_0)$ is equal to $D_u f(x_0,y_0) + D_v f(x_0,y_0)$ where $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors.

Solution:

The equality holds if $f(x,y)$ is differentiable at P. Otherwise the equality does not necessarily hold. For example, the function $f(x,y)$ of part (a) is constant along $x-$ and $y-$axes so that $D_i f(0,0) = D_j f(0,0) = 0$ but $D_{i+j} f(0,0) = 2\sqrt{2}$.

2. Find the point(s) on the graph of $x^2+4y^2+z^2 = 12$ where the tangent plane is perpendicular to the line $\dfrac{1-x}{2} = 1-y = \dfrac{z-3}{-2}$.

Solution:

$(2x, 8y, 2z) \parallel (2,1,2) \Rightarrow (2x,8y,2z) = \alpha(2,1,2)$. Solving for x, y and z we get $x = \alpha$, $y = \alpha/8$, $z = \alpha$ which should determine a point on the surface so that:

$$\alpha^2 + \frac{\alpha^2}{16} + \alpha^2 = 12 \Rightarrow \alpha^2 = \frac{192}{33} \Rightarrow \alpha = \pm\frac{8}{\sqrt{11}}.$$

So we get the points $(\frac{8}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{8}{\sqrt{11}})$ and $(-\frac{8}{\sqrt{11}}, -\frac{1}{\sqrt{11}}, -\frac{8}{\sqrt{11}})$.

3. **a)** Let $f(x,y)$ and $g(x,y)$ be differentiable functions. Show that $\nabla(fg) = f\nabla g + g\nabla f$.

Solution:

$$\nabla(fg) = (fg_x + gf_x)\boldsymbol{i} + (fg_y + gf_y)\boldsymbol{j} = f(g_x\boldsymbol{i} + g_y\boldsymbol{j}) + g(f_x\boldsymbol{i} + f_y\boldsymbol{j}) = f\nabla g + g\nabla f$$

b) The plane $2y - 3z = 8$ intersects the cone $z^2 = 4x^2 + 4y^2$ in an ellipse. Find the highest and lowest points of intersection.

Solution:

$2y - 3z = 8 \Rightarrow z = \frac{2y-8}{3}$. We wish to find the maximum and minimum values for the function $z = f(x, y) = \frac{2y-8}{3}$ subject to the constraint $z^2 = 4x^2 + 4y^2$. Hence $(\dfrac{2y-8}{3})^2 = 4x^2 + 4y^2 \Rightarrow y^2 - 8y + 16 = 9x^2 + 9y^2$.

$$\begin{aligned} g(x,y) &= 9x^2 + 8y^2 + 8y - 16 = 0 \\ \nabla f(x,y) &= \lambda \nabla g(x,y) \\ \frac{2}{3}\boldsymbol{j} &= \lambda(18x\boldsymbol{i} + (16y+8)\boldsymbol{j}), \end{aligned}$$

which means

$$18\lambda x = 0 \quad \text{and} \quad (16y+8)\lambda = \frac{2}{3}.$$

So, either $\lambda = 0$ or $x = 0$. Since $\lambda = 0$ is inconsistent with the second equation it follows that $x = 0$. Substituting in $g(x, y)$, we get $8y^2 + 8y - 16 = 0 \Rightarrow y^2 + y - 2 = 0 \Rightarrow y = 1, y = -2. \Rightarrow z = f(0, 1) = -2$ and $z = f(0, -2) = -4$. The highest point of intersection is $(0, 1, -2)$ and the lowest point of intersection is $(0, -2, -4)$.

4. Find $\displaystyle\iint_R \frac{\sin x}{x} dA$, where R is the triangle with vertices $(0, 0), (2, 0)$ and $(2, 2)$. (Sketch the region R.)

Solution:

$$\iint_R \frac{\sin x}{x} dA = \int_0^2 \int_0^x \frac{\sin x}{x} dy dx = \int_0^2 y \left.\frac{\sin x}{x}\right|_{y=0}^{y=x} dx =$$

$$\int_0^2 \sin x \, dx = -\cos x|_0^2 = 1 - \cos 2.$$

B U Department of Mathematics

Math 102 Calculus II

Fall 2002 Second Midterm

1. Let $w = F(xz, yz)$ be a differentiable function of independent variables x, y and z. Show that w satisfies the partial differential equation:

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = z\frac{\partial w}{\partial z}.$$

Solution:

Let us write the function as $w = F(u, v)$ where $u = xz$ and $v = yz$. By the chain rule

$$\begin{aligned}\frac{\partial w}{\partial x} &= \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} &= z\frac{\partial F}{\partial u}\\ \frac{\partial w}{\partial y} &= \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} &= z\frac{\partial F}{\partial v}\end{aligned}$$

which implies that

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = xz\frac{\partial F}{\partial u} + yz\frac{\partial F}{\partial v}.$$

On the other hand,

$$\begin{aligned}z\frac{\partial w}{\partial z} &= z\left(\frac{\partial F}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial z}\right)\\ &= zx\frac{\partial F}{\partial u} + zy\frac{\partial F}{\partial v}.\end{aligned}$$

Thus, the given partial differential equation is obviously satisfied.

2. **(a)** Do the surfaces $z = \sqrt{x^2 + y^2}$ and $z = \frac{1}{10}(x^2 + y^2) + \frac{5}{2}$ intersect? If so, describe the curve of intersection.

(b) Show that these surfaces have a common tangent plane at their point(s) of intersection, and that the equation of this common tangent plane at a point $P_0(x_0, y_0, z_0)$ on the curve of intersection is given by $xx_0 + yy_0 = 5z$.

Solution:

(a) The projection onto $xy-$ plane of the curve of intersection is given by

$$\sqrt{x^2 + y^2} = \frac{1}{10}(x^2 + y^2) + \frac{5}{2}.$$

Let $\sqrt{x^2 + y^2} = u$. Then:

$$\begin{aligned}&10u = u^2 + 25 \Rightarrow u^2 - 10u + 25 = 0\\ \Rightarrow\ &(u-5)^2 = 0 \Rightarrow u = 5\\ \Rightarrow\ &x^2 + y^2 = 25 \quad \text{and} \quad z = 5.\end{aligned}$$

Therefore they intersect at the circle $x^2+y^2=25$ lying on the plane $z=5$.

(b) Let us find normal vectors for the tangent planes to these surfaces:

First, $z^2=x^2+y^2 \Rightarrow 2z\dfrac{\partial z}{\partial x}=2x \Rightarrow \dfrac{\partial z}{\partial x}=\dfrac{x}{z}$ and similarly $\dfrac{\partial z}{\partial y}=\dfrac{y}{z}$, by implicit differentiation. So we get:

$$\boldsymbol{N}_1=\frac{x_0}{z_0}\boldsymbol{i}+\frac{y_0}{z_0}\boldsymbol{j}-\boldsymbol{k} \quad \text{at} \quad P_0(x_0,y_0,z_0).$$

Since $z_0=5$, we substitute that in the normal vector:

$$\boldsymbol{N}_1=\frac{x_0}{5}\boldsymbol{i}+\frac{y_0}{5}\boldsymbol{j}-\boldsymbol{k}.$$

For the paraboloid $z=\dfrac{1}{10}(x^2+y^2)+\dfrac{5}{2}$, the partial derivatives are $\dfrac{\partial z}{\partial x}=\dfrac{x}{5}$ and $\dfrac{\partial z}{\partial y}=\dfrac{y}{5}$. Hence, a normal vector is:

$$\boldsymbol{N}_2=\frac{x_0}{5}\boldsymbol{i}+\frac{y_0}{5}\boldsymbol{j}-\boldsymbol{k} \quad \text{at} \quad P_0(x_0,y_0,z_0).$$

Since normal vectors are the same, the surfaces share a common tangent plane at all points P_0 on the curve of intersection. The equation of this common tangent plane is determined by:

$$\frac{x_0}{5}(x-x_0)+\frac{y_0}{5}(y-y_0)=(z-z_0).$$

Since $z_0=5$,

$$\begin{aligned} x_0(x-x_0)+y_0(y-y_0) &= 5z-25 \\ x_0x+y_0y-(x_0^2+y_0^2) &= 5z-25, \quad \text{where } x_0^2+y_0^2=25 \\ \Rightarrow x_0x+y_0y &= 5z. \end{aligned}$$

3. Find the point(s) on the cone $z=\sqrt{x^2+y^2}$ that are nearest to the point $(3,1,0)$.

Solution:

We are to minimize $(\text{distance})^2=(x-3)^2+(y-1)^2+z^2$ where $z^2=x^2+y^2$. Let $F(x,y)=(x-3)^2+(y-1)^2+x^2+y^2$ where x and y are independent variables. Critical points of $F(x,y)$ are determined by:

$$\left.\begin{array}{l} F_x=0=2(x-3)+2x \\ F_y=0=2(y-1)+2y \end{array}\right\} \Rightarrow 2x=3 \Rightarrow x=3/2 \text{ and } y=1/2.$$

One observes geometrically that this point gives a minimum. So the minimum distance is:

$$\sqrt{\left(\frac{3}{2}-3\right)^2+\left(\frac{1}{2}-1\right)^2+\frac{9}{4}+\frac{1}{4}}=\sqrt{\left(\frac{9}{4}\right)+\left(\frac{1}{4}\right)+\frac{10}{4}}=\sqrt{\frac{20}{4}}=\sqrt{5}.$$

Alternatively, by the method of Lagrange multipliers, minimize $(x-3)^2+(y-1)^2+z^2$ subject to the constraint $x^2+y^2-z^2=0$:

Let $H(x,y,z,\lambda)=(x-3)^2+(y-1)^2+z^2-\lambda(x^2+y^2-z^2)$

$$\begin{array}{rlcl} (1)\ H_x & = & 0 & = & 2(x-3)-2\lambda x, \\ (2)\ H_y & = & 0 & = & 2(y-1)-2\lambda y, \\ (3)\ H_z & = & 0 & = & 2z(1+\lambda), \\ (4)\ H_\lambda & = & 0 & = & -(x^2+y^2-z^2). \end{array}$$

From (3), either $z=0$ or $\lambda=-1$. If $z=0$, from (4)

$$x^2+y^2=0 \Rightarrow x=y=z=0$$

This gives the origin, with a distance that is equal to $\sqrt{9+1}=\sqrt{10}$. If $\lambda=-1$, from (1) and (2) it follows that:

$$\begin{array}{rcl} 4x-6=0 & \Rightarrow & x=3/2 \\ 4y-2=0 & \Rightarrow & y=1/2, \end{array}$$

and from (4),

$$z^2=\frac{9}{4}+\frac{1}{4}=\frac{10}{4}\Rightarrow z=\frac{\sqrt{10}}{2}.$$

Hence the point is $(3/2,1/2,\sqrt{10}/2)$, with the distance:

$$\sqrt{\left(\frac{3}{2}-3\right)^2+\left(\frac{1}{2}-1\right)^2+\frac{10}{4}}=\sqrt{5}.$$

Since this is smaller than $\sqrt{10}$, the point must be $(3/2,1/2,\sqrt{10}/2)$.

4. The following expression represents the volume of a solid bounded above by a surface and below by a region R of the $xy-$plane :

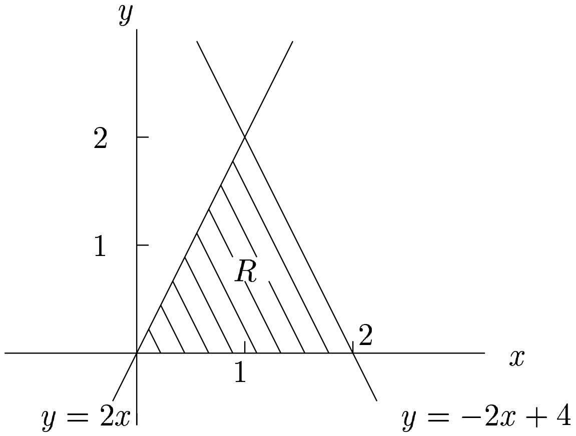
$$\int_0^1\int_0^{2x}\frac{1}{1+y^2}dydx+\int_1^2\int_0^{-2x+4}\frac{1}{1+y^2}dydx$$

(a) What is the equation of this surface?
(b) Skech the region R over which the integration is done.
(c) Write an equivalent double integral for this volume, with the order of integration reversed (DO NOT EVALUATE).

Solution:

(a) The surface is given by $z=\dfrac{1}{1+y^2}$.

(b)



(c) Volume $= \displaystyle\int_0^2 \int_{y/2}^{2-y/2} \frac{1}{1+y^2} dxdy.$

B U Department of Mathematics

Math 102 Calculus II

Fall 2003 Second Midterm

1. A particle moves along a helix with position function $\boldsymbol{r}(t)=\cos t\boldsymbol{i}+\sin t\boldsymbol{j}+t\boldsymbol{k}$. (a) Find the function $D(x,y,z)$ giving the distance from the particle to the origin. (b) Find the rate $\frac{dD}{dt}$ at which the distance from the particle to the origin changes as a function of time. (c) Find $\lim_{t\to\infty}\frac{dD}{dt}$ and $\lim_{t\to-\infty}\frac{dD}{dt}$

Solution:

(a) $D(x,y,z)=\sqrt{x^2+y^2+z^2}$ where $x=\cos t, y=\sin t, z=t$ $D(t)=\sqrt{\cos^2 t+\sin^2 t+t^2}=\sqrt{1+t^2}$

(b) $\frac{dD}{dt}=\frac{\partial D}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial D}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial D}{\partial z}\cdot\frac{dz}{dt}$

$=\frac{2x}{2\sqrt{x^2+y^2+z^2}}(-\sin t)+\frac{2y}{2\sqrt{x^2+y^2+z^2}}(\cos t)+\frac{2z}{2\sqrt{x^2+y^2+z^2}}$

$=\frac{-x\sin t+y\cos t+z}{\sqrt{x^2+y^2+z^2}}=\frac{-\cos t\sin t+\sin t\cos t+t}{\sqrt{1+t^2}}=\frac{t}{\sqrt{1+t^2}}$

(c) For $t>0$, $\lim_{t\to\infty}\frac{dD}{dt}=\lim_{t\to\infty}\frac{t}{\sqrt{1+t^2}}=\lim_{t\to\infty}\frac{t}{|t|\sqrt{\frac{1}{t^2}+1}}=1$ For $t<0$, $\lim_{t\to-\infty}\frac{dD}{dt}=$

$\lim_{t\to-\infty}\frac{t}{\sqrt{1+t^2}}=\lim_{t\to-\infty}\frac{t}{|t|\sqrt{\frac{1}{t^2}+1}}=-1$

2. Let $f(x,y)=xe^y$.
(a) Find the rate of change of f at the point $P(2,0)$ in the direction from P to $Q(1/2,2)$.
(b) In what direction does f have the maximum rate of change?

Solution:

(a) $\nabla f(x,y)=e^y\boldsymbol{i}+xe^y\boldsymbol{j}\to\nabla f(2,0)=\boldsymbol{i}+2\boldsymbol{j}$. The unit vector in the direction of $=-3/2\boldsymbol{i}+2\boldsymbol{j}$ is $\boldsymbol{u}=\frac{-3/2\boldsymbol{i}+2\boldsymbol{j}}{||PQ||}=\frac{-3/2\boldsymbol{i}+2\boldsymbol{j}}{5/2}=-3/5\boldsymbol{i}+4/5\boldsymbol{j}$. So, the rate of change of f in the direction from P to Q is $D_{\boldsymbol{u}}f(2,0)=\nabla f(2,0)\boldsymbol{u}=(\boldsymbol{i}+2\boldsymbol{j})(-3/5\boldsymbol{i}+4/5\boldsymbol{j})=-3/5+8/5=1$.

(b) f increases fastest in the direction of the gradient vector $\nabla f(2,0)=\boldsymbol{i}+2\boldsymbol{j}$.
The maximum rate of change is $||\nabla f(2,0)||=\sqrt{1+4}=\sqrt{5}$.

3. Find parametric equations for the tangent line to the curve that is determined by the intersection of the cone $z=\sqrt{x^2+y^2}$ and the plane $x+2y+2z=20$ at the point $P(4,3,5)$.

Solution:

The tangent line to the common curve is along the direction which is perpendicular to the normal vectors of each surface at the given point. Let $f(x,y,z)=\sqrt{x^2+y^2}-z$. Normal vector of f is $\boldsymbol{n}_1=\nabla f(4,3,5)=\left(\frac{2x}{2\sqrt{x^2+y^2}}\boldsymbol{i}+\frac{2y}{2\sqrt{x^2+y^2}}\boldsymbol{j}-\boldsymbol{k}\right)\Bigg|_{(4,3,5)}=4/5\boldsymbol{i}+$

$3/5\boldsymbol{j} - \boldsymbol{k}$ or $\boldsymbol{n}_1 = 4\boldsymbol{i} + 3\boldsymbol{j} - 5\boldsymbol{k}$. Normal vector of the plane is $\boldsymbol{n}_2 = \boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$.

Hence, $\boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 4 & 3 & -5 \\ 1 & 2 & 2 \end{vmatrix} = 16\boldsymbol{i} - 13\boldsymbol{j} + 5\boldsymbol{k}$ is tangent to the line; the line is determined by: $x(t) = 4 + 16t, y(t) = 3 - 13t, z(t) = 5 + 5t$.

4. Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

Solution:

$g(x, y) = x^2 + y^2 - 1$ is the constraint curve. The equation $\nabla f = \lambda \nabla g$ must be solved for (x, y, z, λ):
$2x\boldsymbol{i} + 4y\boldsymbol{j} = \lambda(2x\boldsymbol{i} + 2y\boldsymbol{j})$
$2x = 2\lambda x$
$4y = 2\lambda y$
$2x = 2\lambda x \Rightarrow x = 0$ or $\lambda = 1$,
$x = 0 \Rightarrow x^2 + y^2 = 1,\ y^2 = 1 \Rightarrow y = \mp 1$
$\lambda = 1 \Rightarrow y = 0$, so $x = \mp 1$
Maximum value is $f(0, \mp 1) = 2$, and minimum value is $f(\mp 1, 0) = 1$.

B U Department of Mathematics

Math 102 Calculus II

Fall 2004 Second Midterm

1.) Find the point(s) on the hyperbolic paraboloid $z = (y-2)^2 - (x+1)^2 + 1$ nearest to the point (-1,2,3).

Solution:

We want to minimize the function $F(x,y) = (x+1)^2 + (y-2)^2 + (z-3)^2$ with the constraint $g(x,y) = z - (y-2)^2 + (x+1)^2 - 1 = 0$. Using Lagrange's method

$$\vec{\nabla} F = \lambda \vec{\nabla} g$$

we get

$$\begin{aligned} 2(x+1) &= 2\lambda(x+1) \\ 2(y-2) &= -2\lambda(y-2) \\ 2(z-3) &= \lambda \end{aligned}$$

Investigating these equations we see that there are two sets of solutions:

Case I: $\lambda = 1$. This implies $y = 2$ and $z = 7/2$. However for these y and z values the constraint equation is not solvable for any x. Therefore, this case does not give any solution.

Case II: $\lambda = -1$. This gives $x = -1$ and $z = 5/2$. Using the constraint equation we get $(y-2)^2 = 3/2$. Therefore, the closest points are

$$\left(-1, 2 + \sqrt{\frac{3}{2}}, \frac{5}{2}\right) \quad and \quad \left(-1, 2 - \sqrt{\frac{3}{2}}, \frac{5}{2}\right).$$

2.) a) Show that the following limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^4}$$

Solution:

Approaching (0,0) along $y = mx$ we find,

$$\lim_{x\to 0} \frac{m^2x^3}{x^2 + m^4x^4} = 0.$$

Approaching (0,0) along $x = y^2$ we have,

$$\lim_{y \to 0} \frac{y^4}{y^4 + y^4} = \frac{1}{2}.$$

Therefore, the limit does not exist.

b) If $F(x, y) = 0$ find d^2y/dx^2 in terms of the partial derivatives of $F(x, y)$.

Solution:

Taking a partial derivative of $F(x, y) = 0$ with respect to x we get

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Differentiating this equation with respect to x we obtain

$$\frac{d^2y}{dx^2} = \frac{-F_{xx}F_y^2 + 2F_{xy}F_xF_y - F_{yy}F_x^2}{F_y^3}.$$

c) Find constant(s) "b" such that at any point of intersection of two spheres $(x - b)^2 + y^2 + z^2 = 5$ and $x^2 + y^2 + (z - 2)^2 = 3$, their tangent planes will be perpendicular to each other.

Solution:

The normals of the spheres are:

$$\begin{aligned} \vec{N_1} &= 2(x - b)\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \\ \vec{N_2} &= 2x\mathbf{i} + 2y\mathbf{j} + 2(z - 2)\mathbf{k} \end{aligned}$$

For perpendicular tangent planes $\vec{N_1}.\vec{N_2} = 0$ from which we obtain

$$x^2 + y^2 + z^2 = 2z + bx\,.$$

Intersection points should also satisfy the sphere equations simultaneously which are:

$$\begin{aligned} x^2 + y^2 + z^2 &= 5 - b^2 + 2bx \\ x^2 + y^2 + z^2 &= 4z - 1 \end{aligned}$$

Comparing the right hand sides of these three equations we find $b^2 = 4$, i.e, $b = \pm 2$.

d) Approximate $\sqrt{(2.95)^2 + (4.01)^2}$.

Solution:

We can use partial derivatives to approximate the value of a function at a certain point using

$$F(x,y) \approx F(x_0,y_0) + F_x(x_0,y_0)\Delta x + F_y(x_0,y_0)\Delta y\,.$$

For this problem $F(x,y) = \sqrt{x^2+y^2}$ and $x_0 = 3, y_0 = 4$. Moreover, $\Delta x = -0.05$ and $\Delta y = 0.01$ and the partial derivatives are

$$F_x = \frac{x}{\sqrt{x^2+y^2}}\,, \qquad F_y = \frac{y}{\sqrt{x^2+y^2}}\,.$$

Therefore,

$$F(2.95, 4.01) \approx F(3,4) + F_x(3,4)(-0.05) + F_y(3,4)(0.01) = 5 - 0.03 + 0.008 = 4.978\,.$$

3.) Find the volume of the solid region $x^2+y^2 = x$ that lies inside the solid sphere $x^2+y^2+z^2 \leq 1$.

Solution:

We want to find the volume of the circular cylinder $x^2+y^2 = x$ lying inside the unit sphere. First, note that z can be both positive and negative, i.e.,

$$z = \pm\sqrt{1-x^2-y^2}\,.$$

The required volume can be found from the following double integral,

$$V = 2\int\int_R \sqrt{1-x^2-y^2}dA\,.$$

The factor of 2 appears since the integrand has to be positive for a volume. The region R is the circle $x^2+y^2 = x$. Transforming this double integral into polar coordinates we get

$$V = 4\int_0^{\pi/2}\int_0^{\cos\theta} \sqrt{1-r^2}\,rdrd\theta\,.$$

Here we got another factor of 2 using symmetry around x-axis. Performing the radial integration we find

$$V = \frac{4}{3}\int_0^{\pi/2}(1-\sin^3\theta)d\theta = \frac{4}{3}\int_0^{\pi/2}(1-\sin\theta+\sin\theta\cos^2\theta)d\theta = \frac{6\pi-8}{9}\,.$$

4.) a) Reverse the order of integration of the following integration and then evaluate it.

$$I = \int_1^2\int_x^{2x} ydydx$$

Solution:

Reversing the order we find

$$I = \int_1^2 \int_1^y y dx dy + \int_2^4 \int_{y/2}^2 y dx dy\,.$$

Evaluating it we get,

$$I = \int_1^2 (y^2 - y) dy + \int_2^4 (2y - \frac{y^2}{2}) dy = \frac{7}{2}\,.$$

b) Transform the following iterated integral from rectangular to polar coordinates.

$$I = \int_0^2 \int_0^2 F(\sqrt{x^2 + y^2}) dx dy$$

Solution:

In polar coordinates this integral becomes,

$$I = \int_0^{\pi/4} \int_0^{2/\cos\theta} F(r)\, r dr d\theta + \int_{\pi/4}^{\pi/2} \int_0^{2/\sin\theta} F(r)\, r dr d\theta\,.$$

c) Which of the double integrals is larger?

$$I_1 = \int\int_R (x^4 + 6x^2y^2 + y^4) dA\ , \qquad I_2 = \int\int_R (4x^3y + 4xy^3) dA$$

Solution:

Note that

$$I = I_1 - I_2 = \int\int_R (x - y)^4 dA\,.$$

Since $(x-y)^4$ is always positive, $I = I_1 - I_2$ represents a volume and therefore it is always positive. This implies that I_1 is larger than I_2 for any region R.

B U Department of Mathematics

Math 102 Calculus II

Fall 2005 Second Midterm

1. Find the derivative of $f(x, y, z) = x^3 + y^3 z$ at $(-1, 2, 1)$ in the direction toward (0,3,3).

Solution:

A vector in the indicated direction is $(0, 3, 3) - (-1, 2, 1) =< 1, 1, 2 >$.
The unit vector in that direction is $\vec{u} = \frac{1}{\sqrt{6}} < 1, 1, 2 >$
$\vec{\nabla} f(x, y, z) =< 3x^2, 3y^2 z, y^3 >\Rightarrow \vec{\nabla} f(-1, 2, 1) =< 3, 12, 8 >$

Hence, the derivative of $f(x, y, z) = x^3 + y^3 z$ at $(-1, 2, 1)$ in the direction toward (0,3,3) is

$$\vec{\nabla} f(-1, 2, 1) \cdot \vec{u} =< 3, 12, 8 > \cdot \frac{1}{\sqrt{6}} < 1, 1, 2 >= \frac{1}{\sqrt{6}} \cdot 31$$

2. Find the point on the sphere $x^2 + y^2 + z^2 = 1$ furthest from the point $(2, 1, 2)$ using Lagrange multipliers.

Solution:

We want to maximize $f(x, y, z) = (x - 2)^2 + (y - 1)^2 + (z - 2)^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$.

$$\begin{aligned} \vec{\nabla} f &= < 2(x-2), 2(y-1), 2(z-2 > \\ \vec{\nabla} g &= < 2x, 2y, 2z > \end{aligned}$$

Let $\vec{\nabla} f = \lambda \vec{\nabla} g$. Then $< 2(x - 2), 2(y - 1), 2(z - 2) >= \lambda < 2x, 2y, 2z >$.
Thus

$$\left.\begin{aligned} 2(x-2) &= 2\lambda x \\ 2(y-1) &= 2\lambda y \\ 2(z-2) &= 2\lambda z \end{aligned}\right\} \Rightarrow \begin{aligned} x(1-\lambda) &= 2 \\ y(1-\lambda) &= 1 \\ z(1-\lambda) &= 2 \end{aligned}$$

Hence $1 - \lambda \neq 0$.
Substitute $x = \frac{2}{1-\lambda}$, $y = \frac{1}{1-\lambda}$, $z = \frac{2}{1-\lambda}$ in the constraint equation:

$$\begin{aligned} \frac{4}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} + \frac{4}{(1-\lambda)^2} = 1 \quad &\Rightarrow \quad (1-\lambda)^2 = 9 \\ &\Rightarrow \quad 1 - \lambda = \pm 3 \\ &\Rightarrow \quad \lambda = -2 \text{ or } \lambda = 4 \end{aligned}$$

case1: $\lambda = -2$.
Then $x = \frac{2}{3}$, $y = \frac{1}{3}$, $z = \frac{2}{3}$ and

$$\sqrt{(x-2)^2 + (y-1)^2 + (z-2)^2} = \sqrt{\frac{16}{9} + \frac{4}{9} + \frac{16}{9}} = \sqrt{\frac{36}{9}} = \sqrt{4} = 2$$

case2: $\lambda = 4$.

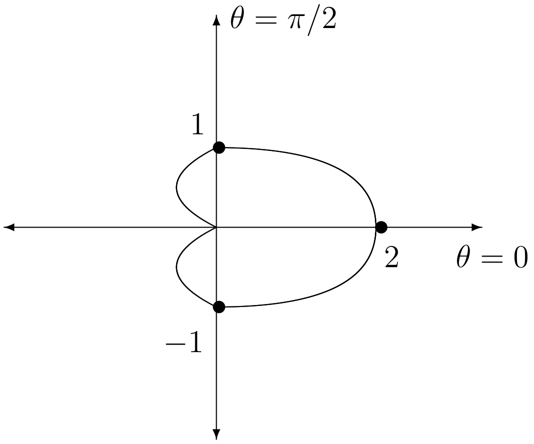
Then $x = -\frac{2}{3},\ y = -\frac{1}{3},\ z = -\frac{2}{3}$ and

$$\sqrt{(x-2)^2 + (y-1)^2 + (z-2)^2} = \sqrt{\frac{64}{9} + \frac{16}{9} + \frac{64}{9}} = \sqrt{\frac{144}{9}} = \sqrt{16} = 4$$

So the point on the sphere furthest from $(2, 1, 2)$ is $(-\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3})$

3. Find the area of the region enclosed by the cardioid $r = 1 + \cos\theta$ using double integrals.

Solution:



$$\begin{aligned} A &= \iint_R 1dA = \int_0^{2\pi} \int_0^{1+\cos\theta} rdrd\theta = \int_0^{2\pi} \frac{1}{2}r^2 \Big|_0^{1+\cos\theta} d\theta = \\ &= \frac{1}{2}\int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta)d\theta = \\ &= \frac{1}{2}\int_0^{2\pi} (1 + 2\cos\theta + \frac{1+\cos 2\theta}{2})d\theta = \\ &= \frac{1}{2}(\frac{3}{2}\theta + 2\sin\theta + \frac{1}{2}\sin 2\theta)\Big|_0^{2\pi} = \\ &= \frac{1}{2}(\frac{3}{2}2\pi) = \frac{3}{2}\pi \end{aligned}$$

4. If $z = e^{x/y}\sin(\frac{x}{y}) + e^{y/x}\cos(\frac{y}{x})$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$

Solution:

Let $u = \frac{x}{y}$.

Then

$$\begin{aligned} z &= e^u \sin(u) + e^{1/u}\cos(\tfrac{1}{u}), \\ \frac{\partial z}{\partial x} &= \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du} \cdot \frac{1}{y} \\ \frac{\partial z}{\partial y} &= \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = -\frac{dz}{du} \cdot \frac{x}{y^2} \end{aligned}$$

By addition,

$$\begin{aligned}
x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} &= \frac{dz}{du}\cdot\frac{\partial u}{\partial x}\cdot x + \frac{dz}{du}\cdot\frac{\partial u}{\partial y}\cdot y \\
&= \frac{dz}{du}\cdot\frac{x}{y} - \frac{dz}{du}\cdot\frac{x}{y^2}\cdot y \\
&= \frac{dz}{du}\cdot\left(\frac{x}{y} - \frac{x}{y}\right) = 0
\end{aligned}$$

B U Department of Mathematics

Math 102 Calculus II

Fall 2006 Midterm 2

1. Let C_1 and C_2 be two parametrized curves in $\mathbb{R}^3$ which intersect at point P. The angle between C_1 and C_2 at P is defined to be the angle between their tangent vectors at P. Now suppose C_1 and C_2 are the curves given by the graphs of $\mathbf{r_1}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \mathbf{k}$ and $\mathbf{r_2}(t) = \mathbf{i} + t\,\mathbf{k}$ $(t \in \mathbb{R})$ respectively.
(a) [5] Find the point(s) of intersection of C_1 and C_2. If they intersect at all, find the angle between C_1 and C_2 at the point(s) they intersect.
(b) [2] What are the curves C_1 and C_2? Describe how they look like.

Solution:

(a) We need to solve
$$r_1(t) = r_2(s)$$
for t and s. We get $\cos t = 1$, $\sin t = 0$ and $1 = s$. For intersection, $s = 1$ and $t = k2\pi$, $k \in \mathbb{Z}$. All the solutions for t and s give the same point: $P = (1, 0, 1)$. Note that C_1 and C_2 passes through P at different times. Now, the angle between C_1 and C_2 at P is by definition the angle between $\mathbf{r_1}'(k2\pi) = \langle -\sin t, \cos t, 0\rangle_{t=0} = \langle 0, 1, 0\rangle$ and $\mathbf{r_2}'(1) = \langle 0, 0, 1\rangle$. But observe that their dot product is 0. Hence the angle between them is $\frac{\pi}{2}$.
(b) C_1 is a circle on the $z = 1$ plane with radius 1 and centered at $(0, 0, 1)$ while C_2 is a line parallel to z-axis and passing through $(1, 0, 1)$.

2. Consider the parametrized curve $\mathbf{r}(t) = \sin e^t\,\mathbf{i} + \cos e^t\,\mathbf{j} + \sqrt{3}e^t\mathbf{k}$.
(a) [5] Find an arc-length parametrization of the curve that has the same orientation as the given curve and has $t = 0$ as the reference point.
(b) [3] Calculate the curvature of the curve at the point corresponding to $t = 1$.

Solution:

(a)
$$\begin{aligned} s = s(t) &= \int_0^t ||\mathbf{r}'(\tau)||d\tau \\ &= \int_0^t \left[(e^\tau \cos e^\tau)^2 + (-e^\tau \sin e^\tau)^2 + (\sqrt{3}e^\tau)^2\right]^{\frac{1}{2}} d\tau \\ &= \int_0^t 2e^\tau d\tau \\ &= 2(e^t - 1). \end{aligned}$$

Then $t = t(s) = \ln\left(\frac{s}{2} + 1\right)$ and the arc-length parametrization is:

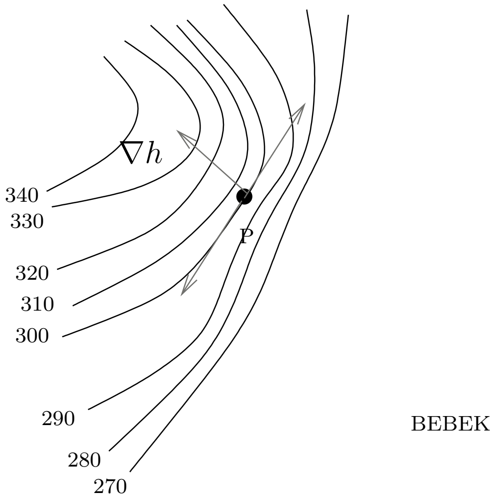
$$\mathbf{r}(s) = \sin e^{\ln(\frac{s}{2}+1)}\,\mathbf{i} + \cos e^{\ln(\frac{s}{2}+1)}\,\mathbf{j} + \sqrt{3}e^{\ln(\frac{s}{2}+1)}\mathbf{k} = \sin(\frac{s}{2}+1)\,\mathbf{i} + \cos(\frac{s}{2}+1)\,\mathbf{j} + \sqrt{3}(\frac{s}{2}+1)\mathbf{k}.$$

(b) The curvature of the arc-length parametrized curve $\mathbf{r}(s)$ is $\kappa(s) = ||\mathbf{N}(s)|| = ||\mathbf{r}''(s)||$. We compute

$$\mathbf{r}''(s) = -\frac{1}{4}\langle \sin(\frac{s}{2} + 1), \cos(\frac{s}{2} + 1), 0\rangle$$

and $\kappa(s) = ||\mathbf{r}''(s)|| = \dfrac{1}{4}$ for all s. This is not surprising. In fact, the curve is nothing but a circular helix.

3. [3] In the sketch of the campus below, the marked point P is the place where you can sit down and see the beautiful Bebek bay. One can see the level curves of the height function $h(x, y)$ from the sea level and the corresponding values of h on the left. Roughly sketch the direction of the gradient of h at P. Also draw the direction(s) at P along which the directional derivative of h is zero.



4. [5] Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and $z = f(x^2 + y^2)$. Calculate

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y}.$$

Solution:

Let $u = x^2 + y^2$. Then

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = y\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} - x\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = 2xy\frac{\partial z}{\partial u} - 2xy\frac{\partial z}{\partial u} = 0.$$

5. [6] Find the local linear approximation L of $f(x, y, z) = \dfrac{x + y}{y + z}$ at $P = (-1, 1, 1)$. Approximate f by L at point $Q = (-0.99, 0.99, 1.01)$.

Solution:

$$f_x(x, y, z) = \frac{1}{y + z}, f_y(x, y, z) = \frac{z - x}{(y + z)^2}, f_z(x, y, z) = -\frac{x + y}{(y + z)^2}.$$

$$L(Q) = f(P) + f_x(P)\Delta x + f_y(P)\Delta y + f_z(P)\Delta z = 0 + \frac{1}{2}0.01 - \frac{1}{2}0.01 + 0 = 0.$$

6. As you know from the class, the following is a photograph of the Schools of Sagrada Familia of architect Antoní Gaudi. Its remarkable roof can be considered as the graph of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$f(x, y) = x \cos y.$$

(a) [6] Find all critical points of f and determine explicitly if they are local maximum, local minimum or saddle point.

(b) **Bonus** [2] Draw the x, y and z-axes and the local extrema of part (a) on the picture. Only a complete answer takes the bonus!

Solution:

(a) f is differentiable everywhere. So, its critical points are those at which ∇f vanishes; i.e

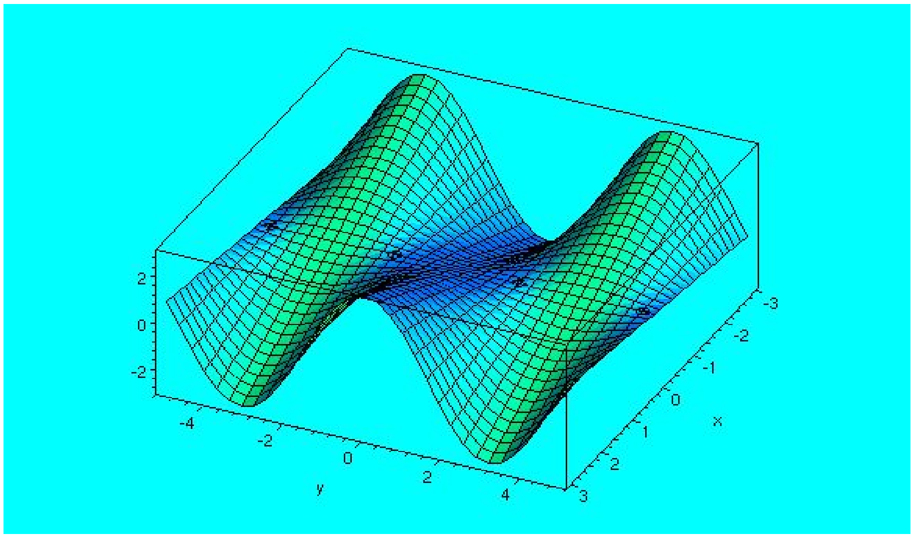
$$f_x(x,y) = \cos y = 0, f_y(x,y) = -x\sin y = 0.$$

Therefore the critical points are $y = \frac{\pi}{2} + k\pi$ $(k \in \mathbb{Z})$ and $x = 0$. To classify these points, we employ the second derivative test. At a critical point $(0, \frac{\pi}{2} + k\pi)$:

$$D = f_{xx}f_{yy} - f_{xy}^2 = 0 \cdot f_{yy} - (-\sin y)^2 = -\sin^2(\frac{\pi}{2} + k\pi) = -(\pm 1)^2 = -1 < 0.$$

Hence, all critical points are saddle points.

(b)



B U Department of Mathematics

Math 102 Calculus II

Fall 1999 Second Midterm

1. Find a unit normal vector and an equation for the tangent plane for the surface $x^2 + y^2 = 3z$ at the point $(1, 3, 10/3)$

Solution:

Let $F(x, y, z) = x^2 + y^2 - 3z$
We know that ∇F is normal to the level surface $F = 0$
$\nabla F = 2x\hat{\boldsymbol{i}} + 2y\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$
At $(1, 3, 10/3)$ we get $\nabla F = 2\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$, $|\nabla F| = 7$
$\hat{N} = \frac{1}{7}(2\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}})$
Eqn. for the tangent plane $\nabla F(\vec{\boldsymbol{x}}_0) \cdot (\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_0) = 0$
$\vec{\boldsymbol{x}}_0 = (1, 3, 10/3)$, $\vec{\boldsymbol{x}} = (x, y, z)$
$2(x - 1) + 6(y - 3) - 3(z - 10/3) = 0$
$2x + 6y - 3z = 10$

2. Let $f(x, y) = (x^2 + y - 2)^4 + (x - y + 2)^3$

(a) Find the total differential df at the point $(1, -2)$

Solution:

$df(\vec{\boldsymbol{x}}_0) = f_x(\vec{\boldsymbol{x}}_0)dx + f_y(\vec{\boldsymbol{x}}_0)dy$, $\vec{\boldsymbol{x}}_0 = (1, -2)$
$f_x = 8(x^2 + y - 2)^3 x + 3(x - y + 2)^2$
$f_y = 4(x^2 + y - 2)^3 - 3(x - y + 2)^2$
$f_x(1, -2) = 8(-27) + 3(25) = -141$
$f_y(1, -2) = 4(-27) - 3(25) = -183$
$df(1, -2) = -141dx - 183dy$

(b) Let $x = u - 2v + 1$, $y = 2u + v - 2$. Using the chain rule find $\dfrac{\partial f}{\partial v}$ when $u = 0$, $v = 0$

Solution:

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v};\ f_v = f_x x_v + f_y y_v$$

$u = 0,\ v = 0 \Rightarrow x = 1,\ y = -2$

$x_v = -2,\ y_v = 1$

$$\begin{aligned}\frac{\partial f}{\partial v}(u = 0,\ v = 0) &= -2f_x(1, -2) + f_y(1, -2)\\ &= -2(-141) - 183\\ &= 99\end{aligned}$$

3. A flat circular plate has the shape of the region $x^2 + y^2 \leq 1$. The plate, including the boundary where $x^2+y^2 = 1$, is heated so that the temperature at any point (x, y) is $T(x, y) = x^2+2y^2-x$.

   (a) In which direction $T(x, y)$ decreases fastest at the origin?

Solution:

$\nabla T = (2x - 1)\hat{\boldsymbol{\imath}} + 4y\hat{\boldsymbol{\jmath}},\ \nabla T(0, 0) = -\hat{\boldsymbol{\imath}}$

T decreases fastest in the $-\nabla T$ direction. So $-\nabla T = \hat{\boldsymbol{\imath}} \Rightarrow$ along the $x - axis$

   (b) Find the hottest and the coldest points on the plate and the temperature at each of these points.

Solution:

Critical points: $T_x = 0,\ T_y = 0 \Rightarrow x = 1/2,\ y = 0$

$T_x = 2x - 1,\ T_{xx} = 2,\ T_y = 4y,\ T_{yy} = 4,\ T_{xy} = 0$

$\Rightarrow A = 2,\ B = 0,\ C = 4,\ D = B^2 - AC = -8$

$D < 0,\ A > 0 \Rightarrow T(1/2, 0) = -1/4$, relative minimum

Must check the behavior at the boundary:

$T(x) = T(x, y(x)) = 2 - x^2 - x,\ (y(x) = \pm\sqrt{1 - x^2})$

$T'(x) = -2x - 1 = 0 \Rightarrow x = -1/2 \Rightarrow y = \pm\frac{\sqrt{3}}{2}$

$T''(x) = -2 \Rightarrow$ maxima

$T(-1/2) = 9/4$

Hottest points: $(-1/2, \pm\frac{\sqrt{3}}{2}) \Rightarrow T = 9/4$

Coldest points: $(1/2, 0) \Rightarrow T = -1/4$

(Boundary would be handled by Lagrange multiplier method which gives $(\pm1, 0)$ as minima at boundary)

4. **(a)** Find the volume of the solid bounded by the surface $z = 6 - xy$ and the planes $x = 2$, $x = -2$, $y = 0$, $y = 3$, and $z = 0$

Solution:

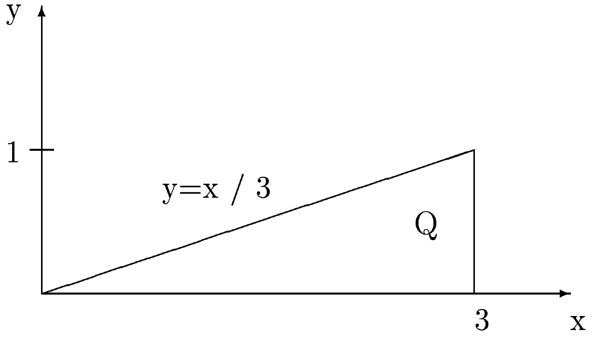
$$V = \int_{-2}^{2}\int_{0}^{3}(6 - xy)dy\,dx = \int_{-2}^{2}(18 - \frac{9}{2}x)dx$$

$$= \left[18x - \frac{9}{4}x^2\right]_{-2}^{+2} = 72$$

(b) Evaluate the double integral: $I = \int\limits_{0}^{1}\int\limits_{3y}^{3} e^{x^2}\,dx\,dy$

Solution:

(b)



So $dx\,dy$ can be converted by the help of figure above, to $dy\,dx$

$$I = \int\limits_{0}^{3}\int\limits_{0}^{x/3} e^{x^2}\,dy\,dx$$

$$I = \frac{1}{3}\int\limits_{0}^{3} xe^{x^2}\,dx = \frac{1}{6}\left[e^{x^2}\right]_0^3$$

$$I = \frac{1}{6}(e^9 - 1)$$

B U Department of Mathematics

Math 102 Calculus II

Spring 2000 Second Midterm

1. Suppose $x = 2r - s$, $y = r + 2s$ and f is a function of x, y such that first order and second order partial derivatives exist and these partial derivatives are continuous. Express $\dfrac{\partial^2 f}{\partial y\,\partial x} = f_{xy}$ in terms of f_{rr}, f_{rs}, f_{ss}

Solution:

$$\left.\begin{array}{r} x = 2r - s \\ y = r + 2s \end{array}\right\} \Rightarrow \begin{array}{r} r = \frac{1}{5}(2x + y) \\ s = \frac{1}{5}(2y - x) \end{array}$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(f_r\frac{2}{5} + f_s\frac{-1}{5}\right) = \frac{2}{5}\frac{\partial}{\partial y}f_r - \frac{1}{5}\frac{\partial}{\partial y}f_s$$

$$= \frac{2}{5}\left(f_{rr}\frac{\partial r}{\partial y} + f_{rs}\frac{\partial s}{\partial y}\right) - \frac{1}{5}\left(f_{sr}\frac{\partial r}{\partial y} + f_{ss}\frac{\partial s}{\partial y}\right) = \frac{2}{25}f_{rr} + \frac{4}{25}f_{rs} - \frac{1}{25}f_{sr} - \frac{2}{25}f_{ss}$$

$$= \frac{1}{25}(2f_{rr} + 3f_{rs} - 2f_{ss}) \text{ since } f_{rs} = f_{sr}.$$

2. Let $f(r)$ be some given function and define $g(x, y, z) = f(r)$ where $r = \sqrt{x^2 + y^2 + z^2}$

(a) Show that $\nabla g(x, y, z) = \frac{f\,'(r)}{r}(x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}})$

(b) Show that ∇g is always perpendicular to circles centered at the origin provided $f'(r) \neq 0$.

Solution:

(a) $\dfrac{\partial r}{\partial x} = x(x^2 + y^2 + z^2)^{-1/2} = \dfrac{x}{r}$ similarly $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$, $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$\begin{aligned} \nabla g(x, y, z) &= \frac{\partial g}{\partial x}\hat{\boldsymbol{i}} + \frac{\partial g}{\partial y}\hat{\boldsymbol{j}} + \frac{\partial g}{\partial z}\hat{\boldsymbol{k}} \\ &= \frac{\partial g}{\partial r}\frac{\partial r}{\partial x}\hat{\boldsymbol{i}} + \frac{\partial g}{\partial r}\frac{\partial r}{\partial y}\hat{\boldsymbol{j}} + \frac{\partial g}{\partial r}\frac{\partial r}{\partial z}\hat{\boldsymbol{k}} \\ &= f\,'(r)\left[\frac{\partial r}{\partial x}\hat{\boldsymbol{i}} + \frac{\partial r}{\partial y}\hat{\boldsymbol{j}} + \frac{\partial r}{\partial z}\hat{\boldsymbol{k}}\right] \\ &= \frac{f\,'(r)}{r}\left[x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}\right]. \end{aligned}$$

(b) ∇g is perpendicular to level curves of g, so it is enough to show circles at the origin are the level curves of g. Level curve of g with value c is the set $= \{(x, y, z) : g(x, y, z) = c\} = \{(x, y, z) : f(\sqrt{x^2 + y^2 + z^2}) = c\}$ which is a circle with radius $\sqrt{c}$ centered at the origin. Therefore the vector field ∇g is perpendicular to circles centered at the origin

3. Find the dimensions of the rectangular box of maximum volume that can be inscribed in the ellipsoid $x^2 + 4y^2 + 2z^2 = 8$

Solution:

Maximize Volume of the box $f = V = 8xyz$ but we can consider $f = xyz$
Constraint $g = x^2 + 4y^2 + 2z^2$
$\nabla f = yz\hat{\boldsymbol{\imath}} + xz\hat{\boldsymbol{\jmath}} + xy\hat{\boldsymbol{k}}$
$\nabla g = 2z\hat{\boldsymbol{\imath}} + 8y\hat{\boldsymbol{\jmath}} + 4z\hat{\boldsymbol{k}} \neq 0$ since x, y, z are non zero. We have a non-negative volume
We take $\nabla f = \lambda \nabla g$ so we get
$yz = \lambda 2x,\ xz = \lambda 8y,\ xy = \lambda 4z$
$\lambda = \dfrac{yz}{2x} = \dfrac{xz}{8y} = \dfrac{xy}{4z}$
Solving these equations yields to $x = \pm 2y,\ z = \pm\sqrt{2}y$
Writing constraint in terms of y gives us
$4y^2 + 4y^2 + 4y^2 = 8 \Rightarrow y = \sqrt{\frac{2}{3}}$
Hence we get $y = \sqrt{\frac{2}{3}},\ x = 2\sqrt{\frac{2}{3}},\ z = \frac{2}{\sqrt{3}}$

4. By means of a double integral, find the volume of the tetrahedron whose vertices are at the points $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ where a, b, c are positive real numbers.

Solution:

First we find the equation of the surface, then we will find volume under that surface
$\vec{\mathbf{AB}} = (-a, b, 0)$ and $\vec{\mathbf{AC}} = (-a, 0, c)$

$$\vec{\mathbf{AB}} \times \vec{\mathbf{AC}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = bc\hat{\boldsymbol{\imath}} + ac\hat{\boldsymbol{\jmath}} + ab\hat{\boldsymbol{k}}$$

So the equation of the surface is: $bc\ x + ac\ y + ab\ z - abc = 0$

$$\begin{aligned} V &= \int\limits_0^a \int\limits_0^{\frac{-b}{a}x+b} \frac{abc - bcx - acy}{ab}\, dy\, dx \\ &= \int\limits_0^a \left[cy - \frac{c}{a}xy - \frac{cy^2}{2b} \right]_{y=0}^{y=\frac{-b}{a}x+b} dx \\ &= \int\limits_0^a \left[c\left(-\frac{b}{a}x + b\right) - \frac{cx}{a}\left(-\frac{b}{a}x + b\right) - \frac{c}{2b}\left(-\frac{b}{a}x + b\right)^2 \right] dx \end{aligned}$$

$$
\begin{aligned}
&= \int_0^a \left[-\frac{cbx}{a} + cb + \frac{cbx^2}{a^2} - \frac{cbx}{a} - \frac{c}{2b}\left(\frac{b^2x^2}{a^2} - \frac{2b^2x}{a} + b^2\right) \right] dx \\
&= \int_0^a \left[-\frac{cbx}{a} + cb + \frac{cbx^2}{a^2} - \frac{cbx}{a} - \frac{cbx^2}{2a^2} + \frac{cbx}{a} - \frac{cb}{2} \right] dx \\
&= \int_0^a \left[-\frac{cbx}{a} + \frac{cbx^2}{2a^2} + \frac{cb}{2} \right] dx \\
&= -\frac{cbx^2}{2a} + \frac{cbx^3}{6a^2} + \frac{cbx}{2}\bigg|_{x=0}^{x=a} \\
&= \frac{abc}{6}.
\end{aligned}
$$

B U Department of Mathematics

Math 102 Calculus II

Spring 2001 Second Midterm

1. The function f is defined by

$$f(x,y) := \begin{cases} \frac{x^2y}{x^3+y^3} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Is f continuous at $(0,0)$?

(b) By using the definition of partial derivative, check whether f has a partial derivative with respect to x at $(0,0)$.

Solution:

(a) As $(x,y) \to (0,0)$ on a straight line $y = mx$, the function tends to

$$\begin{aligned} \lim_{\substack{(x,y)\to(0,0) \\ y=mx}} f(x,y) &= \lim_{x\to 0} \frac{x^2 \cdot mx}{x^3 + m^3x^3} \\ &= \lim_{x\to 0} \frac{m}{1+m^3} \\ &= \frac{m}{1+m^3}, \end{aligned}$$

and this limit depends on m. Thus f has no limit as $(x,y) \to (0,0)$ and f is not continuous at $(0,0)$.

(b) Since

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{f(h,0) - 0}{h} \\ &= \lim_{h\to 0} \frac{\frac{h^2\cdot 0}{h^3+0^3}}{h} \\ &= \lim_{h\to 0} 0 \end{aligned}$$

exists (as a real number) and equals 0, the partial derivative of f with respect to x exists at $(0,0)$ and is equal to 0.

2. Find the point of the graph of $z = -x^2 + xy + 2y^2$ where the tangent plane is parallel to the plane with the equation $x - 14y + z = 4$.

Solution:

The partial derivatives of $f(x,y) := -x^2 + xy + 2y^2$ at any point (a,b) are

$$\begin{aligned} \frac{\partial f}{\partial x} &= -2x + y, & \frac{\partial f}{\partial y} &= x + 4y, \\ \frac{\partial f}{\partial x}(a,b) &= -2a + b, & \frac{\partial f}{\partial y}(a,b) &= a + 4b. \end{aligned}$$

At any point $(a, b, z(a,b))$ the tangent plane to the surface $z = f(x,y) = -x^2 + xy + 2y^2$ has the equation

$$\begin{aligned} z &= f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) \\ &= f(a,b) + (-2a+b)(x-a) + (a+4b)(y-b), \\ (-2a+b)x + (a+4b)y - z + \text{constant} &= 0 \end{aligned}$$

and a normal to this tangent plane is the vector $(-2a+b)\vec{\imath} + (a+4b)\vec{\jmath} - \vec{k}$. This normal should be parallel to the normal $\vec{\imath} - 14\vec{\jmath} + \vec{k}$ of the plane $x - 14y + z = 4$. Hence $(-2a+b)\vec{\imath} + (a+4b)\vec{\jmath} - \vec{k} = \lambda(\vec{\imath} - 14\vec{\jmath} + \vec{k})$ for some λ, and solving we find

$$\begin{gathered} \lambda = -1, \\ -2a + b = -1, \quad a + 4b = 14, \\ a = 2, \quad b = 3, \\ f(2,3) = -2^2 + 2 \cdot 3 + 2 \cdot 3^2 = 20. \end{gathered}$$

Thus at the point $(2,3,20)$, the tangent plane to the surface is parallel to the plane $x - 14y + z = 4$.

3. The radius of a right circular cylinder is decreasing at a rate of 1 cm/sec and its height is increasing at a rate of 3 cm/sec. Is the volume increasing, or decreasing, when the height is 10 cm and the radius is 2 cm.

Solution:

We have

$$\begin{aligned} V &= \pi r^2 h, \\ V(t) &= \pi [r(t)]^2 h(t), \qquad \text{and by Chain Rule,} \\ \frac{dV}{dt} &= \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = 2\pi r(t) h(t) \frac{dr}{dt} + \pi [r(t)]^2 \frac{dh}{dt} \end{aligned}$$

(as it follows also from the product rule for derivatives). In the problem we are given $\frac{dr}{dt} = -1$ cm/sec and $\frac{dh}{dt} = +3$ cm/sec, so

$$\frac{dV}{dt}(t) = -2\pi r(t) h(t) + 3\pi [r(t)]^2$$

(in cm/sec) and at the moment t_0 when $r = 2$ cm/sec and $h = 10$ cm/sec, we have

$$\frac{dV}{dt}(t_0) = -2\pi \cdot 2 \cdot 10 + 3\pi [2]^2 = -40 + 12\pi$$

(in cm^3/sec). Since $-40 + 12\pi = 4(-10 + 3\pi) = 4 \cdot 3 \cdot (-3.3333... + 3.14159...)$ is negative, the volume is decreasing at that instant.

4. A space shuttle in the shape of the ellipsoid $4x^2 + y^2 + 4z^2 = 16$ enters the earth's atmosphere and its surface begins to heat. After one hour, the temperature at the point (x, y, z) on the shuttle's surface is $T(x, y, z) = 8x^2 + 4yz - 16z + 600$. Find the hottest point(s) on the shuttle's surface.

Solution:

The problem is to maximize $T(x, y, z)$ subject to the constraint $g(x, y, z) := 4x^2 + y^2 + 4z^2 - 16 = 0$. We use the Lagrange's method of multipliers. There is only one constraint, so we we introduce only one multiplier λ. We have to solve the equations

$$\begin{aligned} \frac{\partial T}{\partial x} &= \lambda \frac{\partial g}{\partial x}, \\ \frac{\partial T}{\partial y} &= \lambda \frac{\partial g}{\partial y}, \\ \frac{\partial T}{\partial z} &= \lambda \frac{\partial g}{\partial z}, \\ g &= 0, \end{aligned}$$

for x, y, z, λ. In the present problem, these equations read as follows:

$$\begin{aligned} 16x &= \lambda \cdot 8x, \\ 4z &= \lambda \cdot 2y, \\ 4y - 16 &= \lambda \cdot 8z, \\ 4x^2 + y^2 + 4z^2 - 16 &= 0, \end{aligned}$$

or, more simply

$$\begin{aligned} 2x &= \lambda x, \\ 2z &= \lambda y, \\ y - 4 &= 2\lambda z, \\ 4x^2 + y^2 + 4z^2 - 16 &= 0. \end{aligned}$$

We shall now solve these equations. We distinguish some cases.
Case I. $\lambda = 0$. Then we successively find $x = 0$, $z = 0$, $y = 4$; and observe that $(0, 4, 0)$ satisfies the constraint $g(x, y, z) = 0$.
Case II. $\lambda \neq 0$, $x \neq 0$. Then we get $\lambda = 2$, $z = y$, $y - 4 = 4y$; so $y = -\frac{4}{3}$, $z = -\frac{4}{3}$. From $0 = g(x, -\frac{4}{3}, -\frac{4}{3}) = 4x^2 + (-\frac{4}{3})^2 + 4(-\frac{4}{3})^2 - 16 = 4x^2 + \frac{16}{9} + 4\frac{16}{9} - 16 = 4(x^2 + \frac{20}{9} - 4) = 4(x^2 - \frac{16}{9})$, we get $x = \pm\frac{4}{3}$. Hence the solutions are $(\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3})$ and $(-\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3})$.
Case III. $\lambda \neq 0$, $x = 0$. In this case our equations are

$$\begin{aligned} 2z &= \lambda y, \\ y - 4 &= 2\lambda z, \\ y^2 + 4z^2 &= 16. \end{aligned}$$

Thus

$$\begin{gathered}
y - 4 = 2z \cdot \lambda = \lambda y \cdot \lambda = \lambda^2 y, \\
y = \frac{4}{1-\lambda^2}, \\
z = \frac{\lambda}{2} y = \frac{\lambda}{2} \frac{4}{1-\lambda^2} = \frac{2\lambda}{1-\lambda^2}, \\
16 = y^2 + 4z^2 = (\frac{4}{1-\lambda^2})^2 + 4(\frac{2\lambda}{1-\lambda^2})^2 = \frac{16(1+\lambda^2)}{(1-\lambda^2)^2}, \\
1 + \lambda^2 = (1-\lambda^2)^2 = 1 - 2\lambda^2 + \lambda^4, \\
\lambda^4 = 3\lambda^2, \\
\lambda = \pm\sqrt{3} \qquad \text{(since } \lambda \neq 0 \text{ in case III)}, \\
y = \frac{4}{1-\lambda^2} = \frac{4}{1-3} = -2, \\
z = \frac{2\lambda}{1-\lambda^2} = \frac{2 \cdot (\pm 3)}{1-3} = \mp\sqrt{3}
\end{gathered}$$

and the solutions are $(0, -2, \sqrt{3})$ and $(0, -2, -\sqrt{3})$.
Thus the points at which T is maximum are among$(0, 4, 0)$, $(\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3})$, $(-\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3})$, $(0, -2, \sqrt{3})$ and $(0, -2, -\sqrt{3})$. We calculate T at these points:

$$\begin{aligned}
T(0,4,0) &= 8 \cdot 0^2 + 4 \cdot 4 \cdot 0 - 16 \cdot 0 + 600 \\
&= 600, \\
T(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}) &= 8 \cdot (\pm\frac{4}{3})^2 + 4 \cdot (-\frac{4}{3})(-\frac{4}{3}) - 16(-\frac{4}{3}) + 600 \\
&= 12\frac{16}{9} + 16\frac{4}{3} + 600 = 600 + \frac{128}{3}, \\
T(0, -2, \sqrt{3}) &= 8 \cdot 0^2 + 4 \cdot (-2) \cdot \sqrt{3} - 16\sqrt{3} + 600 \\
&= 600 - 24\sqrt{3}, \\
T(0, -2, -\sqrt{3}) &= 8 \cdot 0^2 + 4 \cdot (-2) \cdot (-\sqrt{3}) - 16 \cdot (-\sqrt{3}) + 600 \\
&= 600 + 24\sqrt{3}.
\end{aligned}$$

Since $243 < 256$, or $9\sqrt{3} < 16$, or $3\sqrt{3} < \frac{16}{3}$, or $24\sqrt{3} < \frac{128}{3}$, the largest of these numbers is $600 + \frac{128}{3}$, so there are two hottest points, namely the points with coordinates $(\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3})$ and $(\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3})$.

B U Department of Mathematics

Math 102 Calculus II

Spring 2002 Second Midterm

1. Determine the equations of two planes tangent to the ellipsoid

$$\frac{x^2}{2}+\frac{y^2}{4}+\frac{z^2}{9}=1$$

at two different points, but both having the same normal vector $\vec{N}=3\vec{\imath}+\vec{k}$.

Solution:

Let $F(x,y,z)=\dfrac{x^2}{2}+\dfrac{y^2}{4}+\dfrac{z^2}{9}-1$.
Let $P_0(x_0,y_0,z_0)$ be a point of tangency.
$\nabla F(P_0)=x_0\vec{\imath}+\dfrac{y_0}{2}\vec{\jmath}+\dfrac{2z_0}{9}\vec{k}$ is parallel to $\vec{N}$
$\Rightarrow \nabla F(P_0)=t\vec{N}$ for some $t\in\mathbb{R}$ $\Rightarrow$ $x_0=3t$, $\dfrac{y_0}{2}=0$ and $\dfrac{2z_0}{9}=t$.
Hence, $x_0=3t$, $y_0=0$ and $z_0=\dfrac{9t}{2}$. $P_0(x_0,y_0,z_0)$ is on the ellipsoid.
So, $\dfrac{9t^2}{2}+0+\dfrac{81t^2}{36}=1 \Rightarrow 27t^2=4 \Rightarrow t=\mp\dfrac{2}{3\sqrt{3}}$
Thus, the points of tangency are $\left(\dfrac{2}{\sqrt{3}},0,\sqrt{3}\right)$ and $\left(-\dfrac{2}{\sqrt{3}},0,-\sqrt{3}\right)$
with tangent planes

$$3\cdot\left(x-\frac{2}{\sqrt{3}}\right)+0\cdot(y-0)+1\cdot(z-\sqrt{3})=0 \quad\text{and}\quad 3\cdot\left(x+\frac{2}{\sqrt{3}}\right)+0\cdot(y-0)+1\cdot(z+\sqrt{3})=0$$

that is, $\boxed{z=-3x+3\sqrt{3}}$ and $\boxed{z=-3x-3\sqrt{3}}$.

2. Let $f(x, y)$ be a function of the cartesian coordinates x and y.

   a) Express ∇f in terms of the polar coordinates r and θ.

Solution:

$$r^2 = x^2 + y^2 \Rightarrow r\frac{\partial r}{\partial x} = x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} \Rightarrow \frac{\partial r}{\partial x} = \cos\theta$$
$$r^2 = x^2 + y^2 \Rightarrow r\frac{\partial r}{\partial y} = y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r} = \frac{r\sin\theta}{r} \Rightarrow \frac{\partial r}{\partial y} = \sin\theta$$
$$\tan\theta = \frac{y}{x} \Rightarrow \sec^2\theta\frac{\partial\theta}{\partial x} = -\frac{y}{x^2} = -\frac{r\sin\theta}{r^2\cos^2\theta} \Rightarrow \frac{\partial\theta}{\partial x} = -\frac{\sin\theta}{r}$$
$$\tan\theta = \frac{y}{x} \Rightarrow \sec^2\theta\frac{\partial\theta}{\partial y} = \frac{1}{x} = \frac{1}{r\cos\theta} \Rightarrow \frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r}$$

By the Chain Rule;

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial\theta}\frac{\partial\theta}{\partial x} = \frac{\partial f}{\partial r}\cos\theta + \frac{\partial f}{\partial\theta}\frac{-\sin\theta}{r}$$
$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial\theta}\frac{\partial\theta}{\partial y} = \frac{\partial f}{\partial r}\sin\theta + \frac{\partial f}{\partial\theta}\frac{\cos\theta}{r}.$$

Hence, the gradient is

$$\nabla f = \frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} = \left(\frac{\partial f}{\partial r}\cos\theta - \frac{\partial f}{\partial\theta}\frac{\sin\theta}{r}\right)\vec{\imath} + \left(\frac{\partial f}{\partial r}\sin\theta + \frac{\partial f}{\partial\theta}\frac{\cos\theta}{r}\right)\vec{\jmath}.$$

   b) If $h = h(r)$ is a function of the polar coordinate r, show that the maximum rate of change of the function $h(r)$ is $|h_r|$.

Solution:

$h = h(r) \Rightarrow h_\theta = 0$

So from (a),

$$\nabla h = h_r\cos\theta\,\vec{\imath} + h_r\sin\theta\,\vec{\jmath} = h_r\left(\cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath}\right)$$

Thus the maximum rate of change of h is

$\|\nabla h\| = |h_r|\sqrt{\cos^2\theta + \sin^2\theta} = |h_r|.$

3. A man is at the point $(2, 4, 28)$ on a hill whose altitude is given by the function

$$z = 100 - 2x^2 - 4y^2.$$

   a) In which direction should he travel in order to descend the hill most rapidly? Also find the rate of change in this direction.

Solution:

Let P_0 be the point $(2, 4, 28)$.

He must travel in the direction of $-\nabla z(P_0)$.

$$\left.\frac{\partial z}{\partial x}\right|_{P_0} = -4x\,|_{P_0} = -8 \quad\text{and}\quad \left.\frac{\partial z}{\partial y}\right|_{P_0} = -8y\,|_{P_0} = -32$$

Thus, $\nabla z(P_0) = -8\vec{\imath} - 32\vec{\jmath}$.

$\therefore$ He must travel in the direction of $8\vec{\imath} + 32\vec{\jmath}$ or simply, in the direction of $\vec{\imath} + 4\vec{\jmath}$.

In this direction the rate of change is $-|\nabla z(P_0)| = -8\sqrt{1 + 16} = -8\sqrt{17}$.

b) In which direction should he travel, so that he remains at the same altitude?

Solution:

He should travel in a direction perpendicular to $\nabla z(P_0)$, hence in directions of $\mp(4\vec{\imath}-\vec{\jmath})$ or equivalently, he should travel in the tangent direction of the level curve $100-2x^2-4y^2=28$ i.e. $x^2+2y^2=36$

$\Rightarrow\ 2x+4yy'=0$

$\Rightarrow\ y'\left|_{P_0}=-\frac{x}{2y}\right|_{P_0}=-\frac{1}{4}$

$\therefore$ He should travel in the direction of $\mp(4\vec{\imath}-\vec{\jmath})$.

4. Determine the points on the surface $x^2+y^2+z^2=25$ where the function $f(x,y,z)=x+2y+3z$ is

a) a minimum; also find the minimum value,

Solution:

Let $\omega=x+2y+3z-\lambda(x^2+y^2+z^2-25)$.

$\omega_x=1-2\lambda x=0$

$\omega_y=2-2\lambda y=0$

$\omega_z=3-2\lambda z=0$

Thus, $x=\frac{1}{2\lambda}$, $y=\frac{2}{2\lambda}$ and $z=\frac{3}{2\lambda}$. Put these into $x^2+y^2+z^2=25$.

$\therefore\ \frac{1}{4\lambda^2}(1+4+9)=25$

$\Rightarrow\ 4\lambda^2=\frac{14}{25}\ \Rightarrow\ \lambda=\mp\frac{\sqrt{14}}{10}$

giving points $P_1\left(\frac{5}{\sqrt{14}},\frac{10}{\sqrt{14}},\frac{15}{\sqrt{14}}\right)$ and $P_2\left(\frac{-5}{\sqrt{14}},\frac{-10}{\sqrt{14}},\frac{-15}{\sqrt{14}}\right)$.

Now, $f(P_1)=\frac{1}{\sqrt{14}}(5+20+45)=5\sqrt{14}$ and $f(P_2)=\frac{1}{\sqrt{14}}(-5-20-45)=-5\sqrt{14}.$

Thus, f has a minimum at P_2 and the minimum value is $-5\sqrt{14}$.

b) a maximum; also find the maximum value.

Solution:

By part (a), f has a maximum at P_1 and the maximum value is $5\sqrt{14}$.

B U Department of Mathematics

Math 102 Calculus II

Spring 2003 Second Midterm

1. **(a)** Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{5x^2-2xy+5y^2}{x^2+y^2}$ does not exist.

Solution:

Approaching $(0,0)$ along different curves, say $y = mx$:

$$\lim_{x\to 0}\frac{5x^2-2mx^2+5m^2x^2}{x^2+m^2x^2} = \frac{5-2m+5m^2}{1+m^2}$$

which depends upon m.

Choosing $m = 0$ we get lim=5, and $m = 1$ we get lim=4. Since they are different the limit does not exist.

(b) Given $u = \cos(x-y) + \ln(x+y)$, for $x+y>0$, compute $u_{xx} - u_{yy}$.

Solution:

We compute the necessary partial derivatives:

$$\begin{aligned} u_x &= -\sin(x-y) + \frac{1}{x+y} \\ u_y &= \sin(x-y) + \frac{1}{x+y} \\ u_{xx} &= -\cos(x-y) - \frac{1}{(x+y)^2} \\ u_{yy} &= -\cos(x-y) - \frac{1}{(x+y)^2}. \end{aligned}$$

Clearly $u_{xx} - u_{yy} = 0$.

2. Find the real number c, if the directional derivative of $f(x,y,z) = z^c \arctan(x+y)$ at the point $P_0(0,0,4)$ in direction of $\boldsymbol{u} = \langle 1,1,0\rangle$ is 2.

Solution:

We need the gradient of $f(x, y, z)$:

$$\boldsymbol{\nabla} f = \langle f_x, f_y, f_z \rangle = \left\langle \frac{z^c}{1+(x+y)^2}, \frac{z^c}{1+(x+y)^2}, cz^{c-1}\arctan(x+y) \right\rangle.$$

Evaluating at P_0: $\boldsymbol{\nabla} f(0, 0, 4) = \langle 4^c, 4^c, 0 \rangle$.
The unit vector in the direction of $\boldsymbol{u}$ is:

$$\boldsymbol{v} = \frac{\boldsymbol{u}}{||\boldsymbol{u}||} = \frac{1}{\sqrt{2}}\langle 1, 1, 0 \rangle.$$

Then the directional derivative is easily written to be:

$$D_{\boldsymbol{u}} f(0, 0, 4) = \boldsymbol{\nabla} f(0, 0, 4) \cdot \boldsymbol{v} = 4^c\sqrt{2}.$$

Given that $D_{\boldsymbol{u}} f(0, 0, 4) = 2$ we find that $4^c\sqrt{2} = 2 \Rightarrow c = \dfrac{1}{4}$.

3. Find and classify the critical points (as local maximum, local minimum or saddle point) of $f(x, y) = x^2y + x^2 + y^2 - xy - x$.

Solution:

1st derivative test: $f_x = 2xy + 2x - y - 1 = 0 \Rightarrow (2x - 1)(y + 1) = 0$. Hence $x = 1/2$ or $y = -1$.
$f_y = x^2 + 2y - x = 0$. We now use the previous values of x and y:

$$\begin{aligned} &x = 1/2 \Rightarrow 1/4 + 2y - 1/2 = 0 \Rightarrow y = 1/8 \\ &y = -1 \Rightarrow x^2 - x - 2 = 0 \Rightarrow x = 2,\ x = -1. \end{aligned}$$

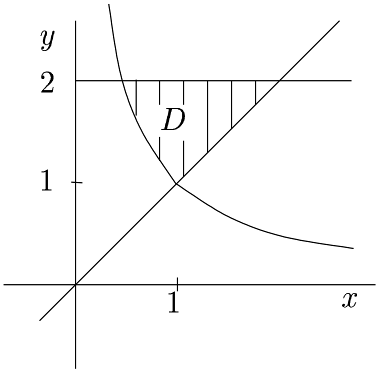
Thus, the critical points are found to be $P(1/2, 1/8)$, $Q(2, -1)$ and $R(-1, -1)$ (note that f is a polynomial, hence everywhere differentiable).
2nd derivative test: $f_{xx} = 2(y + 1)$, $f_{yy} = 2$ and $f_{xy} = 2x - 1$. So:

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 4(y + 1) - (2x - 1)^2.$$

We treat each critical point separately:
$D(1/2, 1/8) = 9/2 > 0$ and $f_{xx}(1/2, 1/8) > 0$, so that $P(1/2, 1/8)$ is a local minimum.
$D(2, -1) = -9 < 0$, so $Q(2, -1)$ is a saddle point.
$D(-1, -1) = -9 < 0$, so $R(-1, -1)$ is a saddle point.

4. **(a)** Express $\displaystyle\iint_D 2xy\,dA$ as an iterated double integral in $dxdy$ and $dydx$ orders, if D is the region bounded on the left by $xy = 1$, on the right by $y = x$ and above by $y = 2$. Evaluate one of the integrals.

Solution:



As seen from the graph:

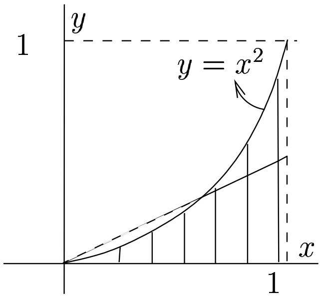
$$\begin{aligned}\iint_D 2xy\,dA &= \int_{1/2}^{1}\int_{1/x}^{2} 2xydydx + \int_{1}^{2}\int_{x}^{2} 2xydydx \\ &= \int_{1}^{2}\int_{1/y}^{y} 2xydxdy.\end{aligned}$$

Evaluating the last integral:

$$\int_{1}^{2}\int_{1/y}^{y} 2xydxdy = \int_{1}^{2}\left(y^3 - \frac{1}{y}\right)dy = \frac{15}{4} - \ln 2.$$

(b) Let $f(x,y) \geq 0$ be a continuous function. Express the volume under the graph of f and above the region $D = \{(x,y)|0 \leq y \leq x^2,\ 0 \leq x \leq 1\}$ as a polar double integral (do not try to evaluate the integral).

Solution:



We first express boundary functions in polar coordinates:

$$\begin{aligned}&y = x^2 \longrightarrow r\sin\theta = r^2\cos^2\theta \Rightarrow r = 0 \text{ or } r = \tan\theta\sec\theta \\ &x = 1 \longrightarrow r\cos\theta = 1 \Rightarrow r = \sec\theta.\end{aligned}$$

Now intersecting these two boundaries (note that $r = 0$ is not the actual boundary):

$$\sec\theta = \sec\theta\tan\theta \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}.$$

Polar boundaries have become: $\sec\theta\tan\theta \leq r \leq \sec\theta$ and $0 \leq \theta \leq \pi/4$. We now write the volume:

$$V = \iint_D f(x,y)dA = \int_0^{\frac{\pi}{4}}\int_{\sec\theta\tan\theta}^{\sec\theta} f(r\cos\theta, r\sin\theta)rdrd\theta.$$

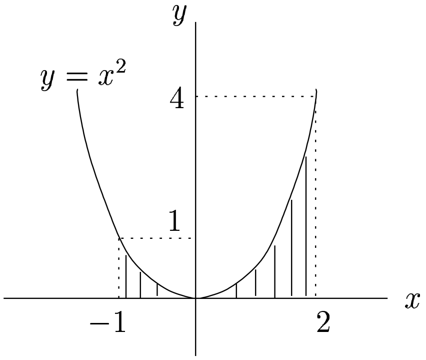
B U Department of Mathematics

Math 102 Calculus II

Spring 2004 Second Midterm

1. Evaluate $\displaystyle\int_0^1\int_{-1}^{-\sqrt{y}} \cos 3x^3 dxdy + \int_0^4\int_{\sqrt{y}}^2 \cos 3x^3 dxdy$.

   Solution:



Changing the order of iteration over the shaded region in the above figure, one obtains:

$$\begin{aligned}\left(\int_0^1\int_{-1}^{-\sqrt{y}} + \int_0^4\int_{\sqrt{y}}^2\right) \cos 3x^3 dxdy &= \int_{-1}^2\int_0^{x^2} \cos 3x^3 dydx \\ &= \int_{-1}^2 x^2\cos 3x^3 dx \\ &= \frac{1}{9}\sin 3x^3\Big|_{-1}^2 \\ &= \frac{1}{9}\left(\sin(24) - \sin(-3)\right).\end{aligned}$$

2. Test the function $f(x,y) = x\sin y$ for relative maxima, minima and saddle points.

   Solution:

Since f is differentiable everywhere, the critical points are where $\nabla f = 0$, i.e. when $f_x = f_y = 0$. For this,

$$f_x(x,y) = \sin y = 0; \quad f_y(x,y) = x\cos y = 0.$$

Then, (1) $y = k\pi$, $k \in Z$; (2) either $x = 0$ or $\cos y = 0$. $\cos y$ cannot be zero since $\sin y = 0$ at the same time. Then the set of all critical points is $\{(0,k\pi) | \, k \in Z\}$. To classify these points, compute

$$\begin{aligned}D(0,k\pi) &= f_{xx}\cdot f_{yy} - f_{xy}^2\big|_{(0,k\pi)} \\ &= 0\cdot(-x\sin y) - \cos^2 y\big|_{(0,k\pi)} \\ &= -\cos^2(k\pi)\end{aligned}$$

which is always negative. Therefore all the critical points of f are saddle points.

3. (a) Sketch the level curve of $f(x, y) = \dfrac{x}{y^2}$ that passes through the point $(4, -4)$. Find and draw, reasonably accurately, the gradient vector of f at that point.

Solution:

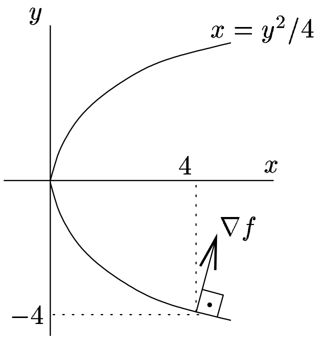
Since $f(4, -4) = \frac{1}{4}$, the level curve of f passing through $(4, -4)$ is the $(\frac{1}{4})$-level curve. This curve is given by:

$$f(x, y) = \frac{x}{y^2} = \frac{1}{4} \quad \Rightarrow \quad x = \frac{y^2}{4}.$$

It is a parabola. Furthermore,

$$\nabla f(4, -4) = \left\langle \frac{1}{y^2}, \frac{-2x}{y^3} \right\rangle \bigg|_{(4,-4)} = \left\langle \frac{1}{16}, \frac{1}{8} \right\rangle = \frac{1}{16} \langle 1, 2\rangle$$

is a vector perpendicular to the level curve at $(4, -4)$:



(b) Let $f(x, y, z) = z \ln(x^2 + y^2 - 1)$, $g(x, y, z) = x^2 + y^2 + z^2$. Find a **unit** vector $\vec{u}$ such that the directional derivatives of f and g at $(1, 1, 1)$ in the direction of $\vec{u}$ are both equal to zero.

Solution:

Since $D_{\vec{u}} f(x, y) = \vec{u} \cdot \nabla f(1, 1, 1) = 0$ and $D_{\vec{u}} g(x, y) = \vec{u} \cdot \nabla g(1, 1, 1) = 0$, $\vec{u}$ is a unit vector normal to both $\nabla f(1, 1, 1)$ and $\nabla g(1, 1, 1)$. In other words, $\vec{u}$ is parallel to $\nabla f(1, 1, 1) \times \nabla g(1, 1, 1)$. Since

$$\nabla f(1, 1, 1) = \left\langle \frac{2xz}{x^2 + y^2 + 1}, \frac{2yz}{x^2 + y^2 + 1}, \ln(x^2 + y^2 + 1) \right\rangle \bigg|_{(1,1,1)} = \langle 2, 2, 0\rangle \text{ and}$$

$$\nabla g(1, 1, 1) = \langle 2x, 2y, 2z\rangle \big|_{(1,1,1)} = \langle 2, 2, 2\rangle,$$

it follows that $\vec{u} \parallel \langle 1, 1, 0\rangle \times \langle 1, 1, 1\rangle = \langle 1, -1, 0\rangle$. Hence $\vec{u}$ is either $\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\rangle$ or $\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\rangle$.

4. A rectangular box, open at the top (without upper lid) has volume 18cm^3. It is constructed of material costing 3 TLs/cm^2 for the base, 2 TLs/cm^2 for the front face and 1 TLs/cm^2 for the sides and back. Use the Lagrange multiplier method to find the dimensions of the box for which the cost of construction is a minimum (no other method except Lagrange method is allowed. TL stands for Turkish Lira).

Solution:

The cost K in terms of edge lengths a, b and c is

$$K(a,b,c) = 3ab + 2ac + (2bc + ac) = 3ab + 2bc + 3ac.$$

We are to minimise K subject to the constraint $V(a,b,c) = abc = 18$. If there is a relative minimum, at that point one has $\nabla f = \lambda \nabla g$ for some real number λ (provided that $\nabla g \neq 0$). Note that,

$$\nabla f = \langle 3b + 3c, 3a + 2c, 3a + 2b \rangle \quad \text{and} \quad \nabla g = \langle bc, ac, ab \rangle.$$

Now solve

$$\begin{aligned} 3b + 3c &= \lambda bc, \\ 3a + 2c &= \lambda ac, \\ 3a + 2b &= \lambda ab, \\ abc &= 18 \end{aligned}$$

simultaneously. Multiplying the first three rows by a, b and c (since all are nonzero) respectively, one obtains

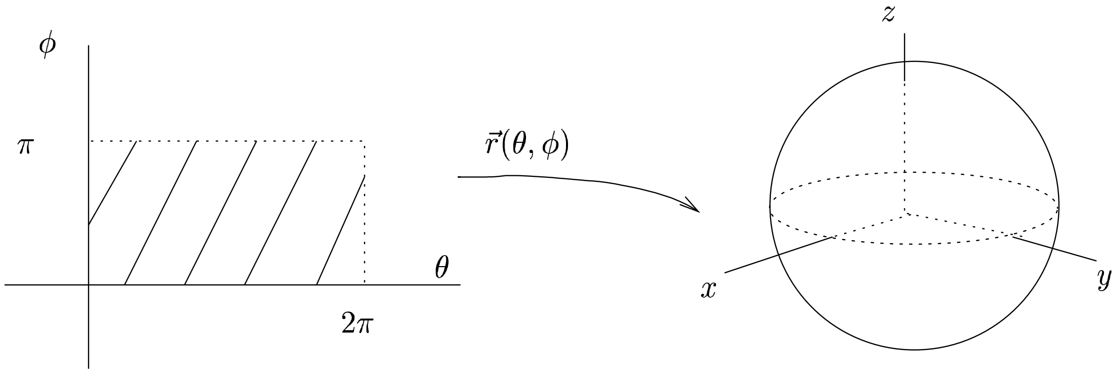
$$3ab + 3ac = 3ab + 2bc = 3ac + 2bc = 18\lambda.$$

First equation gives $3a = 2b$ and second gives $b = c$. The identity $abc = 18$ implies $\frac{2b}{3} \cdot b \cdot b = 18$. Therefore $b = 3 = c$ and $a = 2$. By the nature of the problem, it is easy to observe that the point $(2,3,3)$ gives a minimum of the cost function.

Bonus Consider the parametrisation

$$x = \sin\phi\cos\theta, \quad y = \sin\phi\sin\theta, \quad z = \cos\phi; \quad 0 \le \theta \le 2\pi, \ 0 \le \phi \le \pi;$$

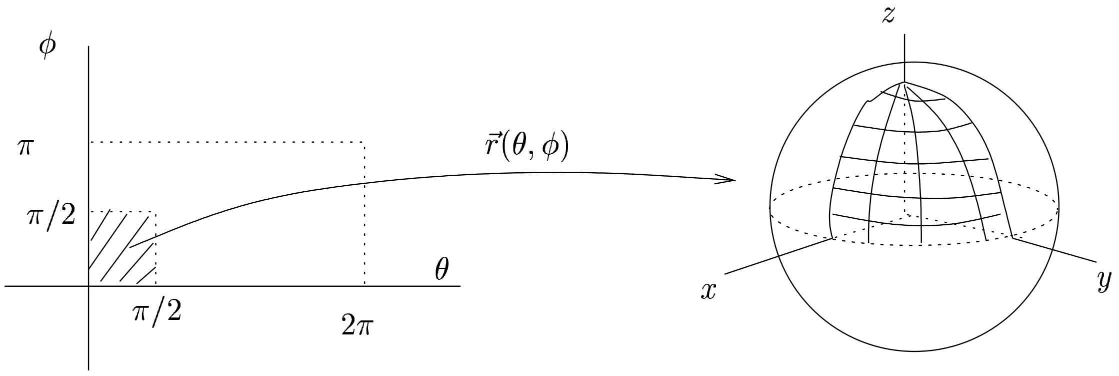
of the unit sphere (centered at 0, with radius 1).



Sketch exactly that portion of the rectangle $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$ which is mapped by this parametrisation to the part of the unit sphere in the first octant.

Solution:

In the first octant θ ranges from 0 up to $\pi/2$ while for each θ, ϕ ranges from 0 to $\pi/2$.



Boğaziçi University
Department of Mathematics
Math 102 Calculus II

1	2	3	4	$\sum$
25 points each				**100**

Date: May 7, 2008 Time: 18:10-19:10	Full Name	: BEHGET NECATİGİL
	Math 102 Number	:
	Student ID	:

Spring 2008 – Second Midterm Examination

IMPORTANT

1. Write your name, surname on top of each page. **2.** The exam consists of 4 questions some of which have more than one part. **3.** Read the questions carefully and write your answers neatly under the corresponding questions. **4.** Show all your work. Correct answers without sufficient explanation might not get full credit. **5.** Calculators are not allowed.

---

1. Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s if

$$w = x + 2y + z^2 \; , x = \frac{r}{s} \; , y = r^2 + \ln s \; , z = 2r.$$

Soln

We have,

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$= \frac{1}{s} + 4r + 2(2r) \cdot 2 = 12r + \frac{1}{s}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= -\frac{r}{s^2} + \frac{2}{s}$$

2. **(a)** Find the derivative of $f(x,y,z) = x^3 - xy^2 - z$ at $P_0(1,1,0)$ in the direction of $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

2. **(b)** In what direction does f change most rapidly, what are the rates of change in these directions?

Soln a) let $u = \frac{A}{\|A\|} = \frac{1}{7}(2i - 3j + 6k)$

and we have $\nabla f(x,y,z) = \langle 3x^2 - y^2, -2xy, -1 \rangle$

so, $\nabla f(1,1,0) = \langle 2, -2, -1 \rangle$

hence

$$D_u f(1,1,0) = \langle 2,-2,-1 \rangle \cdot \langle \tfrac{2}{7}, -\tfrac{3}{7}, \tfrac{6}{7} \rangle$$

$$= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}$$

b) • f increases most rapidly in the direction

$$\nabla f(1,1,0) = \langle 2,-2,-1 \rangle$$

and $$D_{\frac{\nabla f(1,1,0)}{\|\nabla f(1,1,0)\|}} f(1,1,0) = \|\nabla f(1,1,0)\| \cos 0 = \underline{\underline{3}}$$

• f decreases most rapidly in the direction

$$-\nabla f(1,1,0) = \langle -2, 2, 1 \rangle$$

and $$D_{\frac{-\nabla f(1,1,0)}{\|-\nabla f(1,1,0)\|}} f(1,1,0) = \|\nabla f(1,1,0)\| \cos \pi = \underline{\underline{-3}}$$

(x,y)	$(1,1)$	$(1,0)$	$(0,0)$	$(9,0)$	$(0,1)$	$(0,9)$	$(9/2,9/2)$
f	4	3	0	-61	3	-61	$-41/2$

↘ abs. max

abs. min

3. Find the absolute maximum and minimum values of

$$f(x,y) = 2 + 2x + 2y - x^2 - y^2$$

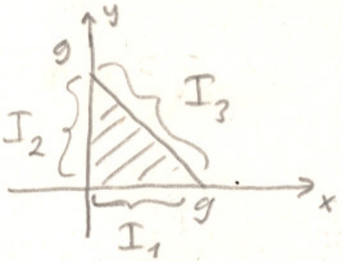
on the triangular plate in the first quadrant bounded by the lines $x = 0$, $y = 0$, $y = 9 - x$.

Soln Since f is cont. and the plate is "closed" and bounded f attains its absolute max. and abs. minimum on the plate.

i) we first check for interior critical points:

$$\left.\begin{aligned} f_x &= 2-2x = 0 \\ f_y &= 2-2y = 0 \end{aligned}\right\} \Rightarrow \quad (x,y) = (1,1) \text{ is the only critical point in the interior.}$$

ii)



now we check the critical points on the "boundary" of the region

I_1) we have $y=0$ and $x \in [0,9]$ and we put

$u(x) = f(x,0) = 2+2x-x^2$, $u'(x) = 2-2x = 0 \Rightarrow x = 1$

we should check the endpoints of I_1, so we consider the points $(1,0)$, $(0,0)$, $(9,0)$

I_2) we have $x=0$ and $y \in [0,9]$ and we put

$v(y) = f(0,y) = 2+2y-y^2$, $v'(y) = 2-2y = 0 \Rightarrow y = 1$

So we check $(0,1)$, $(0,0)$, $(0,9)$

I_3) we have $y = 9-x$ and $x \in [0,9]$ and we put

$w(x) = 2+2x+2(9-x) - x^2 - (9-x)^2 = -2x^2+18x-61$

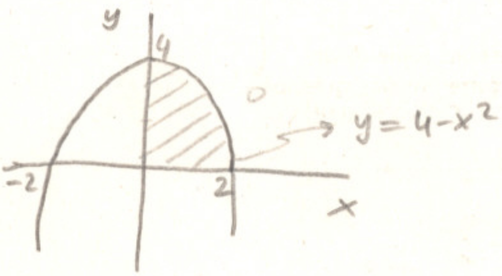
$\Rightarrow \quad w'(x) = -4x+18 = 0 \Rightarrow x = \frac{9}{2}$ So we check $(9/2, 9/2)$, $(9,0)$, $(0,9)$

4. (a) Evaluate the following double integral $\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dydx$.

4. (b) Write the integral $\int\int_R (y-2x^2)\, dA$, as iterated integrals $\int\int (y-2x^2)\, dydx$ **and** $\int\int (y-2x^2)\, dxdy$ with suitable limits of integration, where R is the square with vertices $(0,1)$, $(1,0)$, $(0,-1)$, $(-1,0)$. **Do not evaluate them.**

Soln a) we rewrite

$$\int_0^4 \int_0^{\sqrt{4-y}} \frac{x e^{2y}}{4-y}\, dxdy$$

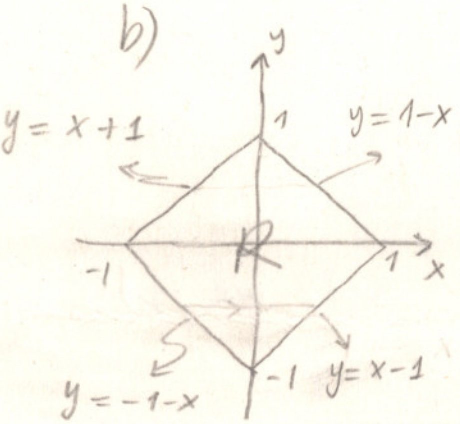


$$= \int_0^4 \left[\frac{x^2}{2} \frac{e^{2y}}{4-y} \right]_0^{\sqrt{4-y}} dy$$

$$= \int_0^4 \frac{\cancel{4-y}}{2} \cdot \frac{e^{2y}}{\cancel{4-y}}\, dy$$

$$= \int_0^4 \frac{e^{2y}}{2}\, dy = \frac{e^{2y}}{4} \Big|_0^4 = \frac{e^8}{4} - \frac{1}{4}$$

b)



i)

$$\iint_R (\ )dA = \int_{-1}^{0} \int_{-1-x}^{x+1} (y-2x^2)\, dydx + \int_0^1 \int_{x-1}^{1-x} (y-2x^2)\, dydx$$

ii)

$$\iint_R (y-2x^2)\, dA = \int_{-1}^{0} \int_{-1-y}^{y+1} (y-2x^2)\, dxdy + \int_0^1 \int_{y-1}^{1-y} (y-2x^2)\, dxdy$$

B U Department of Mathematics

Math 102 Calculus II

Summer 2001 Second Midterm

1. The surface of a mountain is described by the function $h(x,y) = 3\,e^{-(3x^4+y^2)}$, where h, x, y are in km; and the x-axis points towards east, the y-axis points towards north. A mountaineer is at the point $P(1,1)$. As seen from P;

(a) In which direction does the height increase the fastest?

(b) What is the rate of increase of height with respect to horizontal distance for the direction found in (a)?

(c) What is the rate of increase of height with respect to horizontal distance for the NW direction?

Solution:

(a) The direction of the gradient vector gives the fastest increase in height. $\nabla h(x,y) = -36x^3 e^{-(3x^4+y^2)}\mathbf{i} - 6ye^{-(3x^4+y^2)}\mathbf{j}$, so the vector $\nabla h(1,1) = -36e^{-4}\mathbf{i} - 6e^{-4}\mathbf{j}$ gives the direction of the fastest increase.

(b)The equation for the line passing through $P(1,1)$ in the direction of $\nabla h(1,1)$ is

$$\frac{x-1}{-36e^{-4}} = \frac{y-1}{-6e^{-4}},$$

from which we obtain $y = (x+5)/6$.

$\dfrac{d}{dx}h(x, y(x))$ will give the rate of increase in the given direction with respect to x, where $y(x) = (x+5)/6$ is the equation of the line.

Using the chain rule, we have:

$$\begin{aligned}\frac{dh}{dx} &= \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y}\frac{dy}{dx} \\ &= -36x^3 e^{-(3x^4+y^2)} - 6y^2 e^{-(3x^4+y^2)}\frac{1}{6}.\end{aligned}$$

Thus,

$$\left.\frac{d}{dx}h(x,y(x))\right|_{x=1} = -36e^{-4} - 6e^{-4}\frac{1}{6} = -37e^{-4}$$

(c) Similar to part (b), we use $y(x) = 2 - x$, which is the equation of the line passing through $P(1,1)$ in the NW direction, and we get

$$\frac{dh}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y}\frac{dy}{dx} = -36x^3 e^{-(3x^4+y^2)} - 6y^2 e^{-(3x^4+y^2)}(-1)$$

$$\left.\frac{d}{dx}h(x,y(x))\right|_{\substack{x=1\\y=1}} = -36e^{-4} - 6e^{-4}(-1) = -30e^{-4}$$

2. Using the method of Lagrange multipliers, find the greatest and smallest values that the function $z = f(x, y) = xy$ takes on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$.

Solution:

The constraint curve is $g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$. We solve $\nabla f(x, y) = \lambda \nabla g(x, y)$:

$$y\,\mathbf{i} + x\,\mathbf{j} = \lambda \left(\frac{x}{4}\,\mathbf{i} + y\,\mathbf{j}\right)$$

So,

$$y = \lambda \frac{x}{4}; \qquad x = \lambda y.$$

It follows that $y = \lambda^2(y/4)$, which implies $\lambda^2 = \pm 2$ or $y = 0$. If $y = 0$, then $x = \lambda y = 0$, thus $(x, y) = (0, 0)$, so this choice ends up with a point that is not on the curve. If $\lambda = 2$, then $x = 2y$, and putting this in $g(x, y) = 0$, we find $(x, y) = \pm(2, 1)$. Similarly if $\lambda = -2$ then $(x, y) = \pm(2, -1)$.

We also need to check the points on the constraint curve, where $\nabla g(x, y) = \mathbf{0}$. The only point satisfying $\nabla g(x, y) = \mathbf{0}$ is $(x, y) = (0, 0)$, and it is not on the constraint curve $g(x, y) = 0$. So there is no such point.

We now evaluate the function $f(x, y)$ at the points we have found: $f(2, 1) = 2, f(-2, -1) = 2, f(2, -1) = -2, f(-2, 1) = 2$. Therefore, the maximum and minimum values of f are 2 and -2, respectively, on the given ellipse.

3. Find the volume bounded by the surfaces $x + y = 1$, $\sqrt{x} + \sqrt{y} = 1$, $z = 0$, $z = 10$.

Solution:

We solve the equations $x + y = 1$, $\sqrt{x} + \sqrt{y} = 1$ together, to find their intersection points. Squaring the second equation gives $x + 2\sqrt{xy} + y = 1$. Plugging the first equation, $x + y = 1$, into this result, we get $1 + 2\sqrt{xy} = 1$, or $\sqrt{xy} = 0$. So, the points of intersection are $(x, y) = (0, 1)$ and $(x, y) = (1, 0)$

Rearranging the equations, we get that the region in the plane bounded by the curves $x + y = 1$ and $\sqrt{x} + \sqrt{y} = 1$ is the region bounded by the curves $y = 1 - x$ and $y = 1 + x - 2\sqrt{x}$.

Thus we obtain the volume by the triple integral:

$$\begin{aligned}
\int_0^{10} \int_0^1 \int_{x-1}^{1+x-2\sqrt{x}} dydxdz &= \int_0^{10} \int_0^1 2 - 2\sqrt{x}\,dxdz \\
&= \int_0^{10} \left[2x - \frac{4}{3}x^{3/2}\right]_0^1 dz \\
&= \int_0^{10} \frac{2}{3} dz = \frac{20}{3}.
\end{aligned}$$

4. Find the area of the region Q that lies <u>inside</u> the cardioid $r = 1 + \cos\theta$ and <u>outside</u> the circle $r = 1$.

Solution:

The region corresponds to $1 \leq r \leq 1 + \cos\theta$ and $-\pi/2 \leq \theta \leq \pi/2$ in the r-θ plane. So the area can be found as follows:

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} r dr d\theta &= \int_{-\pi/2}^{\pi/2} \left[\frac{r^2}{2}\right]_1^{1+\cos\theta} d\theta \\
&= \int_{-\pi/2}^{\pi/2} 2\cos\theta + \cos^2\theta d\theta \\
&= \int_{-\pi/2}^{\pi/2} 2\cos\theta + \frac{1+\cos 2\theta}{2} d\theta \\
&= \left[2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{-\pi/2}^{\pi/2} \\
&= 4 + \frac{\pi}{2}.
\end{aligned}$$

B U Department of Mathematics

Math 102 Calculus II

Summer 2002 Second Midterm

1. The length, width and height of a rectangular box are increasing at rates of 1 cm/sec, 2 cm/sec, and 2 cm/sec, respectively. At what rate is the volume increasing when the length is 3 cm, the width is 5 cm, and the height is 7 cm?

Solution:

The volume function is $V(l,w,h) = lwh$. At the point $P(3,5,7)$ it is given that $\frac{dl}{dt} = 1$ cm/sec, $\frac{dw}{dt} = 2$ cm/sec, $\frac{dh}{dt} = 2$ cm/sec.

$$\begin{aligned} \frac{dV}{dt} &= \frac{\partial V}{\partial l}\frac{dl}{dt} + \frac{\partial V}{\partial w}\frac{dw}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} \\ \frac{dV}{dt} &= 1wh + 2lh + 2lw \\ \left.\frac{dV}{dt}\right|_P &= 5\cdot 7\cdot 1 + 3\cdot 7\cdot 2 + 2\cdot 3\cdot 5 \\ &= 35 + 42 + 30 \\ &= 107 \text{ cm}^3/\text{sec}. \end{aligned}$$

2. **a)** Find the points on the graph of $z = y\cos x$ where the tangent plane is parallel to the plane $x - \sqrt{3}y + 2z = -2$.

   b) Determine the equation(s) of such tangent plane(s).

Solution:

a) Let $f(x,y,z) = y\cos x - z$. Then the graph of $z = y\cos x$ is the surface $f(x,y,z) = y\cos x - z = 0$.
The normal vector to the plane $x - \sqrt{3}y + 2z = -2$ is $\boldsymbol{n} =< 1, -\sqrt{3}, 2 >$. Let P be a point on the graph at which the tangent plane is parallel to the given plane. Then $\boldsymbol{\nabla f}$ at the point P must be in the same direction with the normal vector $\boldsymbol{n}$.
Since $\boldsymbol{\nabla f} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle =< -y\sin x, \cos x, -1 >$
and $\boldsymbol{\nabla f}|_P = \lambda\boldsymbol{n} =< \lambda, -\sqrt{3}\lambda, 2\lambda >$, we get: $-1 = 2\lambda$, hence $\lambda = -\frac{1}{2}$.
So $-y\sin x = -\frac{1}{2}$, $\cos x = \frac{\sqrt{3}}{2}$. and we get two points:

$$P_1 : \left(\frac{\pi}{6}, 1, \frac{\sqrt{3}}{2}\right) \text{ and } P_2 : \left(-\frac{\pi}{6}, -1, -\frac{\sqrt{3}}{2}\right).$$

b) The equation of the tangent plane at P_1 is:

$$1(x - \frac{\pi}{6}) - \sqrt{3}(y - 1) + 2(z - \frac{\sqrt{3}}{2}) = 0.$$

The equation of the tangent plane at P_2 is:

$$1(x+\frac{\pi}{6})+\sqrt{3}(y-1)+2(z+\frac{\sqrt{3}}{2})=0.$$

3. Use the method of Lagrange multipliers to find the volume of the largest rectangular box that can be inscribed in the ellipsoid $2x^2+3y^2+6z^2=18$.

Solution:

The volume function $F(x,y,z)=xyz$ is to be maximized. The constraint equation is $2x^2+3y^2+6z^2=18$, so $G(x,y,z)=2x^2+3y^2+6z^2-18$.
$\nabla F=\left\langle\frac{\partial F}{\partial x},\frac{\partial F}{\partial y},\frac{\partial F}{\partial z}\right\rangle=<yz,xz,xy>$ and $\nabla G=\left\langle\frac{\partial G}{\partial x},\frac{\partial G}{\partial y},\frac{\partial G}{\partial z}\right\rangle=<4x,6y,12z>$.

$$\begin{aligned}\nabla F &= \lambda\nabla G\\ <yz,xz,xy> &= \lambda<4x,6y,12z>\end{aligned}$$

We get 3 equations:

$$yz=\lambda 4x \quad\Rightarrow\quad \lambda=\frac{yz}{4x} \qquad (1)$$

$$xz=\lambda 6y \quad\Rightarrow\quad \lambda=\frac{xz}{6y} \qquad (2)$$

$$xy=\lambda 12z \quad\Rightarrow\quad \lambda=\frac{xy}{12z} \qquad (3)$$

Solving equation (1) and (2) together we get:

$$6y^2z=4x^2z, \qquad (4x^2-6y^2)z=0$$

There are two cases:
Case 1: We have either $z=0$, which implies both $x=0$ and $y=0$. Not possible.
Case 2: We have $4x^2-6y^2=0$. So $x=\frac{\sqrt{3}}{\sqrt{2}}y$ (since $x,y,z\geq 0$).
Solving equation (2) and (3) together we get:

$$12z^2x=6y^2x, \qquad (12z^2-6y^2)x=0.$$

There are two cases:
Case 1: We have either $x=0$, which implies both $y=0$ and $z=0$. Not possible.
Case 2: We have $12z^2-6y^2=0$. So $z=\frac{1}{\sqrt{2}}y$.
Plug these into the constraint equation:

$$2\left(\frac{\sqrt{3}}{\sqrt{2}}\right)^2+3y^2+6\left(\frac{1}{\sqrt{2}}y\right)^2=0$$

$$3y^2+3y^2+3y^2-18=0$$
$$y=\sqrt{2}$$

This implies $x=\sqrt{3}$ and $z=1$. The maximum volume is $V(\sqrt{3},\sqrt{2},1)=\sqrt{6}$.

4. Find the volume of the solid enclosed between the surface $z = \cos x^2$ and the region R on the xy-plane bounded by the lines $y = x/2$, $x = 2$ and the x-axis.

Solution:

First, note that the graph of $z = \cos x^2$ is above the xy-plane for $0 \le x \le (\frac{\pi}{2})^{\frac{1}{2}}$ and below the xy-plane for $(\frac{\pi}{2})^{\frac{1}{2}} \le x \le 2$. Hence, we have two different integrals for volume.

$$V_1 = \int_0^{(\frac{\pi}{2})^{\frac{1}{2}}} \int_0^{x/2} \cos x^2 dy\, dx = \int_0^{(\frac{\pi}{2})^{\frac{1}{2}}} \left(y \cos x^2 \Big|_{y=0}^{y=x/2} \right) dx = \int_0^{(\frac{\pi}{2})^{\frac{1}{2}}} \left(\frac{x}{2} \cos x^2 - 0 \right) dx$$

Using u-substitution the last integral becomes $\frac{1}{2} \int_a^b \cos u \frac{du}{2}$ where $\begin{array}{rcl} u & = & x^2 \\ du & = & 2x\, dx \end{array}$.

$$V_1 = \frac{1}{4} \left(\sin x^2 \Big|_{x=0}^{x=(\frac{\pi}{2})^{\frac{1}{2}}} \right) = \frac{1}{4} \left(\sin \frac{\pi}{2} - \sin 0 \right) = \frac{1}{4}.$$

Similarly,

$$V_2 = -\int_{(\frac{\pi}{2})^{\frac{1}{2}}}^{2} \int_0^{x/2} \cos x^2 dy\, dx = \frac{1}{4} \left(- \sin x^2 \Big|_{x=(\frac{\pi}{2})^{\frac{1}{2}}}^{x=2} \right) = -\frac{1}{4} \left(\sin 4 - \sin \frac{\pi}{2} \right) = -\frac{1}{4} \sin 4 + \frac{1}{4}.$$

Total volume is $V_1 + V_2 = -\frac{1}{4} \sin 4 + \frac{1}{2}$.

B U Department of Mathematics

Math 102 Calculus II

Summer 2003 Second Midterm

1. Find the directions in which $f(x,y)=\dfrac{x^2}{2}+\dfrac{y^2}{2}$
   (a) increases most rapidly at (1,1);
   (b) decreases most rapidly at (1,1).
   (c) What are the directions of zero change in f at (1,1)?

Solution:

Since $\boldsymbol{\nabla} f = x\vec{i}+y\vec{j}$,
(a) the direction of maximum change in f is $\boldsymbol{\nabla} f|_{(1,1)}=\vec{i}+\vec{j}$ and
(b) the direction of minimum change in f is $-\boldsymbol{\nabla} f|_{(1,1)}=-\vec{i}-\vec{j}$.
(c) The direction of the zero change can be found by considering the vectors perpendicular to the vectors computed in (a) and (b). Thus, they are $\vec{i}-\vec{j}$ and $\vec{j}-\vec{i}$.

2. Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible.

Solution:

Let x,y,z be three real numbers whose sum is 9. Define $f(x,y,z)=x^2+y^2+z^2$ and $g(x,y,z)=x+y+z-9$. Now, to find an absolute minimum for f with constraint g, we require

$$\begin{aligned}&\boldsymbol{\nabla} f=\lambda\boldsymbol{\nabla} g\\ &\boldsymbol{\nabla} f=2x\vec{i}+2y\vec{j}+2z\vec{k},\ \ \boldsymbol{\nabla} g=\vec{i}+\vec{j}+\vec{k}\\ &2x\vec{i}+2y\vec{j}+2z\vec{k}=\lambda(\vec{i}+\vec{j}+\vec{k})\end{aligned}$$

So, $2x=\lambda, 2y=\lambda, 2z=\lambda$. It follows that $2(x+y+z)=3\lambda$. Since $x+y+z=9$, we get $\lambda=6$. Thus,

$$x=y=z=3.$$

3. Evaluate:
   (a)

$$\int_0^1\int_y^1 \frac{\sin x}{x}dxdy.$$

(b)

$$\int_{-1}^1\int_{-\sqrt{1-y^2}}^0 \frac{4\sqrt{x^2+y^2}}{1+x^2+y^2}dxdy.$$

Solution:

(a)

$$\begin{aligned}\int_0^1\int_y^1 \frac{\sin x}{x}dxdy &= \int_0^1\int_0^x \frac{\sin x}{x}dydx = \int_0^1 \frac{\sin x}{x}y\Big|_0^x dx\\ &= \int_0^1 \sin x dx = \cos x|_0^1 = -\cos 1+1.\end{aligned}$$

(b) Let us use polar coordinates. So, by putting $x^2 + y^2 = r^2$,

$$\begin{aligned}
\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{0} \frac{4\sqrt{x^2+y^2}}{1+x^2+y^2} dxdy &= \int_{\pi/2}^{3\pi/2}\int_{0}^{1} \frac{4r}{1+r^2} r dr d\theta \\
&= (3\pi/2 - \pi/2)\int_{0}^{1} \frac{4r^2+4-4}{1+r^2} dr d\theta \\
&= \pi \int_{0}^{1} \left(4 - 4\frac{1}{1+r^2}\right) dr d\theta \\
&= \pi \left(4r - 4\tan^{-1} r\right)_{0}^{1} = \pi(4-\pi) \\
&= 4\pi - \pi^2.
\end{aligned}$$

4. Evaluate $\displaystyle\int_{0}^{3}\int_{0}^{4}\int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1} \left(\frac{2x-y}{2} + \frac{z}{3}\right) dxdydz$ by the substitution $u = \dfrac{2x-y}{2}$, $v = \dfrac{y}{2}$, $w = \dfrac{z}{3}$.

Solution:

First note that $x = u + v$.
For $x = \dfrac{y}{2}$, we get $x = v \Rightarrow u + v = v \Rightarrow u = 0$
and for $x = \dfrac{y}{2} + 1$, we get $x = v + 1 \Rightarrow u + v = v + 1 \Rightarrow u = 1$.
Finally $du = dx, 2dv = dy, 3dw = dz$ by the substitutions.
Therefore we have,

$$\int_{0}^{1}\int_{0}^{2}\int_{0}^{1} 6(u+w) dudvdw = 6\int_{0}^{1}\int_{0}^{2} \left(\frac{u^2}{2} + wu)\right|_{0}^{1} dvdw = 6\int_{0}^{1}\int_{0}^{2} (\frac{1}{2} + w) dvdw$$

$$= 6\int_{0}^{1} \left(\frac{v}{2} + wv)\right|_{0}^{2} dw = 6\int_{0}^{1} (1+2w) dw = 6\left.\left(w + w^2\right)\right|_{0}^{1} = 12.$$

5. Let Ω be the region in the first octant bounded by the coordinate planes, $x+y = 4$, $y^2+4z^2 = 16$. Write as many integrals as you can for volume of Ω and evaluate one of them.

Solution:

As we know, $x + y = 4$ is a plane equation and $y^2 + 4z^2 = 16$ is an elliptic cylinder equation. So,

$$\int_{0}^{2}\int_{0}^{\sqrt{16-4z^2}}\int_{0}^{4-y} dxdydz, \int_{0}^{4}\int_{0}^{4-y}\int_{0}^{\frac{\sqrt{16-y^2}}{2}} dzdxdy, \int_{0}^{4}\int_{0}^{4-x}\int_{0}^{\frac{\sqrt{16-y^2}}{2}} dzdydx,$$

$$\int_{0}^{4}\int_{0}^{\frac{\sqrt{16-y^2}}{2}}\int_{0}^{4-y} dxdzdy.$$

Triple integrals with the orders $dydxdz$ and $dydzdx$ are too complicated. Let us evaluate one of the integrals;

$$\begin{aligned}
\int_0^2\int_0^{\sqrt{16-4z^2}}\int_0^{4-y} dxdydz &= \int_0^2\int_0^{\sqrt{16-4z^2}}(4-y)dydz \\
&= \int_0^2 (4y-\frac{y^2}{2})\Big|_0^{\sqrt{16-4z^2}} dz \\
&= \int_0^2 (4\sqrt{16-4z^2}-\frac{16-4z^2}{2})dz \\
&= \int_0^2 8\sqrt{16-4z^2}-8+2z^2dz \\
&= (\frac{2z^3}{3}-8z)\Big|_0^2 + 32\int \cos^2\theta d\theta \text{ (by substitution } z=2\sin\theta) \\
&= -\frac{32}{3}+32\int\frac{\cos 2\theta+1}{2}d\theta \text{ (since } \cos^2\theta=\frac{\cos 2\theta+1}{2}) \\
&= -\frac{32}{3}+8\sin 2\theta+16\theta = -\frac{32}{3} - (16\frac{z}{2}\frac{\sqrt{4-z^2}}{2}+\arcsin\frac{z}{2})\Big|_0^2 \\
&= -\frac{32}{3}+16\arcsin 1 = -\frac{32}{3}+8\pi.
\end{aligned}$$

B U Department of Mathematics

Math 102 Calculus II

Date: July 26, 2004 Time: 16:00-17:00	Full Name	:
	Math 102 Number	:
	Student ID	:

Summer 2004 Second Midterm

IMPORTANT

1. Write your name, surname on top of each page. **2.** The exam consists of 5 questions some of which have more than one part. **3.** Read the questions carefully and write your answers neatly under the corresponding questions. **4.** Show all your work. Correct answers without sufficient explanation might <u>not</u> get full credit. **5.** Calculators are <u>not</u> allowed.

Q1	Q2	Q3	Q4	Q5	TOTAL
20 pts	20 pts	20 pts	20 pts	20 pts	100 pts

1.) Given the function $f(x,y) = \dfrac{8x^2y^2}{x^4+y^4}$

a. [5] Find the domain of $f(x,y)$.

Solution:

$$\text{Domain} = \{(x,y) \in \mathbb{R} \times \mathbb{R} : (x,y) \neq (0,0)\}$$

b. [15] Determine whether the $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists or not; if it does, evaluate the limit.

Solution:

ALONG $x = 0$:

$$\lim_{(x,y)\to(0,0)} \frac{8(0)y^2}{0+y^4} = \lim_{(x,y)\to(0,0)} \frac{0}{y^4} = \lim_{(x,y)\to(0,0)} 0 = 0$$

ALONG $y = x$

$$\lim_{(x,y)\to(0,0)} \frac{8x^2y^2}{x^4+y^4} = \lim_{(x,y)\to(0,0)} \frac{8x^4}{2x^4} = \lim_{(x,y)\to(0,0)} 4 = 4$$

Since the limits along different curves gives different results,

$\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist!

2.) [20] Find all the critical points and determine whether they are local minimum, local maximum or saddle points of the function $f(x,y) = x\sin y$.

Solution:

$f(x, y) = x \sin y$

$$\begin{array}{lllllll} f_x & = & \sin y & \quad f_y & = & x \cos y \\ f_{xx} & = & 0 & \quad f_{yy} & = & -x \sin y \\ f_{xy} & = & \cos y & \quad f_{yx} & = & \cos y \end{array}$$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & \cos y \\ \cos y & -x \sin y \end{vmatrix} = -\cos^2 y < 0$$

$D(x, y)$ is always negative.

The critical points of $f(x, y)$:
$f_x = \sin y = 0, \; y = k\pi$ for $k \in \mathbb{Z}$ i.e. $k = 0, \pm 1, \pm 2, ...$
$f_y = x \cos y = 0, x = 0$ or $\cos y = 0$, but $\cos y$ and $\sin y$ cannot be zero at the same time, so $x = 0$

The critical points are: $(0, k), k \in \mathbb{Z}$
That is: $(0, 0), (0, \pm\pi), (0, \pm 2\pi), (0, \pm 3\pi), ...$

All of them are saddle points as $D(x, y) < 0$.

3.) a. [12] Show that the ellipsoid $3x^2+2y^2+z^2 = 9$ and the sphere $x^2+y^2+z^2-8x-6y-8z+24 = 0$ are tangent to each other at the point $(1, 1, 2)$.

Solution:

Let $g(x, y, z) = 3x^2 + 2y^2 + z^2 - 9$ and $f(x, y, z) = x^2 + y^2 + z^2 - 8x - 6y - 8z + 24$. The point $(1, 1, 1)$ satisfies both surfaces $g(x, y, z) = 0$ and $f(x, y, z) = 0$

Then $\nabla g =< 6x, 4y, 2z >$, so $\nabla g(1, 1, 2) = 6\mathbf{i}+4\mathbf{j}+4\mathbf{k}$; and $\nabla f =< 2x-8, 2y-6, 2z-8 >$, so $\nabla g(1, 1, 2) = -6\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$

Since the gradient vectors of g and f at $(1, 1, 2)$ are parallel, the surfaces are tangent at $(1, 1, 2)$.

b. [8] Find the equation of this common tangent plane.

Solution:

The tangent plane has the normal vector $< 6, 6, 4 >$ and passes through the point $(1, 1, 2)$, hence the equation will be $6(x - 1) + 4(y - 1) + 4(z - 2) = 0$, or $6x + 4y + 4z = 18$.

4.) [20] Show that if $w = f(u, v)$ satisfies the Laplace Equation $f_{uu} + f_{vv} = 0$ and if $u = \frac{x^2-y^2}{2}$, $v = xy$ then w satisfies the Laplace Equation $w_{xx} + w_{yy} = 0$.

Solution:

$$u = \frac{x^2 - y^2}{2} \Rightarrow \begin{array}{ll} u_x = x & u_{xx} = 1 \\ u_y = -y & u_{yy} = -1 \end{array}$$

$$v = xy \Rightarrow \begin{array}{ll} v_x = y & v_{xx} = 0 \\ v_y = x & v_{yy} = 0 \end{array}$$

$$w_x = f_u u_x + f_v v_x,$$
$$w_y = f_u u_y + f_v v_y$$

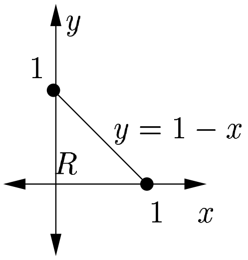
$$\begin{aligned} w_{xx} &= (f_{uu} u_x + f_{vu} v_x) u_x + f_u u_{xx} + (f_{uv} u_x + f_{vv} v_x) v_x + f_v v_{xx} \\ &= f_{uu}(u_x)^2 + f_{vu} v_x u_x + f_u u_{xx} + f_{uv} u_x v_x + f_{vv}(v_x)^2 + f_v v_{xx} \\ &= f_{uu} x^2 + f_{vu} xy + f_u + f_{uv} xy + f_{vv} y^2 + f_v 0 \end{aligned}$$

$$\begin{aligned} w_{yy} &= (f_{uu} u_y + f_{vu} v_y) u_y + f_u u_{yy} + (f_{uv} u_y + f_{vv} v_y) v_y + f_v v_{yy} \\ &= f_{uu}(u_y)^2 + f_{vu} v_y u_y + f_u u_{yy} f_{uv} u_y v_y + f_{vv}(v_y)^2 + f_v v_{yy} \\ &= f_{uu}(-y)^2 + f_{vu}(-xy) + f_u(-1) + f_{uv}(-xy) + f_{vv} x^2 + f_v 0 \end{aligned}$$

$$\begin{aligned} w_{xx} + w_{yy} &= f_{uu} x^2 + f_{vu} xy + f_u + f_{uv} xy + f_{vv} y^2 + f_{uu} y^2 + f_{vu}(-xy) + \\ &\quad (-f_u) + f_{uv}(-xy) + (-f_u) + f_{uv}(-xy) + f_{vv} x^2 \\ &= f_{uu}(x^2 + y^2) + f_{vv}(x^2 + y^2) \\ &= (x^2 + y^2)(f_{uu} + f_{vv}) \\ &= 0 \end{aligned}$$

5.) [20] Find the volume of the solid bounded by the paraboloid $z = x^2 + y^2 + 4$ and the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 1$.

Solution:



$$
\begin{aligned}
V &= \int_0^1\int_0^{1-x}(x^2+y^2+4)dydx \\
&= \int_0^1 \left[x^2y+\frac{y^3}{3}+4y\right]_0^{1-x} dx \\
&= \int_0^1 \left[x^2(1-x)+\frac{(1-x)^3}{3}+4(1-x)\right] dx \\
&= \int_0^1 \left[x^2-x^3+\frac{1}{3}(1-3x+3x^2-x^3)+4-4x\right] dx \\
&= \int_0^1 \left(-\frac{4}{3}x^3+2x^2-5x+\frac{13}{3}\right) dx \\
&= \left[-\frac{4}{3}\frac{x^4}{4}+\frac{2x^3}{3}-\frac{5x^2}{2}+\frac{13x}{3}\right]_0^1 \\
&= -\frac{1}{3}+\frac{2}{3}-\frac{5}{2}+\frac{13}{3} \\
&= \frac{14}{3}-\frac{5}{2} \\
&= \frac{13}{6}
\end{aligned}
$$

B U Department of Mathematics

Math 102 Calculus II

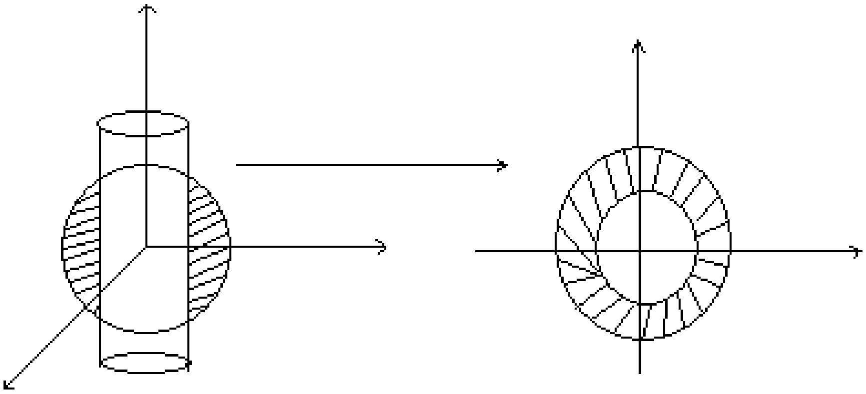
Summer 2005 Second Midterm

1.) Find the volume of the part of the sphere of radius 3 that is left after drilling a cylindrical hole of radius 2 through the center.

Solution:

The sphere has equation: $x^2 + y^2 = 9$
In polar coordinates, $r^2 + z^2 = 9 \Rightarrow z = \pm\sqrt{9 - r^2}$

$$V = \int_0^{2\pi} \int_2^3 [\sqrt{9 - r^2} - (-\sqrt{9 - r^2})] r dr d\theta$$



Let $u = 9 - r^2 \Rightarrow du = -2rdr$

$$\Rightarrow V = -\frac{1}{2} \int_0^{2\pi} \int_5^0 2u^{1/2} du d\theta = -\frac{2}{3} \int_0^{2\pi} -5^{3/2} d\theta = \frac{20\sqrt{5}\pi}{3}$$

2.) Let S denote the set of topless and bottomless right circular cylinders of fixed non-zero surface area A. Use Lagrange multipliers to prove or disprove that the set S has an element with maximal volume. If your answer is positive then compute the maximal possible volume in terms of only A. If your answer is negative tell us precisely the lack of which properties of the solution set of $A = 2\pi rh$ is responsible for the non-existance. (Note that the surface area of the cylinder of radius r and height h with no top and bottom is $2\pi rh$.)

Solution:

$$S(r,h) = 2\pi rh = A \Rightarrow g(r,h) = 2\pi rh - A$$

$$V(r,h) = \pi r^2 h \Rightarrow f(r,h) = \pi r^2 h$$

$$\nabla f = 2\pi rh\mathbf{i} + \pi r^2 h\mathbf{j}$$

$$\nabla g = 2\pi h\mathbf{i} + 2\pi r\mathbf{j}$$

$$\nabla f = \lambda \nabla g \Rightarrow 2\pi rh = \lambda \cdot 2\pi h \text{ and } \pi r^2 = \lambda \cdot 2\pi r$$

$$r = \lambda, r = 2\lambda \Rightarrow r = \lambda = 0.$$

But $r = 0$ gives $S(r,h) = 2\pi \cdot 0 \cdot h = 0$. But the surface area is nonzero. So using the Lagrange multiplier method we conclude that NO such maximal volume cylinder exists.

Note that the variables r, h are such that $r > 0, h > 0$. Hence, the solution set $(A = 2\pi rh)$ is an unbounded set. For the existence of extremum, solution set must be closed and bounded. Also, $V = \dfrac{Ar}{2}$ and since $r > 0, V \longrightarrow \infty$ and so no cylinder with maximal volume!

3.) Transform the following integral into spherical coordinates throughly but do not evaluate.

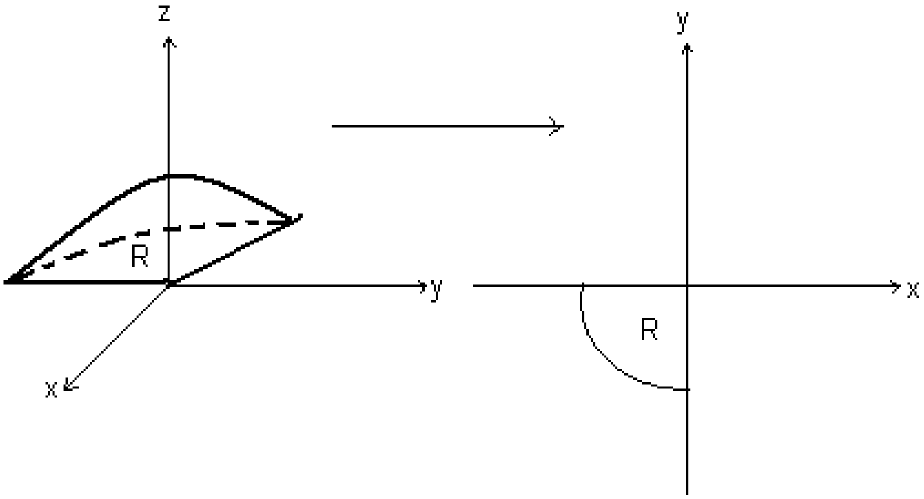
$$\int_{-2}^{0}\int_{-\sqrt{4-x^2}}^{0}\int_{0}^{\sqrt{4-x^2-y^2}} z^3\sqrt{x^2+y^2+z^2}\ \ dzdydx$$

Solution:

Let I be the value of the given integral.

$0 \leq z \leq \sqrt{4-x^2-y^2} \Rightarrow$ The solid over which we integrate is bounded below by $z=0$ (xy-plane) and above by the sphere $x^2+y^2=4$.

$-\sqrt{4-x^2} \leq y \leq 0$ and $-2 \leq x \leq 0$ tell us that the projection of this solid onto xy-plane is the quarter of the circle: $x^2+y^2=4$ in the third quadrant.

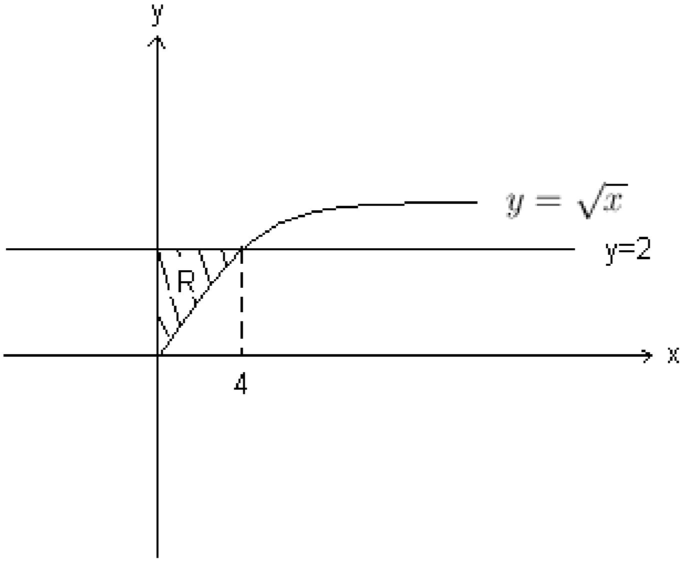


In spherical coordinates: $z=\rho\cdot\cos\phi$ and $\rho=\sqrt{x^2+y^2+z^2}$

$$I=\int_{\pi}^{3\pi/2}\int_{0}^{\pi/2}\int_{0}^{2}(\rho\cdot\cos\phi)^3\cdot(\rho)\cdot\rho^2\sin\phi d\rho d\phi d\theta$$

4.) Evaluate $\int\int_R sin(y^3)\ \ dA$, where R is the region bounded by $y = \sqrt{x}$, $y = 2$ and $x = 0$.

Solution:



Let $I = \displaystyle\int\int_R sin(y^3)\ \ dA = \int_0^4 \int_{\sqrt{x}}^2 \sin y^3 dydx = \int_0^2 \int_0^{y^2} \sin y^3 dydx$

This last integral seems to be more easier to take.

So, $I = \displaystyle\int_0^2 [x \sin y^3]_0^{y^2} dy = \int_0^2 y^2 \sin y^3 dy$

Let $u = y^3$, then $du = 3y^2 dy$ and therefore,

$$I = \int_0^2 \frac{\sin u}{u} du = -\left[\frac{\cos u}{3}\right]_{u_1}^{u_2} = -\left[\frac{\cos y^3}{3}\right]_0^2 = \frac{1 - \cos 8}{3}$$