B U Department of Mathematics

Math 102 Calculus II

Fall 2001 First Midterm

1. **a)** Find an equation for the sphere with center $(0, 0, 2)$ and radius 2 in spherical coordinates.

Solution:

$$x^2 + y^2 + (z-2)^2 = 4 \Rightarrow x^2 + y^2 + z^2 - 4z = 0 \Rightarrow \rho^2 - 4\rho\cos\phi = 0 \Rightarrow \rho = 4\cos\phi$$

b) Describe the graph of $9x^2 - 4y^2 + 36z^2 = -36$.

Solution:

$-9x^2 + 4y^2 - 36z^2 = 36 \Rightarrow -\dfrac{x^2}{4} + \dfrac{y^2}{9} - z^2 = 1$, hence this is a hyperboloid of two sheets.

c) Describe the graph of $\phi = \dfrac{\pi}{4}$.

Solution:

$\phi = \dfrac{\pi}{4} \Leftrightarrow z = \sqrt{x^2 + y^2}$. Hence it is a cone with its vertex at the origin.

2. **a)** Find an equation of the plane which is containing L_1 and is parallel to L_2 where

$$\begin{aligned} L_1 &: \ x = 2 + 8t,\ y = 6 - 8t,\ z = 10t \\ L_2 &: \ x = 3 + 8t,\ y = 5 - 3t,\ z = 6 + t \end{aligned}$$

Solution:

L_1 and L_2 are skew. $A = (2, 6, 0)$ is on L_1 and $B = (3, 5, 6)$ is on L_2. $\boldsymbol{v}_1 = \langle 8, -8, 10\rangle$ is the direction vector of L_1 and $\boldsymbol{v}_2 = \langle 8, -3, 1\rangle$ is the direction vector of L_2. So, the normal of the plane:

$$\boldsymbol{n} = \boldsymbol{v}_1 \times \boldsymbol{v}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 8 & -8 & 10 \\ 8 & -3 & 1 \end{vmatrix} = \langle 22, 72, 40\rangle$$

Hence $22x + 72y + 40z = 476 \Rightarrow 11x + 36y + 20z = 238$.

b) Find the distance between the above lines L_1 and L_2.

Solution:

$$d = \frac{|11 \cdot 3 + 36 \cdot 5 + 20 \cdot 6 - 238|}{\sqrt{11^2 + 36^2 + 20^2}} = \frac{95}{\sqrt{1817}}.$$

Second way:

$$\text{comp}_{\boldsymbol{n}} \boldsymbol{BA} = \frac{||\boldsymbol{BA} \cdot \boldsymbol{n}||}{||\boldsymbol{n}||} = \frac{|-11 + 36 - 120|}{\sqrt{1817}} = \frac{95}{\sqrt{1817}}.$$

3. **a)** Find the parametric equations for the line tangent to the curve $r(t) = e^t\boldsymbol{i}+\sin t\boldsymbol{j}+3\ln(1-t)\boldsymbol{k}$ at $t=0$.

Solution:

We compute a couple of necessary quantities:

$$\begin{aligned}
\boldsymbol{r}(0) &= \boldsymbol{i} \\
\boldsymbol{r}'(t) &= e^t\boldsymbol{i} + \cos t\boldsymbol{j} - \frac{3}{1-t}\boldsymbol{k} \\
\boldsymbol{r}'(0) &= \langle 1, 1, -3\rangle \\
\boldsymbol{r}(w) &= \boldsymbol{r}(0) + w\boldsymbol{r}'(0) \\
&= \langle 1, 0, 0\rangle + w\langle 1, 1, -3\rangle
\end{aligned}$$

Hence $x(w) = 1 + w$, $y(w) = w$, $z(w) = -3w$.

b) Find the length of the section of the curve $\boldsymbol{r}(t) = e^t\cos t\boldsymbol{i} + e^t\sin t\boldsymbol{j} + e^t\boldsymbol{k}$ from $-\pi$ to π.

Solution:

Arclength is given by $L = \displaystyle\int_{-\pi}^{\pi} ||\boldsymbol{r}'(t)||dt$ where
$\boldsymbol{r}'(t) = e^t(\cos t - \sin t)\boldsymbol{i} + e^t(\cos t + \sin t)\boldsymbol{j} + e^t\boldsymbol{k}$ and
$||\boldsymbol{r}'(t)|| = \sqrt{e^{2t}(\cos^2 t - 2\cos t\sin t + \sin^2 t + \cos^2 t + 2\cos t\sin t + \sin^2 t + 1)} = \sqrt{3}e^t$.
So,

$$\int_{-\pi}^{\pi} \sqrt{3}e^t dt = \sqrt{3}e^t\Big|_{-\pi}^{\pi} = \sqrt{3}(e^{\pi} - e^{-\pi}).$$

4. Find the maximum and minimum values of the radius of curvature for the curve $x = \cos t$, $y = \sin t$, $z = \cos t$.

Solution:

To find the curvature we need:

$$\begin{aligned}
\boldsymbol{r}(t) &= \langle \cos t, \sin t, \cos t\rangle \\
\boldsymbol{r}'(t) &= \langle -\sin t, \cos t, -\sin t\rangle \\
\boldsymbol{r}''(t) &= \langle -\cos t, -\sin t, -\cos t\rangle \\
||\boldsymbol{r}'(t)|| &= \sqrt{1+\sin^2 t}
\end{aligned}$$

Then it follows that:

$$\kappa(t) = \frac{||\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)||}{||\boldsymbol{r}'(t)||^3} = \frac{\left|\begin{array}{ccc} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -\sin t & \cos t & -\sin t \\ -\cos t & -\sin t & -\cos t \end{array}\right|}{(\sqrt{1+\sin^2 t})^3}$$

or equivalently:

$$\kappa(t) = \frac{||\langle -1, 0, 1\rangle||}{(\sqrt{1+\sin^2 t})^3} \Rightarrow \frac{1}{\kappa(t)} = \rho(t) = \frac{(1+\sin^2 t)^{3/2}}{\sqrt{2}}.$$

Since $0 \leq \sin^2 t \leq 1$, $\forall t$, when $\sin^2 t = 0$ the radius $\rho(t)$ is minimum, and when $\sin^2 t = 1$ the radius $\rho(t)$ is maximum. So, $\rho = \frac{(1+1)^{3/2}}{\sqrt{2}} = 2$ is the maximum value and $\rho = \frac{(1+0)^{3/2}}{\sqrt{2}} = \sqrt{2}/2$ is the minimum value.

B U Department of Mathematics

Math 102 Calculus II

Fall 2002 First Midterm

1. **(a)** Find parametric equations of the line that passes through the points $A(2,4,-3)$ and $B(3,-1,1)$.

   (b) At what point does this line intersect the $xy-$plane?

Solution:

(a) The direction from A to B is the vector $\boldsymbol{AB} = (3-2)\boldsymbol{i}+(-1-4)\boldsymbol{j}+(1-(-3))\boldsymbol{k} = \boldsymbol{i} - 5\boldsymbol{j} + 4\boldsymbol{k}$. So the parametric equations of this line are

$$\begin{aligned} x &= 2+t \\ y &= 4-5t \\ z &= -3+4t \end{aligned}$$

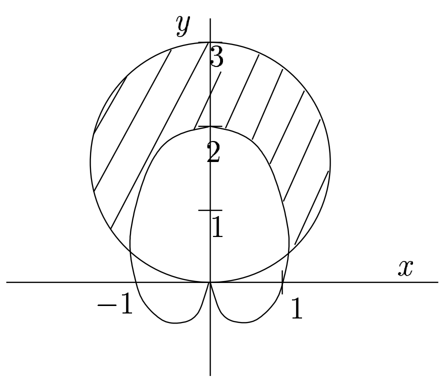
(b) The $xy-$plane has the equation $z=0$. Hence $z=-3+4t=0$ gives $t=3/4$ and so $x=11/4$ and $y=1/4$. Hence the intersection point is $(\frac{11}{4}, \frac{1}{4}, 0)$.

2. **(a)** Sketch the graph of the region R that lies inside the curve $r = 3\sin\theta$ and outside the curve $r = 1+\sin\theta$.

   (b) Find the area of R.

Solution:

(a)



The curves intersect where $3\sin\theta = 1+\sin\theta$, that is, $\sin\theta = 1/2$. So the points of intersection are:

$$(\theta, r) = \left(\frac{\pi}{6}, \frac{3}{2}\right) \quad \text{and} \quad (\theta, r) = \left(\frac{5\pi}{6}, \frac{3}{2}\right).$$

(b) Using symmetry with respect to y-axis

$$
\begin{aligned}
A &= 2 \cdot \tfrac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[(3\sin\theta)^2 - (1+\sin\theta)^2\right] d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[8\sin^2\theta - 2\sin\theta - 1\right] d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[3 - 4\cos 2\theta - 2\sin\theta\right] d\theta \\
&= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\pi/6}^{\pi/2} \\
&= 3\left(\frac{\pi}{2} - \frac{\pi}{6}\right) - 2\left(0 - \frac{\sqrt{3}}{2}\right) + 2\left(0 - \frac{\sqrt{3}}{2}\right) = \pi.
\end{aligned}
$$

3. Find parametric equations of the line in the plane of the points $P(0,0,0)$, $Q(2,2,0)$ and $R(0,1,-2)$ which intersects the line $x = -1+3t$, $y = 1+2t$, $z = \frac{1}{2}t$ in a right angle.

Solution:

First, take a normal vector $\boldsymbol{N}$ to this plane, i.e. a vector parallel to:

$$\boldsymbol{N} = \boldsymbol{PQ} \times \boldsymbol{PR} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 2 & 0 \\ 0 & 1 & -2 \end{vmatrix} = -4\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}.$$

We can take $\boldsymbol{N} = -2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$. The line L perpendicular to the given line L' has a direction vector $\boldsymbol{v}$ which is perpendicular to both $\boldsymbol{N}$ and $(6\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k})$. Hence,

$$\boldsymbol{v} /\!/ \boldsymbol{N} \times (6\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k}) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 2 & 0 \\ 0 & 1 & -1 \end{vmatrix} = -2\boldsymbol{i} + 8\boldsymbol{j} - 20\boldsymbol{k}$$

Take $\boldsymbol{v} = \boldsymbol{i} - 4\boldsymbol{j} + 10\boldsymbol{k}$.
Now we need a point of L. The equation of the plane is $-2x + 2y + z = 0$. The line L passes through the point of intersection of L' and the plane. The point of intersection is where $2(-1+3t) - 2(1+2t) - (\frac{1}{2}t) = 0$ is satisfied, i.e. when $t = \frac{8}{3}$. Thus, the intersection point is $S\left(7, \frac{19}{3}, \frac{4}{3}\right)$. The equations of the line are:

$$L: \begin{cases} x &= 7 + t \\ y &= \dfrac{19}{3} - 4t \\ z &= \dfrac{4}{3} + 10t \end{cases}$$

4. The position vector $\boldsymbol{R}(t) = \ln(t^2+1)\boldsymbol{i} + (t - 2\arctan t)\boldsymbol{j}$ of a moving particle is given below. Find:

(a) the velocity vector at $t = 0$;
(b) the speed at $t = 0$;
(c) the acceleration vector at $t = 0$;
(d) unit tangent vector at $t = 0$;
(e) the curvature at $t = 0$;

(f) the normal and tangential components of acceleration at $t = 0$;
(g) the arc length from $t = 0$ to $t = 2$.

Solution:

(a) Velocity is the variation of position in time:

$$\boldsymbol{V}(t) = \frac{d\boldsymbol{R}(t)}{dt} = \frac{2t}{t^2+1}\boldsymbol{i} + \left(1 - \frac{2}{t^2+1}\right)\boldsymbol{j} = \frac{2t}{t^2+1}\boldsymbol{i} + \left(\frac{t^2-1}{t^2+1}\right)\boldsymbol{j}.$$

Then $\boldsymbol{V}(0) = -\boldsymbol{j}$.

(b) Speed $= \|\boldsymbol{V}(t)\|$, so $\|\boldsymbol{V}(0)\| = \left(\dfrac{ds(t)}{dt}\right)_{t=0} = 1$

(c) Acceleration is the variation of velocity in time:

$$\boldsymbol{a}(t) = \frac{d\boldsymbol{V}(t)}{dt} = \frac{2(t^2+1) - 2t \cdot 2t}{(t^2+1)^2}\boldsymbol{i} + \frac{2t(t^2+1) - 2t(t^2-1)}{(t^2+1)^2}\boldsymbol{j} = \frac{2-2t^2}{(t^2+1)^2}\boldsymbol{i} + \frac{4t}{(t^2+1)^2}\boldsymbol{j}.$$

Then $\boldsymbol{a}(0) = 2\boldsymbol{i}$.

(d) Since $\boldsymbol{V}(0)$ is a unit vector, $\boldsymbol{T}(0) = -\boldsymbol{j}$.

(e) The curvature is given by: $\kappa(t) = \dfrac{\|\boldsymbol{V}(t) \times \boldsymbol{a}(t)\|}{\left|\dfrac{ds(t)}{dt}\right|^3}$.

Hence $\kappa(0) = \dfrac{\|-\boldsymbol{j} \times 2\boldsymbol{i}\|}{1} = \|-2\boldsymbol{k}\| = 2$.

(f) Since $a_N(t) = \kappa(t)\left(\dfrac{ds(t)}{dt}\right)^2$, it follows that $a_N(0) = 2$. Now note that $\|\boldsymbol{a}(t)\|^2 = a_T^2(t) + a_N^2(t)$. At $t = 0$, $4 = a_T^2(0) + 4$. Therefore $a_T(0) = 0$.

(g) In part **(a)** we calculated $\boldsymbol{V}(t)$. Then:

$$\begin{aligned}
\frac{ds(t)}{dt} &= \|\boldsymbol{V}(t)\| = \frac{1}{t^2+1}\sqrt{4t^2 + (t^2-1)^2} = \frac{\sqrt{t^4+2t^2+1}}{t^2+1} \\
&= \frac{\sqrt{(t^2+1)^2}}{t^2+1} = \frac{|t^2+1|}{t^2+1} = 1 \quad (0 \leqslant t \leqslant 2).
\end{aligned}$$

Since speed is constant, $s = \displaystyle\int_0^2 \frac{ds(t)}{dt}dt = \int_0^2 dt = 2$.

B U Department of Mathematics

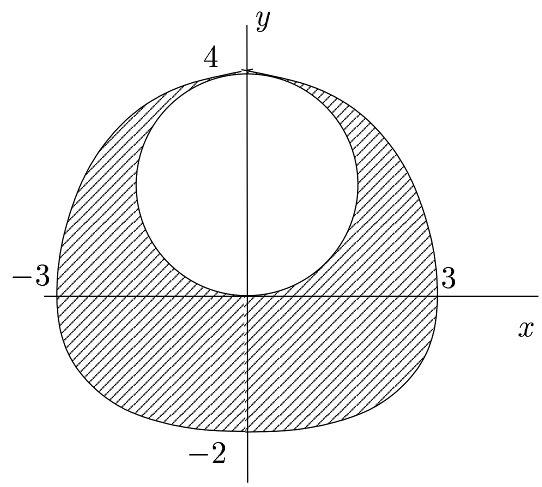
Math 102 Calculus II

Fall 2003 First Midterm

1. **a)** Sketch the region R inside the graph of $r = 3 + \sin\theta$ and outside the graph of $r = 4\sin\theta$.
   b) Find the area of the region R.

Solution:

a)



b) A =area of the limaçon − area of the circle

$$\begin{aligned}
\text{Area of the limaçon} &= \frac{1}{2}\int_0^{2\pi}(3+\sin\theta)^2 d\theta = \frac{1}{2}\int_0^{2\pi}(9+6\sin\theta+\sin^2\theta)d\theta \\
&= \frac{9}{2}\int_0^{2\pi} d\theta + 3\int_0^{2\pi}\sin\theta d\theta + \frac{1}{2}\int_0^{2\pi}\sin^2\theta d\theta \\
&= \left(\frac{9}{2}\theta - 3\cos\theta + \frac{1}{4}\theta - \frac{1}{8}\sin 2\theta\right)\Big|_0^{2\pi} \\
&= 9\pi + \frac{\pi}{2} = 19\frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
\text{Area of the circle} &= \frac{1}{2}\int_0^{\pi}(4\sin\theta)^2 d\theta \\
&= \frac{1}{2}\int_0^{\pi} 16\sin^2\theta d\theta \\
&= 8\int_0^{\pi}\frac{1}{2}(1-\cos 2\theta)d\theta \\
&= (4\theta - 2\sin 2\theta)|_0^{\pi} = 4\pi
\end{aligned}$$

Hence we find $A = 19\frac{\pi}{2} - 4\pi$.

2. **a)** Find parametric equations for the intersection of the planes $2x+y-z=3$ and $x+2y+z=3$.

   b) Find the acute angle between the two planes.

Solution:

a) The normal vectors to the two planes are $\boldsymbol{n}_1 = \langle 2,1,-1\rangle$ and $\boldsymbol{n}_2 = \langle 1,2,1\rangle$ respectively. The line L of intersection of two planes is parallel to $\boldsymbol{n}_1 \times \boldsymbol{n}_2$.

$$\boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3\boldsymbol{i} - 3\boldsymbol{j} + 3\boldsymbol{k},$$

or equivalently a vector in the same direction can be taken as: $\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$.
By inspection the point $(0,2,-1)$ lies on both planes, so the line has an equation in vector form: $\boldsymbol{r}(t) = 2\boldsymbol{j} - \boldsymbol{k} + t(\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$. Parametric equations are: $x = t$, $y = 2 - t$, $z = t - 1$, $(t \in \mathbb{R})$.
b) $\cos\theta = \dfrac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{||\boldsymbol{n}_1|| \cdot ||\boldsymbol{n}_2||} = \dfrac{2+2-1}{\sqrt{6}\sqrt{6}} = \dfrac{1}{2}$. So, $\theta = \dfrac{\pi}{3}$.

3. Let C be the plane curve parametrized by $\boldsymbol{r}(t) = t\boldsymbol{i} + (2t^2+3)\boldsymbol{j}$ and let l be the line parametrized by $\boldsymbol{L}(t) = (t+3)\boldsymbol{i} + (t/4 - 10)\boldsymbol{j}$. Find the point on C where the tangent line is perpendicular to l.

Solution:

The direction vector for $\boldsymbol{L}(t)$ is $\boldsymbol{v} = \boldsymbol{i} + \dfrac{1}{4}\boldsymbol{j}$. The point sought is the point where $\boldsymbol{r}'(t)$ and $\boldsymbol{v}$ are orthogonal. $\boldsymbol{r}'(t) = \boldsymbol{i} + 4t\boldsymbol{j}$, we use $\boldsymbol{v} \cdot \boldsymbol{r}'(t) = 0 \Rightarrow \boldsymbol{v} \cdot \boldsymbol{r}'(t) = 1 + t = 0 \Rightarrow t = -1$. The point occurs at $t = -1$ and is therefore the terminal point of $\boldsymbol{r}(-1) = -\boldsymbol{i} + 5\boldsymbol{j}$ or $(-1, 5)$.

4. **a)** The curve whose vector equation is $\boldsymbol{r}(t) = 2\sqrt{t}\cos t\boldsymbol{i} + 3\sqrt{t}\sin t\boldsymbol{j} + \sqrt{1-t}\boldsymbol{k}$ lies on a quadric surface. Find an equation for this surface and identify it.

   b) Find an equation for the sphere with center $(0,0,2)$ and $r = 2$ in spherical coordinates.

Solution:

a) $x = 2\sqrt{t}\cos t$, $y = 3\sqrt{t}\sin t$, $z = \sqrt{1-t}$.

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + z^2 = t + 1 - t$$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + z^2 = 1,$$

so the surface is an ellipsoid.
b) Cartesian coordinates: $x^2 + y^2 + (z-2)^2 = 4$ or $x^2 + y^2 + z^2 - 4z = 0$. Using the coordinates: $\rho^2 = x^2 + y^2 + z^2$, $z = \rho\cos\phi$

$$\rho^2 - 4\rho\cos\phi = 0 \Rightarrow \rho = 4\cos\phi.$$

B U Department of Mathematics

Math 102 Calculus II

Fall 2004 First Midterm

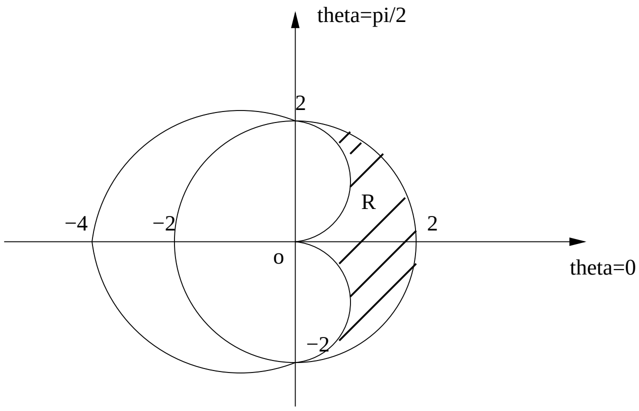
1.) a) Sketch the region R that lies inside the curve $r = 2$ and outside the curve $r = 2 - 2\cos\theta$.
b) Find the area of the region R.

Solution:

a) The curve $r = 2$ is a circle of radius 2 with its center located at the origin. To draw the curve $r = 2 - 2\cos\theta$ we determine some points that this curve passes through such as:

$(\theta = 0, r = 0), (\theta = \pi/2, r = 2), (\theta = \pi, r = 4), (\theta = 3\pi/2, r = 2), (\theta = 2\pi, r = 0)$.

These two curve intersect when $2 = 2 - \cos\theta$, i.e, at $\theta = \pi/2$ and $\theta = 3\pi/2$. So we obtain,



b) The area of the region R can be obtained by the integral

$$\begin{aligned} A &= \int_{-\pi/2}^{\pi/2} \frac{1}{2}[2^2 - (2 - 2\cos\theta)^2]d\theta = 2\int_{-\pi/2}^{\pi/2}[2\cos\theta - \cos^2\theta]d\theta \\ &= 2\int_{-\pi/2}^{\pi/2}[2\cos\theta - (\frac{1+\cos 2\theta}{2})]d\theta = 8 - \pi \end{aligned}$$

Note that since our figure is symmetric with respect to x-axis, this area can also be calculated from A=2 $\int_0^{\pi/2} \frac{1}{2}[2^2 - (2 - 2\cos\theta)^2]d\theta$.

2.) Find an equation of the plane that passes through the points $P_1(0, -3, 2)$ and $P_2(1, 2, 3)$ and parallel to the line of intersection of the planes $2x + y - z = 1$ and $x - 2y + z = 7$.

Solution:

The vector $\vec{N_1} =< 2, 1, -1 >$ is normal to the first plane and $\vec{N_2} =< 1, -2, 1 >$ is normal to the second plane. Therefore the line of intersection of these two planes is parallel to the vector

$\vec{V} = \vec{N_1} \times \vec{N_2}$ = -**i** -3**j** -5 **k**.

The vector $\vec{P_1P_2} =< 1, 5, 1 >$ lies on the required plane. Now, also the vector $\vec{V}$ can be carried onto this plane since this plane is parallel to the line of intersection. Thus,

$\vec{N_3} = \vec{P_1P_2} \times \vec{V}$= -22 **i** + 4 **j** + 2 **k**

is a normal vector for the required plane. To write down the equation of this plane both of the points P_1 and P_2 can be used. Using P_1 we get

$$-22x + 4y + 2z + 8 = 0$$

3.) (a) Find an equation of the plane parallel to the plane $3x - y + 2z + 3 = 0$ if the point (2,2,-1) is equidistant from both planes.

Solution:

Since the required plane is parallel to the plane $3x - y + 2z + 3 = 0$, $< 3, -1, 2 >$ is a normal vector for both of them. Therefore the required plane has an equation of the form $3x - y + 2z + D = 0$ where D is a real number. Now since the point (2,2,-1) is equidistant from both planes we have

$$\frac{|3.2 - 1.2 + 2. - 1 + 3|}{\sqrt{3^2 + (-1)^2 + 2^2}} = \frac{|3.2 - 1.2 + 2. - 1 + D|}{\sqrt{3^2 + (-1)^2 + 2^2}}.$$

This equation implies $|D+2| = 5$ which has two solutions $D = 3$ and $D = -7$. The $D = 3$ solution corresponds to the first plane, therefore the required plane has the equation:

$$3x - y + 2z - 7 = 0$$

(b) Consider the straight line through the point (3,2,3) and perpendicular to the plane given by $2x - y + 3z + 1 = 0$. Find the coordinates of the point of the intersection of that line and that plane.

Solution:

The plane has a normal vector $< 2, -1, 3 >$. Since the required line is perpendicular to the plane, this vector is parallel to the line. Hence, the parametric equation of the line is:

x=3+ 2t , y=2-t , z=3+3t , $-\infty < t < \infty$.

To find the value of t at which the line and the plane intersect, we insert the above result into the plane equation:

2(3+2t) - (2-t) + 3(3+3t) +1 =0,

which gives $t = -1$. The intersection point is obtained by putting $t = -1$ in the parametric equation of the line which gives its coordinates as (1,3,0).

(c) Can a vector (whose initial point is at the origin) have direction angles $\theta_1 = \pi/4$, $\theta_2 = 3\pi/4$, $\theta_3 = \pi/3$ where θ_1, θ_2, θ_3 are the angles between the vector and x,y,z coordinates respectively?

Solution:

For a vector $\vec{V}$ whose direction angles are θ_1, θ_2, θ_3 we have

$$\vec{V}\ .\mathbf{i}= \cos\theta_1|\vec{V}| \quad , \quad \vec{V}\ .\mathbf{j}= \cos\theta_2|\vec{V}| \quad , \quad \vec{V}\ .\mathbf{k}= \cos\theta_3|\vec{V}| \quad ,$$

which leads to,

$$\frac{\vec{V}}{|\vec{V}|} = \cos\theta_1\mathbf{i} + \cos\theta_2\mathbf{j} + \cos\theta_3\mathbf{k}$$

Since this is a unit vector, direction cosines should satisfy

$$\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3=1$$

In our problem this is not satisfied since, $\cos^2(\pi/4) = \cos^2(\pi/4) = 1/2$ and $\cos^2(\pi/3) = 1/4$. Therefore, such a vector does **not** exist.

(d) Find equations in rectangular and cylindrical coordinates for the surface $\rho = 2\sin\phi$ given in spherical coordinates.

Solution:

We first multiply the equation of this surface by ρ which gives $\rho^2 = 2\rho\sin\phi$. To convert this into rectangular coordinates we use $\rho^2 = x^2 + y^2 + z^2$, $\rho\sin\phi = \sqrt{x^2 + y^2}$ and obtain

$$x^2 + y^2 + z^2 = 2\sqrt{x^2 + y^2} \quad \text{(rectangular)}.$$

Now we can convert the final expression into the cylindrical coordinates by applying $x^2 + y^2 = r^2$ which gives

$$r^2 + z^2 = 2r \quad , r \geq 0 \quad \text{(cylindrical)}.$$

4.) Let $\mathbf{r}(t)= \sin 3t\mathbf{i} + \cos 3t\mathbf{j} + 2t\mathbf{k}$ be a vector valued function.

(a) Find the unit tangent vector $\mathbf{T}(t)$ and the principal unit normal vector $\mathbf{N}(t)$.

Solution:

Let ' (prime) denote the derivative with respect to t. Then,

$$\begin{aligned}
\vec{T}(t) &= \frac{\vec{r}(t)'}{|\vec{r}(t)'|} = \frac{3\cos 3t\mathbf{i} - 3\sin 3t\mathbf{j} + 2\mathbf{k}}{\sqrt{13}} \\
\vec{N}(t) &= \frac{\vec{T}(t)'}{|\vec{T}(t)'|} = -\sin 3t\mathbf{i} - \cos 3t\mathbf{j}
\end{aligned}$$

(b) Find the curvature $\kappa(t)$.

Solution:

$$\kappa(t) = \frac{|\vec{T}(t)'|}{|\vec{r}(t)'|} = \frac{9}{13}$$

(c) Find the arc length parametrization of this curve by taking (0,1,0) as the reference point.

Solution:

The point (0,1,0) corresponds to $t = 0$. Taking this point as the reference we can find the arc lengt parametrization in the increasing t direction from the integral

$$s = \int_0^t \left| \frac{d\vec{r}(u)}{du} \right| du = \sqrt{13}t$$

This gives $t = s/\sqrt{13}$. Therefore,

$$\vec{r}(s) = \sin\left(\frac{3s}{\sqrt{13}}\right)\mathbf{i} + \cos\left(\frac{3s}{\sqrt{13}}\right)\mathbf{j} + \frac{2s}{\sqrt{13}}\mathbf{k}$$

(d) Let $\mathbf{r}(t)$ be the position vector of a particle at time t. Find the scalar tangential and normal components of acceleration.

Solution:

From above we have $s = \sqrt{13}t$ and $\kappa(t) = 9/13$. Therefore,

$$\begin{aligned} a_T &= \frac{d^2 s}{dt^2} = 0 \\ a_N &= \kappa\left(\frac{ds}{dt}\right)^2 = 9 \end{aligned}$$

B U Department of Mathematics

Math 102 Calculus II

Fall 2005 First Midterm

1. Let $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ be vectors. Show that

(i) $|\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}}| \leq |\vec{\boldsymbol{a}}||\vec{\boldsymbol{b}}|$.
If we have equality, what can you say about $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$?

Solution:

By the definition of dot product we know that

$$|\vec{a} \cdot \vec{b}| = \left| |\vec{a}||\vec{b}| \cos\theta \right| = |\vec{a}||\vec{b}||\cos\theta|$$

Since $0 \leq |\cos\theta| \leq 1$

$$|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$$

If we have equality, $|\cos\theta| = 1$. Thus, $\theta = 0$ or $\theta = \pi$.
When $\theta = 0$, $\vec{a}$ and $\vec{b}$ are parallel and are in the same direction. On the other hand when $\theta = \pi$, $\vec{a}$ and $\vec{b}$ are parallel and are in the opposite direction.

(ii) $|\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}| \leq |\vec{\boldsymbol{a}}||\vec{\boldsymbol{b}}|$.
If we have equality, what can you say about $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$?

Solution:

By the definition of cross product we know that

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$$

Since $-1 \leq \sin\theta \leq 1$

$$|\vec{a} \times \vec{b}| \leq |\vec{a}||\vec{b}|$$

If we have equality, $\sin\theta = 1$. Thus, $\theta = \pi/2$.
Hence $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

2. Find the parametric equations of the line of intersection of the planes,

$$\begin{array}{ccccccccc} 3x & - & 6y & - & 2z & - & 15 & = & 0 \\ 2x & + & y & - & 2z & - & 5 & = & 0 \end{array}$$

Solution:

Let P_1 be the plane $3x - 6y - 2z = 15$. Then the normal vector for this plane is $n_1 =< 3, -6, -2 >$.

Let P_2 be the plane $2x + y - 2z = 5$. Then the normal vector for this plane is $n_2 =< 2, 1, -2 >$.

The line of the intersection of two planes is perpendicular to the planes' normal vectors n_1 and n_2, and therefore parallel to $n_1 \times n_2$. In other words, the line of intersection is a nonzero scalar multiple of $n_1 \times n_2$. In our case,

$$n_1 \times n_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\vec{i} + 2\vec{j} + 15\vec{k}$$

Now, we need to find a point on the line to write the equation of the line. To find a point on the line, we should take a point common to the two planes.
So we substitute $z = 0$ in the plane equations and solve the equations for x and y simultaneously.

$$\left.\begin{array}{ccccc} 3x & - & 6y & = & 15 \\ 2x & + & y & = & 5 \end{array}\right\} \begin{array}{ccc} x & = & 3 \\ y & = & -1 \end{array}$$

So we found the point $(3, -1, 0)$.
Hence the line is $x(t) = 3 + 14t, \quad y(t) = -1 + 2t, \quad z(t) = 15t$

Note that $(3, -1, 0)$ is not the only intersection point. You may find another intersection point and write the equation of the line according to this point.

3. If $\vec{\boldsymbol{r}}(t)$ is a differentiable vector-valued function of t of constant length, then show that $\vec{\boldsymbol{r}}(t) \perp \vec{\boldsymbol{r}'}(t)$.

Solution:

First consider that $\dfrac{d}{dt}\Big(\vec{r}(t) \cdot \vec{r}(t)\Big) = \vec{r}(t) \cdot \dfrac{dr}{dt} + \dfrac{dr}{dt} \cdot \vec{r}(t) = 2\vec{r}(t)\dfrac{dr}{dt}$

But we know that $\dfrac{d}{dt}\Big(\vec{r}(t) \cdot \vec{r}(t)\Big) = \dfrac{d}{dt}[\|\vec{r}(t)\|^2]$ where $\|\vec{r}(t)\|^2$ is a scalar.

So $\dfrac{d}{dt}[\|\vec{r}(t)\|^2] = 0$

Hence $\vec{r}(t) \cdot \dfrac{dr}{dt} = 0$. In other words $\vec{\boldsymbol{r}}(t) \perp \vec{\boldsymbol{r}'}(t)$.

4. The vector-valued function $\vec{\boldsymbol{r}}(t)$ is given by $\vec{\boldsymbol{r}}(t) =< \cos t, \sin t, t >$.

(i) Find the arc length from $t = 0$ to $t = 2\pi$.

Solution:

$$\begin{aligned} \vec{r}(t) &= < \cos t, \sin t, t > \quad \text{then} \\ \vec{r'}(t) &= < -\sin t, \cos t, 1 > \quad \text{and} \\ \|\vec{r'}(t)\| &= \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2} \end{aligned}$$

Let L be the arc length of $\vec{r}(t)$ from $t = 0$ to $t = 2\pi$

$$L = \int_0^{2\pi} \left\| \frac{dr}{dt} \right\| dt = \int_0^{2\pi} \sqrt{2} dt = \sqrt{2} \int_0^{2\pi} dt = \sqrt{2} \cdot t \Big|_{t=0}^{t=2\pi} = 2\sqrt{2}\pi$$

(ii) Find the principle unit tangent vector $\vec{\boldsymbol{T}}$.

Solution:

The principle unit tangent vector to the graph of $\vec{r}(t)$ at t is $\vec{T}(t) = \dfrac{\vec{r'}(t)}{\|\vec{r'}(t)\|}$.

Since $\vec{r'}(t) =< -\sin t, \cos t, 1 >$ and $\|\vec{r'}(t)\| = \sqrt{2}$ we obtain

$$\vec{T}(t) = \left\langle \frac{-\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

(iii) Find the principle unit normal vector $\vec{\boldsymbol{N}}$.

Solution:

The principle unit normal vector to the graph of $\vec{r}(t)$ at t is $\vec{N}(t) = \dfrac{\vec{T'}(t)}{\|\vec{T'}(t)\|}$.

Since $\vec{T'}(t) = \left\langle \dfrac{-\cos t}{\sqrt{2}}, \dfrac{-\sin t}{\sqrt{2}}, 0 \right\rangle$ and $\|\vec{T'}(t)\| = \sqrt{\dfrac{\cos^2 t}{2} + \dfrac{\sin^2 t}{2}} = \sqrt{\dfrac{1}{2}}$

we obtain

$$\vec{N}(t) =< -\cos t, -\sin t, 0 >= -\cos t\vec{i} - \sin t\vec{j}$$

B U Department of Mathematics

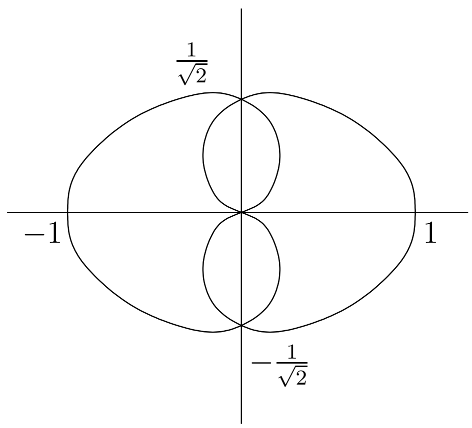
Math 102 Calculus II

Fall 2006 Midterm 1

1. [7] (AntonBivensDavis, p.729, q.51) Sketch the curve given by $r = \cos\dfrac{\theta}{2}$ in polar coordinates. State explicitly the symmetries of the graph and the slopes while approaching the origin. In your sketch, one must observe these easily.

Solution:

The first thing to do is to give values for θ to collect data for r. The reasonable thing to do is to go up to 4π since that is the period for $f(\theta) = \cos\frac{\theta}{2}$. Then it is easy to see that the graph looks like



Then make sure about the symmetries. Since cos is an even function, we get the symmetry with respect to x-axis.

It is not easy to detect the other symmetries. Think as follows: as θ varies from 0 to 2π, the curve traverses the upper half of the figure above. We can then get the symmetry with respect to the origin by observing that

$$f(2\pi + \theta) = -f(\theta).$$

Since there is symmetry with respect to the x-axis and the origin, there is symmetry with respect to the y-axis.

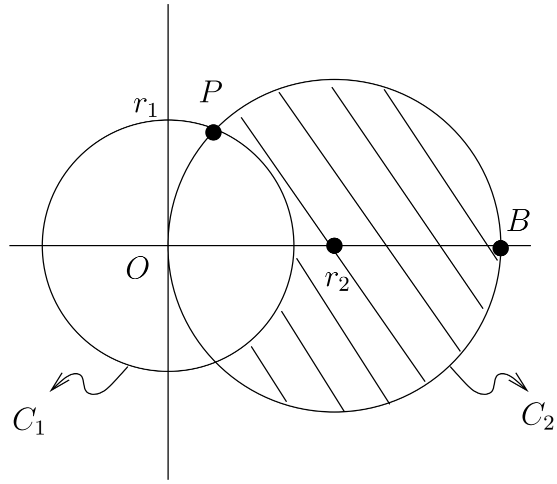
Finally, the slopes while approaching 0 are always 0:

$$\frac{dy}{dx}(\pi) = \frac{dy/d\theta}{dx/d\theta}(\pi) = \frac{r'\sin\theta}{r'\cos\theta}(\pi) = \tan(\pi) = 0.$$

Above computation is valid since at π (or 3π) the denominators are never zero.

2. In the figure below, C_1 and C_2 are circles with radii r_1 and r_2 and with centers at the origin O and at $(r_2, 0)$ respectively. Suppose that $2r_2 > r_1$ so that C_1 and C_2 intersect each other at two distinct points. Let P be the point of intersection in the first quadrant and H be the shaded *crescent*.

(a) [1+1] Write down the two equations in polar coordinates that describe C_1 and C_2.
(b) [2] Find the angle POB in terms of $\frac{r_1}{r_2}$. Easiest way is to use the answer to part (a).
(c) [4] Write down an integral in polar coordinates for the area of H. Then compute the area in terms of $\frac{r_1}{r_2}$.



Solution:

(a) $C_1 : r = r_1, \quad C_2 : (x - r_2)^2 + y^2 = r_2^2 \Rightarrow r^2 = 2r_2 x \Rightarrow r = 2r_2 \cos\theta$
(b) P is a point of intersection of C_1 and C_2. Hence,

$$\theta_0 = \widehat{POB} = \cos^{-1}(\frac{1}{2}\frac{r_1}{r_2}) > 0.$$

(c)

$$\begin{aligned}
\text{area}(H) &= \int_{-\theta_0}^{\theta_0} \frac{1}{2}\big((2r_2\cos\theta)^2 - (r_1)^2)\big)d\theta \\
&= \int_0^{\theta_0} (4r_2^2\frac{1+\cos 2\theta}{2} - r_1^2)d\theta \\
&= (2r_2^2(\theta + \frac{\sin 2\theta}{2}) - r_1^2\theta)_0^{\theta_0} \\
&= 2r_2^2\theta_0 + r_2^2 \sin 2\theta_0 - r_1^2\theta_0 \\
&= (2r_2^2 - r_1^2)\theta_0 + 2r_2^2\cos\theta_0 \sin\theta_0 \\
&= (2r_2^2 - r_1^2)\theta_0 + 2r_2^2 \frac{r_1}{2r_2}\frac{\sqrt{4r_2^2 - r_1^2}}{r_2} \\
&= (2r_2^2 - r_1^2)\theta_0 + r_1\sqrt{4r_2^2 - r_1^2}.
\end{aligned}$$

3. [6] (AntonBivensDavis, p.823, q.44) Let a, b, c and d be four vectors that are parallel to a fixed plane. Show that $(a \times b) \times (c \times d) = 0$. Be as simple as possible but be precise!

Solution:

Let n be a normal to the plane. Given that $a, b, c, d \perp n$, we have $a \times b = kn$ and $c \times d = ln$ for some $k, l \in \mathbb{R}$. So

$$(a \times b) \times (c \times d) = (kn) \times (ln) = (kl)n \times n = 0$$

4. Let D_1 and D_2 be two distinct, non-parallel planes in $\mathbb{R}^3$ and let α be the angle between them. Consider two vectors v_1 and v_2 parallel to D_1 and D_2 respectively. Let θ be the angle between v_1 and v_2. Assume $0 \leq \alpha, \theta \leq \pi$. Determine whether the following statements are true or false. For each, if answer is yes, prove the claim. If no, give an example where the claim is not true.
(a) [2] It is always true that $\theta \geq \alpha$.
(b) [2] It is always true that $\theta \leq \alpha$.

Solution:

Answer is no for both cases. Let L be the line of intersection and v be a vector parallel to L. Then v is parallel to both D_1 and D_2.
(a) Set $v_1 = v_2 = v$. Then $\theta = 0 < \alpha$.
(b) Set $v_1 = v$, $v_2 = -v$. Then $\theta = \pi > \alpha$.

5. [10] Using vector calculus, find the point on the line $x + y - z = 2, 3x + y = 1$ closest to the point $(1, 1, 1)$. You are **not** allowed to answer this question by finding the global minimum of an appropriate function!

Solution:

The case in question is exactly the same as the previous question: two planes intersecting along a line (a line in $\mathbb{R}^3$ is given by two independent linear equations). There are several ways to solve this question. Here, we first find the plane D passing through the point $P = (1, 1, 1)$ and normal to the line. A direction n along the line is:

$$n = \langle 1, 1, -1\rangle \times \langle 3, 1, 0\rangle = \langle 1, -3, -2\rangle$$

The equation for the plane D is

$$D : \ \langle 1, -3, -2\rangle \cdot \langle x - 1, y - 1, z - 1\rangle = x - 3y - 2z + 4 = 0$$

The closest point $Q = (x, y, z)$ on L to P is the point of intersection of L and D; hence Q satisfies:

$$x + y - z = 2, \ \ 3x + y = 1, \ \ -x + 3y + 2z = 4$$

One can then compute that $Q = (x, y, z) = (-\frac{3}{14}, \frac{23}{14}, -\frac{8}{14})$.

B U Department of Mathematics
Math 102 Calculus II

Date: October 31, 2007	Full Name :
Time: 18:10-19:10	Signature :
	Math 102 # :

Fall 2007 First Midterm Examination

KEY

IMPORTANT

1. Write your name, surname on top of each page. **2.** The exam consists of 3 questions some of which have more than one part. **3.** Read the questions carefully and write your answers neatly. **4.** Show all your work. Correct answers without sufficient explanation might not get full credit. **5.** Calculators are not allowed.

Q1	Q2	Q3	TOTAL
35 pts	35 pts	30 pts	100 pts

---

1. Consider the polar curves $C_1 : r = f(\theta) = \sqrt{2}\sin\theta$ and $C_2 : r = g(\theta) = \sqrt{\sin 2\theta}$.

5 pts **a.** Find the $\theta_0 \in (0, \frac{\pi}{2})$ where C_1 and C_2 intersect.

$f(\theta) = g(\theta)$ gives intersections:

$\sin\theta > 0$, $\cos\theta > 0$ on $(0, \pi/2)$

$$\sqrt{2}\,\cancel{\sin\theta} = \sqrt{\sin 2\theta} = \sqrt{2}\sqrt{\sin\theta\cos\theta} \overset{\S}{=} \cancel{\sqrt{2}}\,\sqrt{\sin\theta}\,\sqrt{\cos\theta}$$

$\sin\theta \neq 0$ on $(0, \frac{\pi}{2})$ $\Rightarrow$ $\cancel{\sqrt{2}}\sqrt{\sin\theta} = \cancel{\sqrt{2}}\sqrt{\cos\theta}$

$\Rightarrow$ $\theta_0 = \frac{\pi}{4}$ is the only θ in $(0, \frac{\pi}{2})$

5 pts **b.** Show that $g(\theta) > f(\theta)$ for all $\theta \in (0, \theta_0)$, for θ_0 found above.

On $(0, \frac{\pi}{4})$, obviously $\cos\theta > \sin\theta \Rightarrow \sqrt{\cos\theta} > \sqrt{\sin\theta}$

then $g(\theta) = \sqrt{2}\sqrt{\sin\theta}\sqrt{\cos\theta} > \sqrt{2}\sqrt{\sin\theta}\sqrt{\sin\theta} = \sqrt{2}\sin\theta = f(\theta)$

15 pts **c.** Sketch the graphs of C_1 and C_2 on the same xy–plane. *Hint: Don't hesitate to use the information obtained in part* **a** *and* **b**.

$C_1: \; r = \sqrt{2}\sin\theta = f$

$\theta = 0, \pi, 2\pi \Rightarrow r = 0$

$\theta = \frac{\pi}{2} \Rightarrow r = \sqrt{2}$

$\theta = \frac{3\pi}{2} \Rightarrow r = -\sqrt{2}$

$\pi < \theta < 2\pi \Rightarrow r < 0$

$r \nearrow$ on $(0, \pi/2) \cup (\frac{3\pi}{2}, 2\pi)$

$r \searrow$ on $(\frac{\pi}{2}, \frac{3\pi}{2})$

$C_2: \; r = \sqrt{\sin 2\theta} = g$, $\sin 2\theta \geq 0$ must hold

$\Rightarrow \; 0 \leq 2\theta \leq \pi$ and $2\pi \leq 2\theta \leq 3\pi$

$\Downarrow \qquad\qquad \Downarrow$

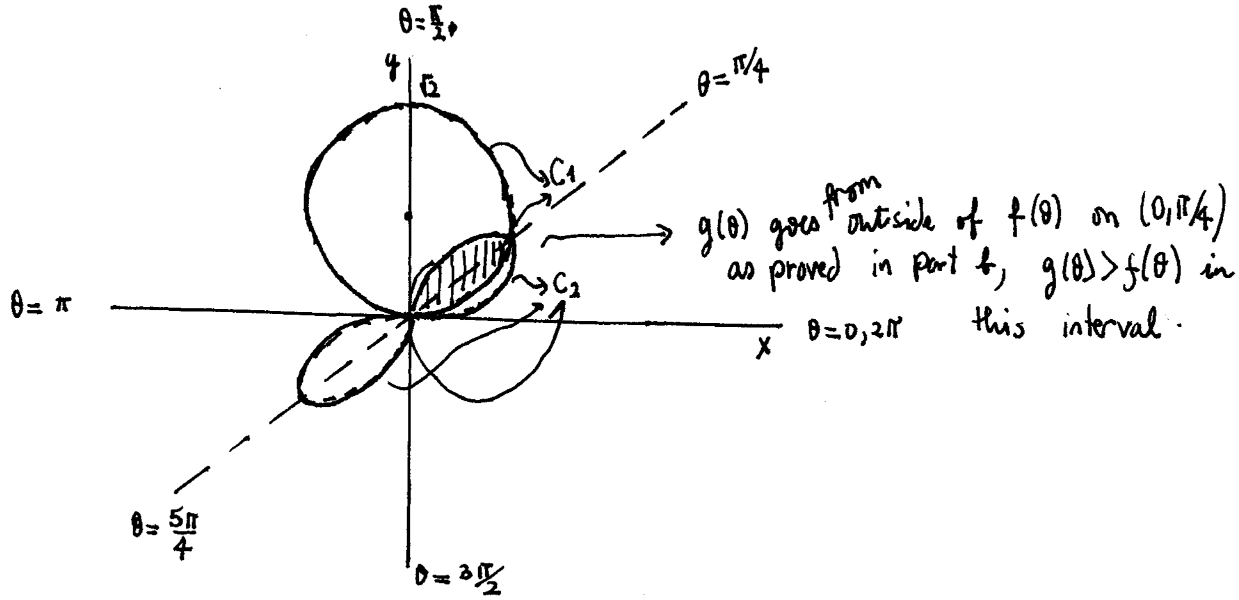
$0 \leq \theta \leq \pi/2$ and $\pi \leq \theta \leq \frac{3\pi}{2}$

no graph in 2nd and 4th quadrants.

$r = 0$ if $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ ← min

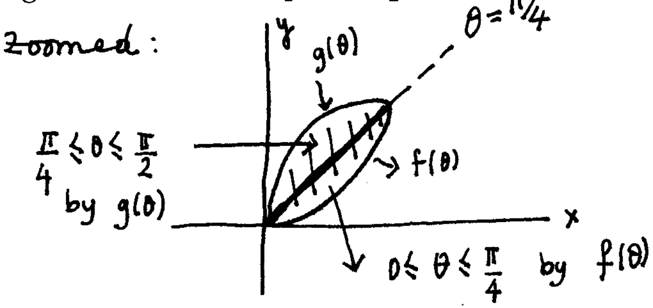
$r = 1$ if $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$, ← max

$r \nearrow$ on $(0, \frac{\pi}{4}) \cup (\pi, \frac{5\pi}{4})$, else $r \searrow$



10 pts **d.** Find the area of the region inside **both** C_1 and C_2.

Desired region zoomed:



Thus,

$$A = \frac{1}{2}\int_0^{\pi/4} 2\underbrace{\sin^2\theta}_{\frac{1-\cos 2\theta}{2}}\,d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2} \sin 2\theta\,d\theta = \frac{1}{2}\int_0^{\pi/4} 1 - \cos 2\theta\,d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2} \sin 2\theta\,d\theta$$

$$= \frac{1}{2}\left\{\theta - \frac{1}{2}\sin 2\theta\right\}\Bigg|_{\theta=0}^{\theta=\pi/4} + \frac{1}{2}\left\{-\frac{1}{2}\cos 2\theta\right\}\Bigg|_{\theta=\pi/4}^{\theta=\pi/2} = \frac{1}{2}\left\{\frac{\pi}{4} - \frac{1}{2}\right\} - \frac{1}{4}\left\{-1\right\} = \frac{\pi}{8}$$

2. Let l_1 be the line $x = 1+t,\ y = -1+t,\ z = t$, and D be the plane through the points $P(0,0,0), Q(3,2,0)$ and $R(0,1,-2)$.

10 pts **a.** Write down an equation for the plane D.

We need a normal vector $\vec{n}$ for D. Construct two vectors on D: $\vec{PQ} = \langle 3,2,0\rangle$ and $\vec{PR} = (0,1,-2)$. Then $\vec{PQ} \times \vec{PR}$ (or $\vec{PR} \times \vec{PQ}$) serves as $\vec{n}$.

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 0 \\ 0 & 1 & -2 \end{vmatrix} = \vec{i}(-4) - \vec{j}(-6) + \vec{k}(3) = -4\vec{i} + 6\vec{j} + 3\vec{k} = \langle -4,6,3\rangle. \\ = \vec{n}.$$

Any vector on D is $\langle x-0, y-0, z-0\rangle$ using P (one can use Q or R as well)

D is described by $\langle x,y,z\rangle \cdot \langle -4,6,3\rangle = 0$

$$\Rightarrow \quad -4x + 6y + 3z = 0.$$

10 pts **b.** Find the intersection of the plane D and the line l_1.

l_1 hits D at a pt if $x = 1+t,\ y = -1+t,\ z = t$ satisfy the plane eqn found above:

$$-4(1+t) + 6(-1+t) + 3t = 0$$

$$\Rightarrow \quad -10 + 5t = 0 \Rightarrow t = 2 \quad \text{yielding the pt } (1+t, -1+t, t)\Big|_{t=2} = (3,1,2).$$

15 pts **c.** Write down an equation for the line l satisfying both of the following:

(i) l lies in the plane D; (ii) l and l_1 intersect perpendicularly.

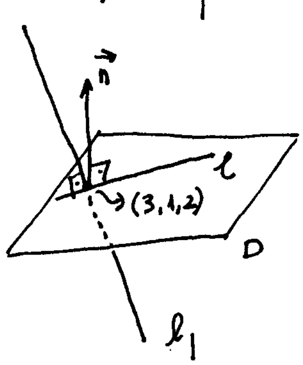
since l lies in D and $l \cap l_1 \neq \phi$ and $l \perp l_1$ we necessarily have $l \cap l_1 = \{(3,1,2)\}$, i.e. $(3,1,2) \in l$.

$l \subset D \Rightarrow \vec{n} \perp l$, $\quad l \perp l_1 \Rightarrow \underbrace{\langle 1,1,1\rangle}_{\text{direction of } l_1} \perp l$

Then $\vec{n} \times \langle 1,1,1\rangle$ can serve as the direction of l

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4 & 6 & 3 \\ 1 & 1 & 1 \end{vmatrix} = \vec{i}(6-3) - \vec{j}(-4-3) + \vec{k}(-4-6) = 3\vec{i} + 7\vec{j} - 10\vec{k} = \langle 3,7,-10\rangle$$

Therefore $l: \langle x(t), y(t), z(t)\rangle = \langle 3,1,2\rangle + t\langle 3,7,-10\rangle$



$$\Rightarrow l: \begin{array}{l} x = 3 + 3t \\ y = 1 + 7t \\ z = 2 - 10t \end{array} \quad \text{is the desired line}$$

3. Let $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$ and $\vec{\varphi}(t) = \langle \sin t, 0, \cos t\rangle$ be parametrisations of two curves in $\mathbb{R}^3$.

15 pts **a.** Find **all** common points of these curves. *Be sure that there is no more!*

$\vec{r}(t)$ and $\vec{\varphi}(t)$ intersect at pt if there exist two numbers t_1 and t_2 st $\vec{r}(t_1) = \vec{\varphi}(t_2)$. Componentwise we have:

$$\cos t_1 = \sin t_2 \qquad \Rightarrow \cos k\pi = \sin t_2$$
$$\sin t_1 = 0 \implies t_1 = k\pi,\ k\in\mathbb{Z}$$
$$t_1 = \cos t_2 \qquad \Rightarrow k\pi = \cos t_2 \Rightarrow \cos t_2 = 0 \Rightarrow t_2 = \frac{\pi}{2} + k\pi$$

but $|\cos t_2| \leq 1$ and $|k\pi| \leq 1$ only when $k=0 \Rightarrow t_1 = 0$

Now lastly we use the 1st eqn: $\cos 0 = \sin\left(\frac{\pi}{2} + k\pi\right)$, $k\in\mathbb{Z}$

$$\Rightarrow 1 = \sin\left(\frac{\pi}{2}+k\pi\right) \Rightarrow k \text{ is even.}$$

$\Rightarrow$ $\vec{r}(t)$ and $\vec{\varphi}(t)$ intersect when $t_1 = 0$, $t_2 = \frac{\pi}{2} + 2k\pi$ and the pts of intersection are: $\left.\begin{matrix}\vec{r}(t_1) = (1,0,0)\\ \vec{\varphi}(t_2) = (1,0,0)\end{matrix}\right\}$ giving only one pt since $\vec{\varphi}$ is periodic with period 2π.

Only one intersection pt: $(1,0,0)$.

15 pts **b.** At each common point, find the angle between the tangent lines to these curves.

The angle between tangent lines is the angle between directions of tangent lines, and directions are given by velocity vectors $\frac{d\vec{r}}{dt}$ and $\frac{d\vec{\varphi}}{dt}$.

$$\frac{d\vec{r}}{dt} = \langle -\sin t, \cos t, 1\rangle \Rightarrow \left.\frac{d\vec{r}}{dt}\right|_{(1,0,0) \text{ or } t=0} = \langle 0,1,1\rangle = \vec{d}_r$$

$$\frac{d\vec{\varphi}}{dt} = \langle \cos t, 0, -\sin t\rangle \Rightarrow \left.\frac{d\vec{\varphi}}{dt}\right|_{(1,0,0) \text{ or } t=\frac{\pi}{2}+2k\pi} = \langle 0,0,-1\rangle = \vec{d}_\varphi$$

Using $\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$ we get:

$$\vec{d}_r \cdot \vec{d}_\varphi = -1 = \sqrt{2}\cdot 1 \cdot \cos\theta \Rightarrow \theta = \frac{3\pi}{4}.$$

<u>Note</u>: since we can take $-\vec{d}_r$ or $-\vec{d}_\varphi$ as direction vectors, obviously $\theta = \frac{\pi}{4}$ is also a correct answer.

B U Department of Mathematics

Math 102 Calculus II

Fall 1999 First Midterm

1. Let $\vec{u} = 2\hat{\imath} - \hat{\jmath} + 2\hat{k}$ and $\vec{w} = 3\hat{\imath} + 4\hat{\jmath} - \hat{k}$. Find a vector $\vec{v}$ such that $\vec{u} \times \vec{v} = \vec{w}$ and $\vec{u} \cdot \vec{v} = 1$. Is $\vec{v}$ unique?

Solution:

Let $\vec{v} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$\vec{u} \times \vec{v} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 2 \\ x & y & z \end{bmatrix} = -(2y+z)\hat{\imath} + (2x-2z)\hat{\jmath} + (x+2y)\hat{k}$$

$$= 3\hat{\imath} + 4\hat{\jmath} - \hat{k} \;\Rightarrow\; \begin{aligned} 2y + z &= -3 \\ 2x - 2z &= 4 \\ x + 2y &= -1 \end{aligned}$$

$\vec{u} \cdot \vec{v} = 2x - y + 2z = 1$

Solving these 4 equations for x, y, z gives the unique soln.

$x = 1$, $y = -1$, $z = -1$

Thus $\vec{v} = \hat{\imath} - \hat{\jmath} - \hat{k}$ and this is the unique vector which satisfies the given two equations

2. A plane has the Cartesian equation $x + 2y - 2z + 7 = 0$. Find:

(a) a normal vector of unit length;

Solution:

$\vec{N} = \hat{\imath} + 2\hat{\jmath} - 2\hat{k}$ is normal to the plane

$|\vec{N}| = 3$. Unit Normal: $\hat{N} = \vec{N}/|\vec{N}|$ So we have $\hat{N} = \frac{1}{3}(\hat{\imath} + 2\hat{\jmath} - 2\hat{k})$

(b) the distance of the plane from the origin;

Solution:

$d = |\vec{\mathbf{OP}} \cdot \hat{N}|$ where $O = (0,0,0)$ and P is any point on the plane.

Let $P = (-7,0,0)$ P is on the plane

Then $\vec{\mathbf{OP}} = -7\hat{\imath}$ by using this vector. If we evaluate d we get $d = \left|\frac{-7}{3}\right|$, $d = \frac{7}{3}$

(c) the coordinates of the point Q on the plane nearest the origin.

Solution:

The line normal to the plane which passes from origin can be parametrized as
$\vec{r}(t) = t\hat{N} \;\Rightarrow x(t) = \frac{1}{3}t,\ y(t) = \frac{2}{3}t,\ z(t) = -\frac{2}{3}t$
This line intersects the plane at the desired point Q
So we put these values in the cartesian equation of the plane
$\frac{1}{3}t + \frac{4}{3}t + \frac{4}{3}t + 7 = 0 \;\Rightarrow\; t = -\frac{7}{3} \quad \Rightarrow x = -\frac{7}{9},\ y = -\frac{14}{9},\ z = \frac{14}{9}$

$$Q = \left(-\frac{7}{9}, -\frac{14}{9}, \frac{14}{9}\right)$$

3. **(a)** Find the parametric equation for the line tangent to the curve
$\vec{r}(t) = e^t\hat{\boldsymbol{i}} + \sin t\hat{\boldsymbol{j}} + 3\ln(1-t)\hat{\boldsymbol{k}}$ at $t = 0$

Solution:

$\vec{r}(0) = \hat{\boldsymbol{i}},\ \vec{r}\,'(t) = e^t\hat{\boldsymbol{i}} + \cos t\hat{\boldsymbol{j}} - \frac{3}{1-t}\hat{\boldsymbol{k}}$, so $\vec{r}\,'(0) = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$
Tangent Line: $\vec{R}(w) = \vec{r}(0) + w\vec{r}\,'(0)$
$\vec{R}(\vec{w}) = (1+w)\hat{\boldsymbol{i}} + w\hat{\boldsymbol{j}} - 3w\hat{\boldsymbol{k}}$, so
$x(w) = 1 + w,\ y(w) = w,\ z(w) = -3w$

(b) Find the length of the section of the curve
$\vec{r}(t) = e^t(\cos t)\hat{\boldsymbol{i}} + e^t(\sin t)\hat{\boldsymbol{j}} + e^t\hat{\boldsymbol{k}}$ from $t = -\pi$ to $t = \pi$

Solution:

$\vec{r}\,'(t) = e^t(\cos t - \sin t)\hat{\boldsymbol{i}} + e^t(\sin t + \cos t)\hat{\boldsymbol{j}} + e^t\hat{\boldsymbol{k}}$
$|\vec{r}\,'(t)|^2 = e^{2t}[(\cos t - \sin t)^2 + (\sin t + \cos t)^2 + 1] = 3e^{2t}$, so $|\vec{r}\,'(t)| = e^t\sqrt{3}$
$L = \displaystyle\int_{-\pi}^{\pi} |\vec{r}\,'(t)|dt = \int_{-\pi}^{\pi} e^t\sqrt{3}dt = \sqrt{3}(e^\pi - e^{-\pi})$

4. **(a)** The curve whose vector equation is $\vec{\mathbf{r}}(t) = 2\sqrt{t}(\cos t)\hat{\boldsymbol{i}} + 3\sqrt{t}(\sin t)\hat{\boldsymbol{j}} + \sqrt{1-t}\ \hat{\boldsymbol{k}}$, $(0 \le t \le 1)$, lies on a quadric surface. Find an equation for this surface and identify it.

Solution:

$x = 2\sqrt{t}\cos t,\ y = 3\sqrt{t}\sin t\ ,\ z = \sqrt{1-t}$
$\Rightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = t = 1 - z^2 \Rightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 + z^2 = 1$
So, the surface is an ellipsoid

(b) Find the equation for the sphere with center $(0, 0, 2)$ and radius 2 in spherical coordinates

Solution:

In Cartesian coordinates, $x^2 + y^2 + (z-2)^2 = 4$
$x^2 + y^2 + z^2 - 4z = 0$
$\rho^2 = x^2 + y^2 + z^2,\ z = \rho\cos\phi$
$\Rightarrow \rho^2 - 4\rho\cos\phi = 0 \;\Rightarrow \rho = 4\cos\phi$

B U Department of Mathematics

Math 102 Calculus II

Spring 2000 First Midterm

1. Is there a plane containing the points $A = (1, 1, -1)\,, B = (0, 1, -1/2)\,, C = (-1, 1, 0)\,,$ $D = (0, 0, 1/4)$. If yes, find this plane.

Solution:

Construct a plane with three points and check whether the fourth is on that plane.
$\overrightarrow{AB} =< -1, 0, 1/2 >, \overrightarrow{AD} =< -1, -1, 5/4 >$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 1/2 \\ -1 & -1 & 5/4 \end{vmatrix} = (1/2)\vec{i} + (3/4)\vec{j} + \vec{k} \text{ which is the normal to the plane}$$

containing the points A, B and D.
So the plane has equation $x/2 + 3y/4 + z = d$. We know D is on the plane, so $d = 1/4$.
$\Rightarrow x/2 + 3y/4 + z = 1/4$
$1/2\,(-1) + 3/4\,(1) + 0 = 1/4$ so C is also in the plane.

OR You can find a vector perpendicular to any of the vectors (by using cross product) and check that the third vector is also perpenducilar to that vector.(by using dot product) then the vector you have found by cross product is the normal vector of the plane.

2. Find the point of intersection of the xz-plane and the line which is tangent to the curve $\vec{r}\,(t) = (-2t + 1)\,\vec{i} + 3\,(t - 1)^2\,\vec{j} + e^t\vec{k}$ at the point$(1, 3, 1)$.

Solution:

xz-plane : $y = 0$.
$\dot{\vec{r}}\,(t) = -2\vec{i} + (6t - 6)\,\vec{j} + e^t\vec{k};\ at\,(1, 3, 1)\ \ t = 0$
$\dot{\vec{r}}\,(0) = -2\vec{i} - 6\vec{j} + \vec{k}$ is the direction vector for the tangent line.
tangent line: $\vec{R}\,(w) =< 1, 3, 1 > +w < -2, -6, 1 >$
$= (1 - 2w)\,\vec{i} + (3 - 6w)\,\vec{j} + (1 + w)\,\vec{k}$.
intersection with $y = 0\ : -6w + 3 = 0 \Rightarrow w = 1/2$
$\vec{R}\,(1/2) =< 0, 0, 3/2 >$ so the point of intersection is $(0, 0, 3/2)$

3. The acceleration of a particle in space is given by $\vec{a}\,(t) = (3t)\,\vec{i} + 4\vec{j} + \vec{k}$ .When $t = 0$, the particle is at the point $(0, 5, 0)$ and its velocity is $4\vec{i}$ .Where is the particle when $t = 1$?

Solution:

$$\vec{v}\,(t) = \frac{3t^2}{2}\vec{i} + 4t\vec{j} + t\vec{k} + \vec{C_1}$$
$$\vec{v}\,(0) = 4\vec{i} = \vec{C_1}$$

$$\Rightarrow \vec{v}(t) = \left(\frac{3t^2}{2}+4\right)\vec{\imath} + 4t\vec{\jmath} + t\vec{k}$$
$$\vec{r}(t) = \left(\frac{t^3}{2}+4t\right)\vec{\imath} + 2t^2\vec{\jmath} + \frac{t^2}{2}\vec{k} + \vec{C_2}$$
$$\vec{r}(0) = 5\vec{\jmath} = \vec{C_2}$$
$$\Rightarrow \vec{r}(t) = \left(\frac{t^3}{2}+4t\right)\vec{\imath} + \left(2t^2+5\right)\vec{\jmath} + \frac{t^2}{2}\vec{k}$$

So, $\vec{r}(1) =< 9/2, 7, 1/2 >$ and it is at the point $(9/2, 7, 1/2)$.

4. Let $\vec{r}(t) = 2\cos(3t)\,\vec{\imath} + 2\sin(3t)\,\vec{\jmath} + 4t\vec{k}$, $0 \le t \le \dfrac{2\pi}{3}$

a) Find $\dfrac{ds}{dt}$ where s is arclength.

Solution:

$$\dot{\vec{r}}(t) = -6\sin(3t)\,\vec{\imath} + 6\cos(3t)\,\vec{\jmath} + 4\vec{k}, \quad \left|\dot{\vec{r}}(t)\right| = \sqrt{52}$$
$$\Rightarrow s = \int_0^t \sqrt{52}\,dw = \sqrt{52}\,w\,\big|_0^t = \sqrt{52}\,t$$
$$\Rightarrow \frac{ds}{dt} = \sqrt{52}$$

b) Is t arclength in $\vec{r}(t)$ above ? If no, find $\vec{r}(s)$.

Solution:

No, since $\left|\dot{\vec{r}}(t)\right| \neq 1$.

Now let us denote $s(0)$ by s_0. Then $s = \sqrt{52}\,t + s_0 \Rightarrow t = \dfrac{s-s_0}{\sqrt{52}}$

$$\Rightarrow \vec{r}(s) = 2\cos\left(\frac{3(s-s_0)}{\sqrt{52}}\right)\vec{\imath} + 2\sin\left(\frac{3(s-s_0)}{\sqrt{52}}\right)\vec{\jmath} + \frac{4(s-s_0)}{\sqrt{52}}\vec{k}, \quad 0 \le s-s_0 \le \frac{2\sqrt{52}\pi}{3}.$$

B U Department of Mathematics

Math 102 Calculus II

Spring 2001 First Midterm

1. Let $\vec{r}(t) = a\cos t\vec{\imath} + a\sin t\vec{\jmath} + bt\vec{k}$, where $a,\, b \in \mathbb{R}$. Let $\theta(t)$ be the angle which the tangent line at a given point of the curve makes with the z-axis.
   (a) Evaluate the velocity and acceleration vector.
   (b) Show that they have constant lengths.
   (c) Show that $\cos\theta(t)$ is constant.

Solution:

(a) We are given

$$\vec{r}(t) = a\cos t\vec{\imath} + a\sin t\vec{\jmath} + bt\vec{k}$$

and differentiating with respect to t, we find the velocity and acceleration vectors:

$$\vec{v}(t) = \vec{r}\,'(t) = -a\sin t\vec{\imath} + a\cos t\vec{\jmath} + b\vec{k},$$
$$\vec{a}(t) = \vec{r}\,''(t) = -a\cos t\vec{\imath} - a\sin t\vec{\jmath} + 0\vec{k}.$$

(b) The squares of the lengths of these vectors are

$$\begin{aligned}\|\vec{v}(t)\|^2 &= (-a\sin t)^2 + (a\cos t)^2 + b^2\\ &= a^2 + b^2,\\ \|\vec{a}(t)\|^2 &= (-a\cos t)^2 + (-a\sin t)^2 + 0^2\\ &= a^2\end{aligned}$$

so $\|\vec{v}(t)\|$ is equal to the constant $\sqrt{a^2+b^2}$, and $\|\vec{a}(t)\|$ is equal to the constant $|a|$.
(c) We have

$$\cos\theta(t) = \frac{\vec{v}(t)\cdot\vec{k}}{\|\vec{v}(t)\|\ \|k\|} = \frac{(-a\sin t\vec{\imath} + a\cos t\vec{\jmath} + b\vec{k})\cdot\vec{k}}{\sqrt{a^2+b^2}\ 1} = \frac{b}{\sqrt{a^2+b^2}},$$

so $\cos\theta(t)$ is a constant.

2. There are three points on axes which are A, B, C. On the x-axis, $A(1,0,0)$, y-axis $B(0,1,0)$ and the z-axis $C(0,0,z_0)$, where $z_0 > 0$. The angle between CA and CB is $\pi/3$. Find the equation of the plane.

Solution:

CAB is an isosceles triangle, its sides CB and CA are equal. Since the angle at C is given to be $\pi/3$, we infer that CAB is an equilateral triangle, so $z_0 = \pm 1$. We are also given that $z_0 > 0$, so $z_0 = 1$.

Thus the problem is to find the equation of the plane that passes through the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. A normal to this plane is

$$CA \times CB = (\vec{i} - \vec{k}) \times (\vec{j} - \vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \vec{k}$$

$$= \vec{i} + \vec{j} + \vec{k}$$

and a point on the plane is, for example, $(1, 0, 0)$. Thus the equation of the plane is

$$[(x\vec{i} + y\vec{i} + z\vec{i}) - (\vec{i} + 0\vec{j} + 0\vec{k})] \cdot (\vec{i} + \vec{j} + \vec{k}) = 0,$$
$$(x - 1) + (y - 0) + (z - 0) = 0,$$

or, better:

$$x + y + z = 1.$$

3. Two particles move from the $x\,z$-plane to the $y\,z$-plane with the trajectories given by $\boldsymbol{r_1}(t) = (t-4)\vec{i} + (2t-4)\vec{j} + (60 - t^3)\vec{k}$ and $\boldsymbol{r_2}(w) = (w-2)\vec{i} + (2w-2)\vec{j} + (8 - w^3)\vec{k}$.

(a) Do their trajectories ever cross each other?

(b) Which path is shorter? (You don't need to evaluate any integral).

Solution:

(a) If there existed t_0, w_0 such that $r_1(t_0) = r_2(w_0)$, then

$$t_0 - 4 = w_0 - 2,$$
$$2t_0 - 4 = 2w_0 - 2,$$
$$60 - t_0^3 = 8 - w_0^3$$

would hold, but the first and second of these equalities are incompatible. So there is no t_0, w_0 with $r_1(t_0) = r_2(w_0)$ and the trajectories do not cross each other.

(b)The $x\,z$-plane is given by the equation $y = 0$, and the $y\,z$-plane by $x = 0$. The trajectories being from the plane $y = 0$ to the plane $x = 0$, the parameter t changes from that value of t for which $2t - 4 = 0$ to that value of t for which $t - 4 = 0$; similarly w changes from the value of w for which $2w - 2 = 0$ to that value of w for which $w - 2 = 0$. In other words, t changes from 2 to 4, and w changes from 1 to 2. The arclength of

$$r_1(t) = t\vec{i} + 2t\vec{j} - t^3\vec{k} + (-4\vec{i} - 4\vec{j} + 60\vec{k}), \quad 2 \le t \le 4$$

is equal to

$$\int_2^4 \|1\vec{i} + 2\vec{j} - 3t^2\vec{k}\|\, dt = \int_2^4 \sqrt{1 + 4 + 9t^4}\, dt = \int_2^4 \sqrt{5 + 9t^4}\, dt$$

and the arclength of

$$\boldsymbol{r_2}(w) = w\vec{i} + 2w\vec{j} - w^3\vec{k} + (-2\vec{i} - 2\vec{j} + 8\vec{k}),; \quad 1 \le t \le 2$$

is equal to

$$\int_1^2 \|1\vec{\imath} + 2\vec{\jmath} - 3w^2\vec{k}\|\, dw = \int_1^2 \sqrt{1+4+9w^4}\, dw = \int_1^2 \sqrt{5+9t^4}\, dt.$$

The inequality

$$\begin{aligned}\int_1^2 \sqrt{5+9t^4}\, dt \le \int_1^2 \sqrt{5+9\cdot 2^4}\, dt &= (2-1)\sqrt{5+9\cdot 2^4} \\ &< (4-2)\sqrt{5+9\cdot 2^4} \le \int_2^4 \sqrt{5+9t^4}\, dt\end{aligned}$$

shows that the path $\boldsymbol{r_2}$ is shorter than $\boldsymbol{r_1}$.

4. A curve is parametrized by $\boldsymbol{r}(t) = \vec{\imath} + 2\vec{\jmath} + 4\vec{k} + \frac{e^{-t^2}}{\sqrt{2}}(-\vec{\imath} + \vec{k})$, where $0 \le t \le 1$.
   (a) Reparametrize the curve by its arclength, starting at $s = 0$ when $t = 0$.
   (b) Evaluate its curvature at any point.

Solution:

(a) We have

$$\begin{aligned}\vec{r}\,(t) &= \vec{\imath} + 2\vec{\jmath} + 4\vec{k} + \frac{e^{-t^2}}{\sqrt{2}}(-\vec{\imath} + \vec{k}), \\ \vec{r}\,'(t) &= \frac{1}{\sqrt{2}}e^{-t^2}(-2t)(-\vec{\imath} + \vec{k}) \\ &= \sqrt{2}te^{-t^2}\vec{\imath} - \sqrt{2}te^{-t^2}\vec{k}, \\ \|\vec{r}\,'(t)\| &= \sqrt{2t^2e^{-2t^2} + 2t^2e^{-2t^2}} \\ &= \sqrt{4t^2e^{-2t^2}} \\ &= 2|t|e^{-t^2} = 2te^{-t^2}\end{aligned}$$

and the arclength from $t = 0$ to $t = t_1$ is equal to

$$s(t_1) = \int_0^{t_1} 2te^{-t^2}\, dt = \left[-e^{-t^2}\right]_0^{t_1} = 1 - e^{-t_1^2}.$$

Thus $e^{-t^2} = 1 - s(t)$ and

$$\vec{r}\,(t(s)) = \vec{\imath} - 2\vec{\jmath} + 4\vec{k} + \frac{1-s}{\sqrt{2}}(-\vec{\imath} + \vec{k}).$$

(b) The curvature is the length of $\frac{d^2}{ds^2}\vec{r}\,(s)$. We have

$$\begin{aligned}\frac{d}{ds}\vec{r}\,(s) &= -\frac{1}{\sqrt{2}}(-\vec{\imath} + \vec{k}) \\ \frac{d^2}{ds^2}\vec{r}\,(s) &= \vec{0}.\end{aligned}$$

Thus the curvature at any point is equal to 0.

The curvature may be computed by using the formula $\kappa = \frac{||\vec{v}\times\vec{a}||}{||\vec{v}||^3}$. We compute

$$\vec{v} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \sqrt{2}te^{-t^2} & 0 & -\sqrt{2}te^{-t^2} \\ \sqrt{2}e^{-t^2}(2t+1) & 0 & -\sqrt{2}e^{-t^2}(2t+1) \end{vmatrix} = \vec{0}$$

and conclude that the curvature is 0 at every point.

B U Department of Mathematics

Math 102 Calculus II

Spring 2002 First Midterm

1. Sketch the region R bounded by the curves $y = x$, $y = \dfrac{1}{\sqrt{3}}x$ and $r = 1 + \cos\theta$.
   Find the area of the region R.

Solution:

$1 + \cos(-\theta) = 1 + \cos\theta = r$.

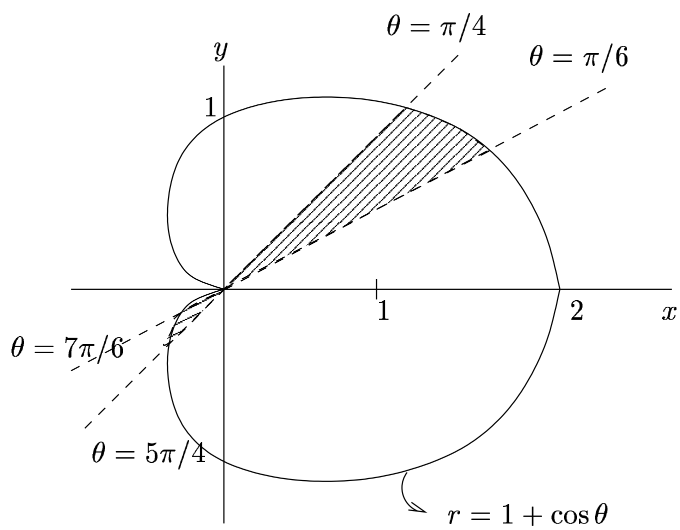
$\therefore$ The graph of $r = 1 + \cos\theta$ is symmetric with respect to x-axis.

$0 \le \theta \le \dfrac{\pi}{2} \Rightarrow r : 2 \searrow 1$

$\dfrac{\pi}{2} \le \theta \le \pi \Rightarrow r : 1 \searrow 0$ and draw the symmetric picture for $\pi \le \theta \le 2\pi$.

When $y = x$, $\tan\theta = \frac{y}{x} = 1 \Rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4}$.

When $y = \dfrac{1}{\sqrt{3}}x$, $\tan\theta = \frac{y}{x} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}, \frac{7\pi}{6}$



$$
\begin{aligned}
A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}(1+\cos\theta)^2 d\theta + \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{5\pi}{4}}(1+\cos\theta)^2 d\theta \\
&= \frac{1}{2}\left[\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}(1+\cos^2\theta+2\cos\theta)d\theta + \int_{\frac{7\pi}{6}}^{\frac{5\pi}{4}}(1+\cos^2\theta+2\cos\theta)d\theta\right] \\
&= \frac{1}{2}\left[\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\left(1+\frac{1+\cos 2\theta}{2}+2\cos\theta\right)d\theta + \int_{\frac{7\pi}{6}}^{\frac{5\pi}{4}}\left(1+\frac{1+\cos 2\theta}{2}+2\cos\theta\right)d\theta\right] \\
&= \frac{1}{2}\left[\left(\frac{3\theta}{2}+\frac{\sin 2\theta}{4}+2\sin\theta\right)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} + \left(\frac{3\theta}{2}+\frac{\sin 2\theta}{4}+2\sin\theta\right)\Big|_{\frac{7\pi}{6}}^{\frac{5\pi}{4}}\right] \\
&= \frac{1}{2}\left[\frac{3\pi}{8}+\frac{1}{4}+\sqrt{2}-\frac{\pi}{4}-\frac{\sqrt{3}}{8}-1+\frac{15\pi}{8}+\frac{1}{4}-\sqrt{2}-\frac{7\pi}{4}-\frac{\sqrt{3}}{8}+1\right] \\
&= \frac{\pi+2-\sqrt{3}}{8}.
\end{aligned}
$$

2. (a) Show that the lines:

$$
\begin{aligned}
L_1 &: \ x = 4t-1, y = t+3, z = 1 \\
L_2 &: \ x = 12t-13, y = 6t+1, z = 3t+2
\end{aligned}
$$

intersect and find the coordinates of point of intersection.

(b) Find an equation of the plane determined by the lines $\mathbf{L_1}$ and $\mathbf{L_2}$.

Solution:

(a) $\left.\begin{aligned} 4t_1-1 &= 12t_2-13 \\ t_1+3 &= 6t_2+1 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} 4t_1-12t_2 &= -12 \\ t_1-6t_2 &= -2 \end{aligned}\right\} \Rightarrow \begin{aligned} t_1 &= -4 \\ t_2 &= -\tfrac{1}{3} \end{aligned}$

Also, $3t_2+2 = 3\left(-\dfrac{1}{3}\right)+2 = 1$.

So, the lines intersect.

$x = 4t_1-1 = 4(-4)-1 = -17$

$y = t_1+3 = -4+3 = -1$

$z = 1$

Hence, $(-17,-1,1)$ is the point of intersection.

(b) Let $\mathcal{P}$ be the plane.

$\vec{u_1} = 4\vec{i}+\vec{j}$ is a direction vector for L_1.

$\vec{u_2} = 12\vec{i}+6\vec{j}+3\vec{k}$ is a direction vector for L_2.

So, $\vec{N} = \vec{u_1}\times\vec{u_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 1 & 0 \\ 12 & 6 & 3 \end{vmatrix} = 3\vec{i}-12\vec{j}+12\vec{k}$ is a normal vector to $\mathcal{P}$.

$(-17,-1,1)$ is on $\mathcal{P}$.

So, $\mathcal{P}:\ 3(x+17)-12(y+1)+12(z-1) = 0 \Leftrightarrow \boxed{x-4y+4z+9=0}$.

3. Find the **distance** between the planes $\mathbf{3x+2y-6z=6}$ and $\mathbf{3x+2y-6z=2}$.

Solution:

$(0,0,-1)$ is on the plane $3x+2y-6z=6$.
So, it is enough to find the distance from $(0,0,-1)$ to the plane $3x+2y-6z=2$.
Hence, by the formula

$$\begin{aligned} d &= \frac{|3\cdot 0+2\cdot 0-6\cdot(-1)-2|}{\sqrt{3^2+2^2+(-6)^2}} \\ &= \frac{4}{\sqrt{9+4+36}} = \frac{4}{\sqrt{49}} = \frac{4}{7}. \end{aligned}$$

4. Consider two particles whose paths are parametrized by
$\vec{r_1}(t) = \dfrac{t}{\sqrt{t^2+1}}\vec{\imath} + \dfrac{1}{\sqrt{t^2+1}}\vec{\jmath},\ -\pi \le t \le \pi$ and $\vec{r_2}(t) = \cos t\,\vec{\imath} + \sin t\,\vec{\jmath},\ -\pi \le t \le \pi$.

(a) Are their motions on the same path? Verify.

(b) Which one is faster at $(0,1)$?

Solution:

(a) No. For $t=0$, $\vec{r_2}(t)$ passes through $(1,0)$.
However, $\vec{r_1}(t)$ does not pass through $(1,0)$ because if $-\pi \le t \le \pi$ then $\frac{1}{\sqrt{t^2+1}} \neq 0$.

(b) $\vec{r_1}'(t) = \dfrac{1}{(t^2+1)^{3/2}}\vec{\imath} + \dfrac{-t}{(t^2+1)^{3/2}}\vec{\jmath}$ and $\vec{r_2}'(t) = -\sin t\,\vec{\imath} + \cos t\,\vec{\jmath}$.

$\vec{r_1}$ passes through $(0,1)$ at $t=0$. $\vec{r_2}$ passes through $(0,1)$ at $t=\dfrac{\pi}{2}$.
$\vec{r_1}'(0) = \vec{\imath} \ \Rightarrow\ \|\vec{r_1}'(0)\| = 1$
$\vec{r_2}'(\dfrac{\pi}{2}) = -\vec{\imath} \ \Rightarrow\ \|\vec{r_2}'(\dfrac{\pi}{2})\| = 1.$
They have the same speed at $(0,1)$.

c) Which path is "curved more" at $(0,1)$?

Solution:

$\vec{r_1}''(t) = \dfrac{-3t}{(t^2+1)^{5/2}}\vec{\imath} + \dfrac{2t^2-1}{(t^2+1)^{5/2}}\vec{\jmath} \Rightarrow \vec{r_1}''(0) = -\vec{\jmath}$
and from part (b), $\vec{r_1}'(0) = \vec{\imath}$.
So, $\kappa_1 = \dfrac{\|\vec{r_1}'(0)\times\vec{r_1}''(0)\|}{\|\vec{r_1}'(0)\|^3} = \dfrac{\|-\vec{k}\|}{\|\vec{\imath}\|^3} = 1.$
$\vec{r_2}''(t) = -\cos t\,\vec{\imath} - \sin t\,\vec{\jmath} \Rightarrow \vec{r_2}''(\dfrac{\pi}{2}) = -\vec{\jmath}$
and from part (b), $\vec{r_2}'(\dfrac{\pi}{2}) = -\vec{\imath}$.
So, $\kappa_2 = \dfrac{\|\vec{r_2}'(\frac{\pi}{2})\times\vec{r_2}''(\frac{\pi}{2})\|}{\|\vec{r_2}'(\frac{\pi}{2})\|^3} = \dfrac{\|\vec{k}\|}{\|-\vec{\imath}\|^3} = 1.$
Thus, they have the same curvature.

d) Calculate the lengths of the paths taken by each of the particles from $t = 0$ to $t = \pi$.

Solution:

$$\|\vec{r_1}'(t)\| = \frac{1}{t^2+1} \;\Rightarrow\; L_1 = \int_0^{\pi} \frac{1}{t^2+1}\,dt = \arctan \pi.$$

$$\|\vec{r_2}'(t)\| = 1 \;\Rightarrow\; L_2 = \int_0^{\pi} dt = \pi.$$

B U Department of Mathematics

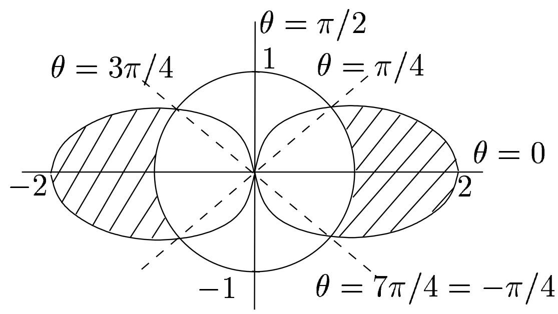
Math 102 Calculus II

Spring 2003 First Midterm

1. Sketch the graphs of the polar curves $r = 1 + \cos 2\theta$ and $r = 1$, and find the area of the region inside $r = 1 + \cos 2\theta$ and outside $r = 1$.

Solution:

Note that $r = 1 + \cos 2\theta$ is symmetric with respect to the map $\theta \to -\theta$ and $r = 1$ is the unit circle with center at the origin.



Let A denote the area of the described region. Then:

$$\begin{aligned} A &= 2\frac{1}{2}\int_{-\pi/4}^{\pi/4} \left[(1+\cos 2\theta)^2 - 1^2\right] d\theta = \int_{-\pi/4}^{\pi/4} (1 + \cos^2 2\theta + 2\cos 2\theta - 1) d\theta \\ &= \int_{-\pi/4}^{\pi/4} \left[\frac{1+\cos 4\theta}{2} + 2\cos 2\theta\right] d\theta = \left[\frac{\theta}{2} + \frac{\sin 4\theta}{8} + \sin 2\theta\right]_{-\pi/4}^{\pi/4} \\ &= \left(\frac{\pi}{8} + 1\right) - \left(-\frac{\pi}{8} - 1\right) = \frac{\pi}{4} + 2. \end{aligned}$$

2. Find the equation of the plane that passes through the points $P(1, 2, 3)$ and $Q(3, 2, 1)$ and is perpendicular to the plane $4x - y + 2z = 7$.

Solution:

Let $\mathcal{P}$ be the plane we are seeking with a normal vector $\boldsymbol{n}$. Since $P, Q \in \mathcal{P}$ the vector connecting P to Q is on our plane, i.e. $\boldsymbol{PQ} = \langle 2, 0, -2\rangle \in \mathcal{P}$. We have also $\boldsymbol{n} \perp \boldsymbol{PQ}$. On the other hand $\mathcal{P}$ is perpendicular to $4x - y + 2z = 7$, this is their normal vectors are perpendicular which is $\boldsymbol{n} \perp \langle 4, -1, 2\rangle$.
We have found two non-parallel vectors both of which are perpendicular to $\boldsymbol{n}$. This implies $\boldsymbol{n}$ may be taken as their cross product, namely $\boldsymbol{n} = \langle 4, -1, 2\rangle \times \boldsymbol{PQ}$. Note that $\boldsymbol{n}$ is not unique.

$$\boldsymbol{n} = \langle 4, -1, 2\rangle \times \langle 2, 0, -2\rangle = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 4 & -1 & 2 \\ 2 & 0 & -2 \end{vmatrix} = 2\boldsymbol{i} + 12\boldsymbol{j} + 2\boldsymbol{k}$$

is a normal vector for $\mathcal{P}$.

Now using a point lying on the plane $\mathcal{P}$, say $P(1,2,3)$, we get:

$$\mathcal{P}: 2(x-1)+12(y-2)+2(z-3)=0 \Leftrightarrow x+6y+z=16$$

as the plane sought.

3. Let $\boldsymbol{r}(t)=\sin t\,\boldsymbol{i}+(t^2+t+2)\boldsymbol{j}+\dfrac{1}{t+1}\boldsymbol{k}$ be a space curve.

(a) Find the unit tangent vector $\boldsymbol{T}$ to $\boldsymbol{r}(t)$ at the point $(0,2,1)$.

Solution:

First observe that $(0,2,1)$ corresponds to the choice $t=0$. Computing the tangent vector of $\boldsymbol{r}(t)$ we get:

$$\boldsymbol{r}'(t)=\left\langle \cos t, 2t+1, -\frac{1}{(t+1)^2}\right\rangle.$$

Evaluating at $t=0$ we find $\boldsymbol{r}'(0)=\langle 1,1,-1\rangle$ and the norm $||\boldsymbol{r}'(0)||=\sqrt{3}$. Then:

$$\boldsymbol{T}(0)=\frac{\boldsymbol{r}'(0)}{||\boldsymbol{r}'(0)||}=\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right\rangle.$$

(b) Find parametric equations of the tangent line to $\boldsymbol{r}(t)$ at $t=0$.

Solution:

$\boldsymbol{r}'(0)=\langle 1,1,-1\rangle$ is parallel to the tangent line and passes through the point $(0,2,1)$. Consequently:

$$\mathcal{L}:\begin{cases} x &= t\\ y &= 2+t\\ z &= 1-t\end{cases}$$

is the tangent line at $(0,2,1)$.

Note that we could use any vector having the same direction with $\langle 1,1,-1\rangle$.

(c) Find the intersection point(s), if any, of the tangent line found in part **(b)** with the plane $3x-2y+z+3=0$.

Solution:

Substituting $x=t$, $y=2+t$ and $z=1-t$ into the equation of the plane $3t-2(2+t)+1-t+3=0$ we see that every term on the left cancels out and we are left with a tautology $0=0$. This means for all t the equality holds, i.e.

$$\mathcal{L}\in\{3x-2y+z+3=0\}.$$

There are infinitely many intersection points formed by $\mathcal{L}$ itself.

4. Let $\boldsymbol{r}(s)=\langle x(s),y(s),z(s)\rangle$, $s\geq 0$ be a space curve parametrized by the arc length measured starting from $s=0$.

(a) Find the value(s) of $z'(1)$ if $x'(1)=y'(1)=1/2$.

Solution:

Since $\boldsymbol{r}(s)$ is parametrized by the arc length s we have $||\boldsymbol{r}'(s)|| = 1$ for every s. In particular $\boldsymbol{r}'(1) = \langle x'(1), y'(1), z'(1)\rangle$ and:

$$||\boldsymbol{r}'(1)|| = \sqrt{x'(1)^2 + y'(1)^2 + z'(1)^2} = \sqrt{\frac{1}{2} + z'(1)^2} = 1.$$

But then easily $z'(1) = \pm 1/\sqrt{2}$.

(b) Find the arc length of the path between $s = 2$ and $s = 5$.

Solution:

$$L = \int_2^5 ||\boldsymbol{r}'(s)|| ds = \int_2^5 ds = 5 - 2 = 3.$$

(c) Given that $x''(s) = 2$, $y''(s) = -1$ and $z''(s) = 3$ at $s = 1/2$, find the curvature $\kappa(1/2)$.

Solution:

$\kappa(s) = ||\boldsymbol{r}''(s)||$ if s is the arc length parameter. Then:

$$\kappa(1/2) = ||\boldsymbol{r}''(1/2)|| = \sqrt{4+1+9} = \sqrt{14}.$$

(d) Find the unit normal vector $\boldsymbol{N}$ at $s = 1/2$.

Solution:

$$\boldsymbol{N}(1/2) = \frac{\boldsymbol{r}''(1/2)}{||\boldsymbol{r}''(1/2)||} = \frac{\langle 2, -1, 3\rangle}{\sqrt{14}} = \left\langle \frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle.$$

B U Department of Mathematics

Math 102 Calculus II

Spring 2004 First Midterm

1. Describe or sketch the surface in $\mathbb{R}^3$ whose equation in spherical coordinates is $\rho = 4\cos\varphi$.

Solution:

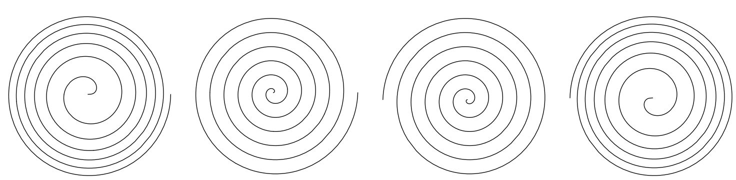
Multiply each side by ρ; in rectangular coordinates, we have:

$$\begin{aligned} & x^2+y^2+z^2 = 4z \\ \Rightarrow\ & x^2+y^2+(z-2)^2 = 4 \end{aligned}$$

which describes a sphere in $\mathbb{R}^3$ with center at (0,0,2) and radius 2.

2. Answer the questions below. In each case, explain your reasoning.

(a) The curves $r=\theta$, $r=-\theta$, $r=\sqrt{\theta}$, $r=-\sqrt{\theta}$ (in a polar coordinate system whose polar axis coincides with the positive x-axis in the usual way) are depicted below in some order. Find out which curve has which equation.



Solution:

Set $f(\theta)=\theta$. Then $f(\theta+2\pi)=f(\theta)+2\pi$, i.e. after each 2π rotation around the origin, $f(\theta)$ increases by 2π. Since in the second figure, for θ close to 0, r is positive, the second figure is the graph of $r=f(\theta)=\theta$ while the third figure is that of $r=-\theta$. First and last figures correspond to $r=\sqrt{\theta}$ and $r=-\sqrt{\theta}$ respectively because 2π rotations around the origin yield non-uniform increase in r.

(b) Are $\vec{u}\cdot\vec{v}$ and $\vec{u}\times\vec{v}$ orthogonal to each other? Why?

Solution:

Orthogonality of a real number and a vector is not defined. This question does not make sense.

(c) Find the vector component of $\vec{u}=\langle 3,-4,4\rangle$ orthogonal to $\vec{a}=\langle 2,2,1\rangle$.

Solution:

Express $\vec{u}$ as a sum of two vectors: $\vec{u} = \vec{u}_a + \vec{u}_\perp$ where $\vec{u}_a$ is the component of $\vec{u}$ along $\vec{a}$ and $\vec{u}_\perp$ the component perpendicular to $\vec{a}$. Then,

$$\begin{aligned}
\vec{u}_\perp &= \vec{u} - \vec{u}_a \\
&= \vec{u} - \operatorname{proj}_{\vec{a}} \vec{u} \\
&= \langle 3, -4, 4 \rangle - \frac{\vec{u} \cdot \vec{a}}{||\vec{a}||^2}\, \vec{a} \\
&= \langle 3, -4, 4 \rangle - \frac{2}{9}\, \langle 2, 2, 1 \rangle \\
&= \langle \frac{23}{9}, -\frac{40}{9}, \frac{34}{9} \rangle
\end{aligned}$$

Observe that $\vec{u}_\perp \cdot \vec{a} = 0$.

(d) Let l be the line with equation $\vec{r} = \langle -1, 1, 0 \rangle + \langle 0, 1, -1 \rangle t$ and P be the plane with equation $x - y - 2z = 3$. Consider the vectors parallel to P. One of these vectors makes the smallest positive angle with l. Find this smallest positive angle.

Solution:

Let $\vec{w}$ be the vector parallel to P that gives the smallest angle with the direction vector $\vec{v}$ of l. Then the three vectors $\vec{w}$, $\vec{v}$ and the normal $\vec{n}$ to P are coplanar. Since

$$\cos(\theta_{\vec{v},\vec{w}}) = \frac{\vec{v} \cdot \vec{n}}{||\vec{v}|| \cdot ||\vec{n}||} = \frac{1}{\sqrt{12}},$$

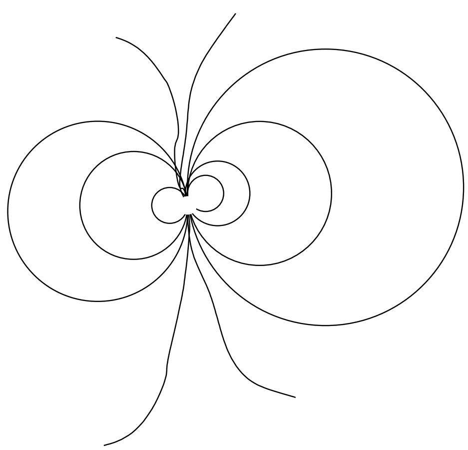
we have:

$$\theta_{\vec{v},\vec{n}} = \sin^{-1} \frac{1}{\sqrt{12}}.$$

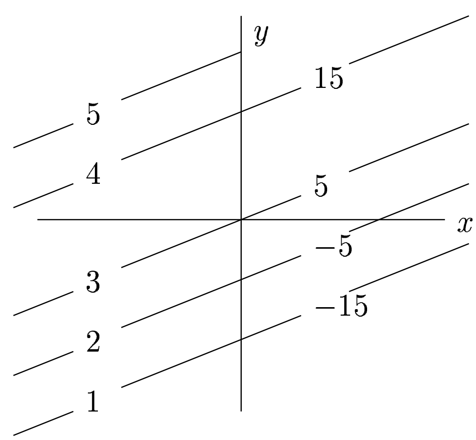
Bonus question. Draw some curves around $(0, 0)$ which can be level curves of a function of two variables that is not continuous at $(0, 0)$.

Solution:

Examples:



all level curves pass through the origin
where function is not defined



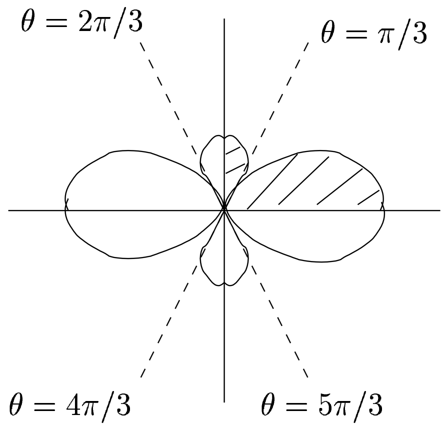
function has a jump along y-axis;
the level curves break

3. A curve C is given in polar coordinates by the equation $r = \frac{1}{2} + \cos 2\theta$. Sketch the curve.

Find the area of the region in the xy-plane in the first quadrant bounded by C.

Find a surface in $\mathbb{R}^3$ that cuts the xy-plane in the curve C.

Solution:



A surface in $\mathbb{R}^3$ that cuts the xy-plane in the above curve C may be a cylinder over C, given in cylindrical coordinates by $r = \frac{1}{2} + \cos 2\theta$.

The shaded region has area A equal to:

$$\begin{aligned}
A &= \frac{1}{2}\int_0^{\pi/3} (\frac{1}{2} + \cos 2\theta)^2 d\theta + \frac{1}{2}\int_{4\pi/3}^{3\pi/2} (\frac{1}{2} + \cos 2\theta)^2 d\theta \\
&= \frac{1}{2}\int_0^{\pi/3} (\frac{1}{2} + \cos 2\theta)^2 d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2} (\frac{1}{2} + \cos 2\theta)^2 d\theta, \text{ (using symmetry with respect to origin)} \\
&= \frac{1}{2}\left\{\left(\frac{\theta}{4} + \frac{\sin 2\theta}{2} + \frac{\sin 4\theta}{8} + \frac{\theta}{2}\right)_0^{\pi/3} + \left(\frac{\theta}{4} + \frac{\sin 2\theta}{2} + \frac{\sin 4\theta}{8} + \frac{\theta}{2}\right)_{\pi/3}^{\pi/2}\right\} \\
&= \frac{1}{2}\left(\frac{\pi}{8} + \frac{\pi}{4}\right) = \frac{3\pi}{16}.
\end{aligned}$$

4. A point moves on a curve according to the vector equation

$$\vec{r}(t) = 4\cos t\vec{i} + 4\sin t\vec{j} + 4\cos t\vec{k} \qquad (0 \le t \le 2\pi)$$

(a) This curve lies on a plane. Find an equation for that plane.

Solution:

Since $x(t) = z(t)$ for all t, the curve lies on the plane $x = z$.

(b) Find the point(s) at which the curvature of this curve is maximum.

Solution:

$$\begin{aligned}
\vec{r}'(t) &= \langle -4\sin t, 4\cos t, -4\sin t\rangle; \\
\vec{r}''(t) &= \langle -4\cos t, -4\sin t, -4\cos t\rangle; \\
\kappa(t) &= \frac{||\vec{r}'(t) \times \vec{r}''(t)||}{||\vec{r}'(t)||^3} \\
&= \frac{16 \cdot ||\langle -1, 0, 1\rangle||}{(1+\sin^2 t)^{3/2}} \\
&= \frac{16\sqrt{2}}{(1+\sin^2 t)^{3/2}}.
\end{aligned}$$

Extrema for $\kappa(t)$ occurs when

$$\begin{aligned}
& \kappa'(t) = 16\sqrt{2} \cdot \frac{-3/2 \cdot 2\sin t\cos t}{(1+\sin^2 t)^{5/2}} = 0 \\
\Rightarrow\ & 2\sin t\cos t = \sin 2t = 0 \ \text{ and } \ 1+\sin^2 t \neq 0 \\
\Rightarrow\ & t \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\} \ \text{ in } \ [0, 2\pi].
\end{aligned}$$

Now, $\kappa''(t) = -24\sqrt{2} \cdot \left[\frac{2\cos 2t}{(1+\sin^2 t)^{5/2}} - \frac{5/2 \cdot \sin^2 2t}{(1+\sin^2 t)^{7/2}}\right]$. Since

$$\kappa''(0),\ \kappa''(\pi) < 0 \ \text{ and } \ \kappa''(\pi/2),\ \kappa''(3\pi/2) > 0,$$

maximum for curvature occurs when $t = 0$ or $t = \pi$, i.e. at points $(4, 0, 4)$ and $(-4, 0, -4)$ of the curve.

5. Let $f(x, y) = \dfrac{x+y}{x^2+y}$ when $x^2 + y \neq 0$.

   Is it possible to define $f(1, -1)$ in such a way that $f(x, y) \to f(1, -1)$ as $(x, y) \to (1, -1)$ along the line $x = 1$?

   Is it possible to define $f(1, -1)$ in such a way that $f(x, y) \to f(1, -1)$ as $(x, y) \to (1, -1)$ along the line $y = -1$?

   Is it possible to define $f(1, -1)$ in such a way that f is continuous at $(1, -1)$? Give reason for your answers.

   Solution:

   $\displaystyle\lim_{\substack{(x,y)\to(1,-1)\\ x=1}} f(x, y) = \lim_{y\to -1} \frac{1+y}{1+y} = 1$. Define $f(1, -1)$ as 1.

   $\displaystyle\lim_{\substack{(x,y)\to(1,-1)\\ y=-1}} f(x, y) = \lim_{x\to 1} \frac{x-1}{x^2-1} = \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}$. Hence define $f(1, -1)$ as $\frac{1}{2}$.

   Finally, since the two limits above are not equal to each other, $f(x, y)$ is by no means continuous at $(1, -1)$ and therefore there is no value for $f(1, -1)$ to make f continuous there.

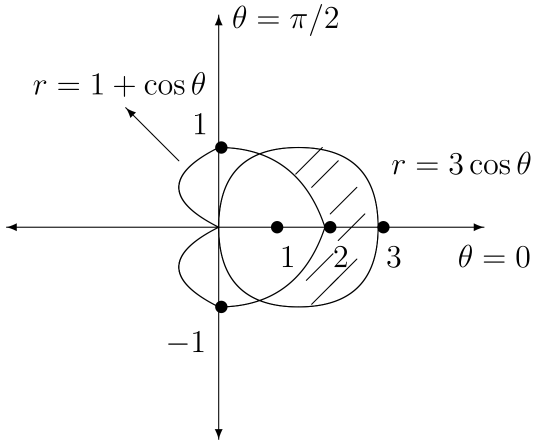
B U Department of Mathematics

Math 102 Calculus II

Spring 2006 First Midterm

1. Find the area of the region inside the circle $r = 3\cos\theta$ and outside the cardioid $r = 1 + \cos\theta$.

Solution:



The first step is to determine the intersections of the cardioid $r = 1 + \cos\theta$ and the circle $r = 3\cos\theta$, since this information is needed for the limits of integration. To find points of intersection, we can equate the two expressions for r.

This yields, $3\cos\theta = 1 + \cos\theta \Rightarrow \cos\theta = \frac{1}{2}$ which is satisfied by the positive angles $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$.

Referring to figure above and using symmetry the area in the first quadrant that is swept out for $0 \le \theta \le \frac{\pi}{3}$ is one-half of the total area. Thus

$$A = 2\left[\int_0^{\pi/3} \frac{1}{2}(3\cos\theta)^2 d\theta - \int_0^{\pi/3} \frac{1}{2}(1+\cos\theta)^2 d\theta\right]$$

which is the area inside the circle minus the area inside the cardioid.

$$
\begin{aligned}
A &= \int_0^{\pi/3} (9\cos^2\theta - 1 - 2\cos\theta - \cos^2\theta)\ d\theta \\
&= \int_0^{\pi/3} (8\cos^2\theta - 2\cos\theta - 1)\ d\theta \\
&= \int_0^{\pi/3} \left[8\left(\frac{\cos 2\theta + 1}{2}\right) - 2\cos\theta - 1\right]\ d\theta \\
&= \int_0^{\pi/3} (4\cos 2\theta - 2\cos\theta + 3)\ d\theta \\
&= 4\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/3} - 2\Big[\sin\theta\Big]_0^{\pi/3} + 3\Big[\theta\Big]_0^{\pi/3} \\
&= 4\left[\frac{\sin 2\pi/3}{2} - \sin 0\right] - 2\Big[\sin(\frac{\pi}{3}) - \sin 0\Big] + 3\Big[\frac{\pi}{3} - 0\Big] \\
&= 4\left[\frac{\sqrt{3}}{4} - 0\right] - 2\left[\frac{\sqrt{3}}{2}) - 0\right] + 3\left[\frac{\pi}{3}\right] = \sqrt{3} - \sqrt{3} + \pi = +\pi
\end{aligned}
$$

2. **(a)** Prove that the sum of two vectors of the same length is always perpendicular to their difference.

Solution:

Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors of the same length. i.e. $||\mathbf{u}|| = ||\mathbf{v}||$.
Then since we know that two vectors are orthogonal if and only if their dot product is zero, let's check

$$
\begin{aligned}
(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) &= \mathbf{u}\cdot(\mathbf{u}-\mathbf{v}) + \mathbf{v}\cdot(\mathbf{u}-\mathbf{v}) \\
&= \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v} \\
&\quad \text{since dot product is distributive over addition} \\
&= \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{v} \\
&\quad \text{since dot product is commutative} \\
&= ||\mathbf{u}||^{\mathbf{2}} - ||\mathbf{v}||^{\mathbf{2}} \\
&\quad \text{by the definition of dot product} \\
&= ||\mathbf{u}||^{\mathbf{2}} - ||\mathbf{u}||^{\mathbf{2}} \\
&\quad \text{since} ||\mathbf{u}|| = ||\mathbf{v}|| \text{is given by the question} \\
&= 0.
\end{aligned}
$$

(b) For which values of a are $< a, 3, -8 >$ and $< a, 4, 2 >$ perpendicular?

Solution:

$< a, 3, -8 >$ and $< a, 4, 2 >$ are perpendicular if and only if their dot product is zero. i.e.
$< a, 3, -8 > \cdot < a, 4, 2 >= 0 \Rightarrow a^2 = 4 \Rightarrow a = \pm 2$

(c) Find the area of the triangle whose vertices are the points $A = (1, 2, 3), B = (4, 5, 6), C = (1, 1, 1)$.

Solution:

$\vec{AB} =< 3, 3, 3 >, \vec{AC} =< 0, -1, -2 >$

$$
\begin{aligned}
\vec{AB} \times \vec{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 3 \\ 0 & -1 & -2 \end{vmatrix} \\
&= (-6+3)\mathbf{i} - (-6)\mathbf{j} + (-3)\mathbf{k} \\
&= -3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}
\end{aligned}
$$

and consequently

$$A = \frac{1}{2}||\vec{AB} \times \vec{AC}|| = \frac{1}{2}\sqrt{9+36+9} = \frac{1}{2}\sqrt{54} = \frac{3\sqrt{6}}{2}$$

3. Find the curvature of the ellipse $x = \cos t, y = 4\sin t \quad (0 \le t \le 2\pi)$ at the points where it intersects the x-axis.

Solution:

We may treat the ellipse as a curve in the xy-plane of an xyz-coordinate system by adding a zero k component and writing its equation as

$$r = 3\cos t\mathbf{i} + 4\sin t\mathbf{j} + 0\mathbf{k}.$$

It is not essential to write zero k component explicitly as long as you assume it to be there when you calculate a cross product. Thus,

$$\begin{aligned} r'(t) &= (-3\sin t)\mathbf{i} + (4\cos t)\mathbf{j} \\ r''(t) &= (-3\cos t)\mathbf{i} + (-4\sin t)\mathbf{j} \end{aligned}$$

$$\begin{aligned} r'(t) \times r''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3\sin t & 4\cos t & 0 \\ -3\cos t & -4\sin t & 0 \end{vmatrix} \\ &= (12\sin^2 t + 12\cos^2 t)\mathbf{k} = 12\mathbf{k}. \end{aligned}$$

Therefore, $||r'(t) \times r''(t)|| = 12$

$$||r'(t)|| = \sqrt{9\sin^2 t + 16\cos^2 t} = \sqrt{9 + 7\cos^2 t}$$

By using the formula for the curvature

$$\kappa(t) = \frac{||r'(t) \times r''(t)||}{||r'(t)||^3}$$

we get

$$\kappa(t) = \frac{12}{(\sqrt{9 + 7\cos^2 t})^3}.$$

The ellipse intersect the x-axis when y=0. i.e. $4\sin t = 0 \Rightarrow t = 0, \pi, 2\pi$ since $0 \le t \le 2\pi$.
Hence

$$\begin{aligned} \kappa(0) &= \frac{12}{64} = \frac{3}{16} \\ \kappa(\pi) &= \frac{12}{64} = \frac{3}{16} \\ \kappa(2\pi) &= \frac{12}{64} = \frac{3}{16} \end{aligned}$$

4. Find the parametric equation of the intersection line of the planes whose equations are $y+2z=2$ and $3x+4y+5z=4$.

Solution:

The given equations yield the normals $n_1 =< 0, 1, 2 >$ and $n_2 =< 3, 4, 5 >$ for planes $y+2z=2$ and $3x+4y+5z=4$, respectively. Let's compute $\mathbf{v} = n_1 \times n_2$.

$$\mathbf{v} = n_1 \times n_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 3 & 4 & 5 \end{vmatrix} = -3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$$

Since $\mathbf{v}$ is orthogonal to n_1 it is parallel to the first plane. Since $\mathbf{v}$ is orthogonal to n_2 it is parallel to the second plane. That is $\mathbf{v}$ is parallel to the intersection line L. To find a point on L, we observe that L must intersect the yz-plane, $x = 0$, since $\mathbf{v}\cdot < 1, 0, 0 >= -3 \neq 0$. Substituting $x = 0$ in the equations of both planes yields

$$y = -\frac{2}{3}, \quad z = \frac{4}{3}$$

Thus $P(0, \frac{-2}{3}, \frac{4}{3})$ is a point on L. A vector equation for L is

$$< x, y, z >=< 0, \frac{-2}{3}, \frac{4}{3} > +t < -3, 6, -3 > .$$

Parametric equations for L are

$$\begin{aligned} x(t) &= -3t \\ y(t) &= -\frac{2}{3} + 6t \\ z(t) &= \frac{4}{3} - 3t \end{aligned}$$

Boğaziçi University
Department of Mathematics
Math 102 Calculus II

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Date: March 26, 2008	Full Name : ASAF HALET ÇELEBİ
Time: 18:10-19:10	Math 102 Number :
	Student ID :

Spring 2008 – First Midterm Examination

IMPORTANT

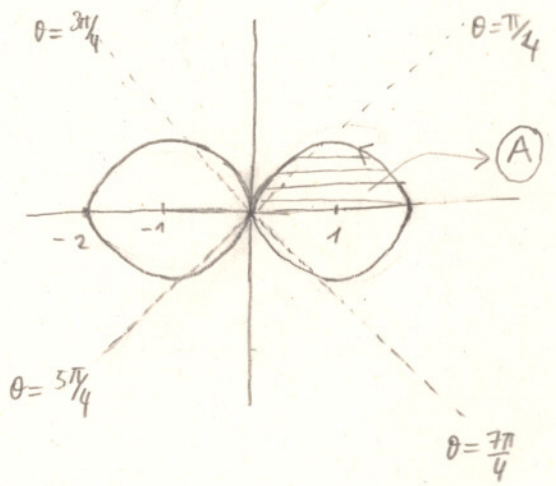
1. Write your name, surname on top of each page. **2.** The exam consists of 4 questions some of which have more than one part. **3.** Read the questions carefully and write your answers neatly under the corresponding questions. **4.** Show all your work. Correct answers without sufficient explanation might not get full credit. **5.** Calculators are not allowed.

---

1. (a) (30 points) Sketch the graph of the polar equation $r = 1 + \cos(2\theta)$ and find the area of the region enclosed by the graph.

We compute:

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
r	2	1	0	1	2	1	0	1	2



Total Area $= 4A$

$$= 4 \int_0^{\pi/2} \frac{1}{2}\left(1+\cos(2\theta)\right)^2 d\theta$$

$$= 2 \int_0^{\pi/2} [1 + 2\cos(2\theta) + \cos^2(2\theta)\, d\theta$$

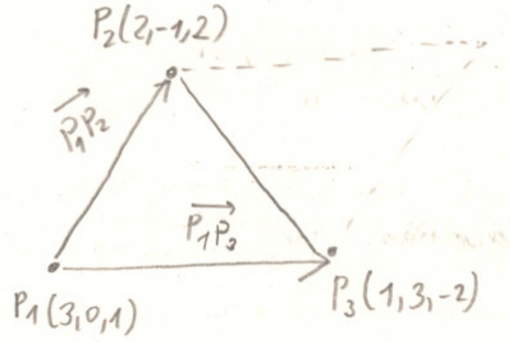
$$= 2\left[\theta + \sin 2\theta + \frac{1}{2}\left(\frac{\sin(8\theta)}{8} + \theta\right)\right]_0^{\pi/2}$$

$$= 2\left(\frac{\pi}{2} + \frac{\pi}{4}\right)$$

$$= \frac{3\pi}{2}$$

2. (a) (20 points) Find the area of the triangle with vertices $P_1(3,0,1)$, $P_2(2,-1,2)$, $P_3(1,3,-2)$ using cross product.

We observe



$\overrightarrow{P_1P_2} = \langle -1, -1, 1 \rangle$

$\overrightarrow{P_1P_3} = \langle -2, 3, -3 \rangle$

the area of the triangle with vertices P_1, P_2, P_3 is the <u>half</u> of the area of the parallelogram generated by $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$. So

$$\text{Area} = \frac{1}{2} \| \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} \|$$

$$= \frac{1}{2} \| \langle -1, -1, 1 \rangle \times \langle -2, 3, -3 \rangle \|$$

$$= \frac{1}{2} \| \langle 0, -5, -5 \rangle \|$$

$$= \frac{5\sqrt{2}}{2}$$

3. Let $\mathbf{L_1} : x = t + 1$, $y = t + 2$, $z = 3$ and $\mathbf{L_2} : x = 3t - 2$, $y = 5t - 3$, $z = t + 2$ be the parametric equations of two lines

(a) (10 points) Find the intersection point of the lines L_1 and L_2.

L_1 and L_2 intersects if there exist $t_1, t_2 \in \mathbb{R}$ such that

$$t_1 + 1 = 3t_2 - 2$$

$$\left.\begin{array}{r} t_1 + 2 = 5t_2 - 3 \\ 3 = t_2 + 2 \end{array}\right\}$$

Solving these we get $t_2 = 1$, $t_1 = 0$ and if we put $t_2 = 1$ and $t_1 = 0$ into the first equation we get

$0 + 1 = 3.1 - 2$ that is $1 = 1$

So the system is consistent and Lines intersect at $(1,2,3)$

(b) (10 points) What is the equation of the plane P containing the lines L_1 and L_2?

See that $\vec{v}_1 = \langle 1, 1, 0 \rangle$ and $\vec{v}_2 = \langle 3, 5, 1 \rangle$ are the direction vectors for L_1 and L_2 respectively. So $\vec{v}_1$ and $\vec{v}_2$ should be parallel to P and therefore $\vec{v}_1 \times \vec{v}_2$ is perpendicular to the plane P. We compute $\vec{v}_1 \times \vec{v}_2 = \langle 1, -1, 2 \rangle$. We know that the point $(1,2,3)$ is on the plane P so we get:

$$(\langle x, y, z \rangle - \langle 1, 2, 3 \rangle) . \langle 1, -1, 2 \rangle = 0 \Leftrightarrow x - y + 2z - 5 = 0$$

(c) (10 points) What is the distance between the above plane P and the origin?

we know that the distance is given by

$$D = \frac{|1.0 + (-1).0 + 2.0 - 5|}{\sqrt{1^2 + (-1)^2 + 2^2}} = \frac{5}{\sqrt{6}}$$

4. (a) (20 points) Find the arc length parametrization of the curve

$$\mathbf{r}(t) = \cos^3 t\,\mathbf{i} + \sin^3 t\,\mathbf{j}, \quad \text{for } 0 \le t \le \pi/2.$$

We take $r(0) = \langle 1, 0 \rangle$ as a reference point and define

$$s(t) = \int_0^t \|r'(\tau)\|\, d\tau$$

$$= \int_0^t \| \langle -3\cos^2\tau \sin\tau ,\ 3\sin^2\tau\cos\tau \rangle \|\, d\tau$$

$$= \int_0^t |3\cos\tau \sin\tau|\, d\tau$$

$$= \int_0^t 3\cos\tau\sin\tau\, d\tau \qquad (t \in [0, \tfrac{\pi}{2}]) \quad (*)$$

$$= \left[\frac{3}{2}\sin^2\tau \right]_0^t$$

So we get $s(t) = \frac{3}{2}\sin^2 t$, and $t = \arcsin\left(\sqrt{\frac{2s}{3}}\right)$

Hence we get

$$r(s) = \cos^3\left(\arcsin\sqrt{\frac{2s}{3}}\right)\hat{\imath} + \sin^3\left(\arcsin\sqrt{\frac{2s}{3}}\right)\hat{\jmath}$$

where $s \in [0, \frac{3}{2}]$.

OR one can continue from $(*)$ as the following:

$$\int_0^t 3\sin\tau\cos\tau\, d\tau = \frac{3}{2}\int_0^t 2\sin\tau\cos\tau\, d\tau = \frac{3}{2}\int_0^t \sin 2\tau\, d\tau$$

$$= \left[-\frac{3}{2}\cdot\frac{\cos 2\tau}{2} \right]_0^t$$

$$= \dots$$

B U Department of Mathematics

Math 102 Calculus II

Summer 2000 First Midterm

1. Let $\vec{v_1}, \vec{v_2}, \vec{v_3}$ be orthogonal vectors in space (which are all nonzero). Show that if $\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} = 0$ then $c_1 = c_2 = c_3 = 0$.

Solution:

$$v_1 \cdot (c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}) = c_1 \left|\vec{v_1}\right|^2 = 0 \Rightarrow c_1 = 0. \text{ (since } v_1 \text{ is assumed to be nonzero).}$$
$$v_2 \cdot (c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}) = c_2 \left|\vec{v_2}\right|^2 = 0 \Rightarrow c_2 = 0. \text{ (since } v_2 \text{ is assumed to be nonzero).}$$
$$v_3 \cdot (c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}) = c_3 \left|\vec{v_3}\right|^2 = 0 \Rightarrow c_3 = 0. \text{ (since } v_3 \text{ is assumed to be nonzero).}$$

2. Find an equation for the plane which contains the line $\vec{r}(t) = (3t+2)\vec{i} + (5t-1)\vec{j} + 4t\vec{k}$ and the point $(3, -1, 3)$.

Solution:

$\vec{r}(0) = 2\vec{i} - \vec{j} \Rightarrow P = (2, -1, 0)$ is on the plane.
Let $Q = (3, -1, 3) \Rightarrow \overrightarrow{PQ} =< 1, 0, 3 >$.
A direction vector of the given line is $< 3, 5, 4 >$.

Then the vector $\overrightarrow{PQ} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 3 \\ 3 & 5 & 4 \end{vmatrix} =< -15, 5, 5 >$ is normal to the plane.

So the plane has equation: $-15x + 5y + 5z = d$.
We know that $(3, -1, 3)$ is on the plane, so, $d = -35$.
$-15x + 5y + 5z = -35$
$\Rightarrow 3x - y - z = 7$

3. Parametrize the curve $y = 3x - 2, -2 \le x \le 0$ with respect to arclength.

Solution:

Let $x = t$. Then a parametrization of the curve is
$\vec{r}(t) = t\vec{i} + (3t-2)\vec{j}$
$\dot{\vec{r}}(t) = \vec{i} + 3\vec{j}, \ \left|\dot{\vec{r}}(t)\right| = \sqrt{10} \neq 1$. So it is not parametrized by arclength.

$$s = \int_0^t \left|\dot{\vec{r}}(w)\right| dw = \int_0^t \sqrt{10}dw = \sqrt{10}\, t.$$

$$s = \sqrt{10}\, t \Rightarrow t = \frac{s}{\sqrt{10}} .$$

$$\vec{r}(s) = \frac{s}{\sqrt{10}}\vec{i} + \left(\frac{3s}{\sqrt{10}} - 2\right)\vec{j}, \ \ -2\sqrt{10} \le s \le 0.$$

4. Two particles are moving on the plane.Their motions are described by $\vec{r_1}(t) = (t+t^3)\vec{i} + (t-t^3)\vec{j}, -4 \le t \le 5$ and $\vec{r_2}(t) = \left[(3t-1) + (3t-1)^3\right]\vec{i} + \left[(3t-1) - (3t-1)^3\right]\vec{j}, -1 \le t \le 2$.

a) Do these particles travel on the same path or on different paths? Justify your answer.

Solution:

Let $u = 3t - 1$. Then $\vec{r_2}(u) = (u + u^3)\,\vec{i} + (u - u^3)\,\vec{j}$ and $-4 \leq u \leq 5$. Hence they travel on the same path.

b) Which particle moves faster at $(0, 0)$? At other points?

Solution:

$\vec{v_1}(t) = \vec{r_1'}(t) = (1 + 3t^2)\,\vec{i} + (1 - 3t^2)\,\vec{j}$; $|\vec{v_1}(t)| = \sqrt{2}\,\sqrt{1 + 9t^4}$.
$\vec{v_2}(t) = \vec{r_2'}(t) = \left[3 + 9\,(3t - 1)^2\right]\vec{i} + \left[3 - 9\,(3t - 1)^2\right]\vec{j}$; $|\vec{v_2}(t)| = 3\sqrt{2}\,\sqrt{1 + 9\,(3t - 1)^4}$.
$|\vec{v_1}(0)| = \sqrt{2}$, $|\vec{v_2}(1/3)| = 3\sqrt{2}$ at $(0, 0)$ and as we see the second one is faster. For the given intervals , the second one is always faster.

B U Department of Mathematics

Math 102 Calculus II

Summer 2001 First Midterm

1. ***(i)*** Let σ_1 be a plane containing the point $\boldsymbol{P}_1 = (1,1,1)$. A vector normal to this plane is $\boldsymbol{n}_1 = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$. Find an equation for this plane. ***(ii)*** Let σ_2 be another plane containing the point $\boldsymbol{P}_2 = (0,1,1)$. A vector normal to this plane is $\boldsymbol{n}_2 = 2\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$. Find an equation for this plane too. ***(iii)*** Find an equation for the line of intersection, ℓ, of the two planes given above.

Solution:

(i) For a point (x,y,z) on the plane σ_1, the vector $(x-1)\hat{\boldsymbol{\imath}} + (y-1)\hat{\boldsymbol{\jmath}} + (z-1)\hat{\boldsymbol{k}}$ is perpendicular to $\boldsymbol{n}_1 = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$. Taking the dot product, we get $(x-1)+(y-1)+(z-1) = 0$, or $x+y+z=3$.
(ii) Similar to the previous part, for a point (x,y,z) on σ_2, the vector $x\hat{\boldsymbol{\imath}} + (y-1)\hat{\boldsymbol{\jmath}} + (z-1)\hat{\boldsymbol{k}}$ is perpendicular to $\boldsymbol{n}_2 = 2\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$. Setting the dot product to 0, we get $2x + (y-1) = 0$, or $2x+y=1$.
(iii) We have the equations for σ_1 and σ_2:

$$\left\{ \begin{array}{lcl} x+y+z & = & 3 \\ 2x+y & = & 1 \end{array} \right.$$

By the second equation $y = 1-2x$. Also, subtracting the second equation from the first, we get $-x+z=2$, or $z = 2+x$.
Setting $x=t$, we obtain the following set of equations for the line of intersection for $t \in \mathbb{R}$:

$$\left\{ \begin{array}{lcl} x & = & t \\ y & = & 1-2t \\ z & = & 2+t \end{array} \right.$$

2. Consider the curve $\boldsymbol{C} : \boldsymbol{r}(t) = e^t \cos t\hat{\boldsymbol{\imath}} + e^t \sin t\hat{\boldsymbol{\jmath}} + e^t\hat{\boldsymbol{k}}$. ***(i)*** Calculate the arclength $s = \int |\boldsymbol{v}(t)|dt$. ***(ii)*** Taking $s=1$ at $t=0$, express $\boldsymbol{r}$ as a function of the arclength s.

Solution:

Differentiating the position vector $\boldsymbol{r}(t)$, we get the velocity vector $\boldsymbol{v}(t) = (e^t\cos t - e^t \sin t)\hat{\boldsymbol{\imath}} + (e^t \sin t + e^t \cos t)\hat{\boldsymbol{\jmath}} + e^t\hat{\boldsymbol{k}}$, and from this, we compute $|\boldsymbol{v}(t)|$ as:

$$|\boldsymbol{v}(t)| = \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2 + e^{2t}} = \sqrt{3e^{2t}} = \sqrt{3}e^t.$$

Integrating $|\boldsymbol{v}(t)|$ gives $s = \int |\boldsymbol{v}(t)|dt = \int \sqrt{3}e^t dt = \sqrt{3}e^t + C$ for some constant C.

(ii) Setting $s=1$ at $t=0$, we find $C = \sqrt{3}-1$. So $s = \sqrt{3}e^t - 1 + \sqrt{3}$, hence $e^t = \dfrac{s-1+\sqrt{3}}{\sqrt{3}}$ and $t = \ln\left(\dfrac{s-1+\sqrt{3}}{\sqrt{3}}\right)$. Putting these values in the equation of $\boldsymbol{r}(t)$ we obtain $\boldsymbol{r}(s) = \left(\frac{s-1+\sqrt{3}}{\sqrt{3}}\right)\cos\left(ln\left(\frac{s-1+\sqrt{3}}{\sqrt{3}}\right)\right)\hat{\boldsymbol{\imath}} + \left(\frac{s-1+\sqrt{3}}{\sqrt{3}}\right)\sin\left(ln\left(\frac{s-1+\sqrt{3}}{\sqrt{3}}\right)\right)\hat{\boldsymbol{\jmath}} + \left(\frac{s-1+\sqrt{3}}{\sqrt{3}}\right)\hat{\boldsymbol{k}}$.

3. Consider the curve $\boldsymbol{C} : \boldsymbol{r}(s) = s\cos(\ln s)\,\hat{\boldsymbol{\imath}} + s\sin(\ln s)\,\hat{\boldsymbol{\jmath}} + s\,\hat{\boldsymbol{k}}$, where s is the arclength. ***(i)*** Find the unit tangent vector $\hat{\boldsymbol{T}}(s)$. ***(ii)*** Find the curvature $\kappa(s)$. ***(iii)*** Find the unit normal vector $\hat{\boldsymbol{N}}(s)$.

Solution:

(i) We differentiate $\boldsymbol{r}(s)$ and we get $\boldsymbol{r}'(s) = (\cos(\ln s) - \sin(\ln s))\,\hat{\boldsymbol{\imath}} + (\sin(\ln s) + \cos(\ln s))\,\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$

Since the curve is parametrized by arclength, $\hat{\boldsymbol{T}}(s) = \boldsymbol{r}'(s) = (\cos(\ln s) - \sin(\ln s))\,\hat{\boldsymbol{\imath}} + (\sin(\ln s) + \cos(\ln s))\,\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$

(ii) $\kappa(s) = |\boldsymbol{r}''(s)|$, because the curve is parametrized by arclength. We compute $\boldsymbol{r}''(s)$:

$$\boldsymbol{r}''(s) = \left(-\frac{1}{s}\sin(\ln s) - \frac{1}{s}\cos(\ln s)\right)\,\hat{\boldsymbol{\imath}} + \left(\frac{1}{s}\cos(\ln s)) - \frac{1}{s}\sin(\ln s)\right)\,\hat{\boldsymbol{\jmath}}$$

So, $\kappa(s) = |\boldsymbol{r}''(s)| = \left[\frac{1}{s^2}(\cos(\ln s) + \sin(\ln s))^2 + \frac{1}{s^2}(\cos(\ln s) - \sin(\ln s))^2\right]^{1/2} = \sqrt{\frac{2}{s^2}} = \frac{\sqrt{2}}{s}$

(iii) For a curve parametrized by arclength, $\hat{\boldsymbol{N}}(s) = \frac{1}{|\boldsymbol{r}''(s)|}\boldsymbol{r}''(s)$. Thus, $\hat{\boldsymbol{N}}(s) = \frac{s}{\sqrt{2}}\boldsymbol{r}''(s) = \left(-\frac{1}{\sqrt{2}}\sin(\ln s) - \frac{1}{\sqrt{2}}\cos(\ln s)\right)\,\hat{\boldsymbol{\imath}} + \left(\frac{1}{\sqrt{2}}\cos(\ln s)) - \frac{1}{\sqrt{2}}\sin(\ln s)\right)\,\hat{\boldsymbol{\jmath}}$

4. A particle, starting from $\boldsymbol{r}(0) = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}$ with velocity $\boldsymbol{v}(0) = 3\,\hat{\boldsymbol{k}}$ at time $t = 0$, accelerates with $\boldsymbol{a}(t) = 2\,\hat{\boldsymbol{\imath}} + 12t^2\,\hat{\boldsymbol{\jmath}} + 4e^{2t}\,\hat{\boldsymbol{k}}$. ***(i)*** Find $\boldsymbol{v}(t)$. ***(ii)*** Find $\boldsymbol{r}(t)$. ***(iii)*** Find the speed of the particle at time $t = 1$ sec.

Solution:

(i) We integrate $\boldsymbol{a}(t)$ to get $\boldsymbol{v}(t)$.

$\boldsymbol{v}(t) = \int \boldsymbol{a}(t)dt = \int 2\,\hat{\boldsymbol{\imath}} + 12t^2\,\hat{\boldsymbol{\jmath}} + 4e^{2t}\,\hat{\boldsymbol{k}}\,dt = 2t\,\hat{\boldsymbol{\imath}} + 4t^3\,\hat{\boldsymbol{\jmath}} + 2e^{2t}\,\hat{\boldsymbol{k}} + \boldsymbol{c}_1$, where $\boldsymbol{c}_1$ is a constant vector to be determined. Using $\boldsymbol{v}(0) = 3\,\hat{\boldsymbol{k}}$, we have $2\,\hat{\boldsymbol{k}} + \boldsymbol{c}_1 = 2\,\hat{\boldsymbol{k}}$, thus $\boldsymbol{c}_1 = \hat{\boldsymbol{k}}$. Hence, $\boldsymbol{v}(t) = 2t\,\hat{\boldsymbol{\imath}} + 4t^3\,\hat{\boldsymbol{\jmath}} + (2e^{2t} + 1)\,\hat{\boldsymbol{k}}$.

(ii)Integration of $\boldsymbol{v}(t)$ gives $\boldsymbol{r}(t)$, so $\boldsymbol{r}(t) = \int \boldsymbol{v}(t)dt = \int 2t\hat{\boldsymbol{\imath}} + 4t^3\hat{\boldsymbol{\jmath}} + (2e^{2t} + 1)\hat{\boldsymbol{k}}\,dt = t^2\,\hat{\boldsymbol{\imath}} + t^4\,\hat{\boldsymbol{\jmath}} + (e^{2t} + t)\,\hat{\boldsymbol{k}} + \boldsymbol{c}_2$, where $\boldsymbol{c}_2$ is the constant vector of integration. We can compute $\boldsymbol{c}_2$ using $\boldsymbol{r}(0) = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}$. We have $\boldsymbol{r}(0) = \hat{\boldsymbol{k}} + \boldsymbol{c}_2 = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}$, so $\boldsymbol{c}_2 = \hat{\boldsymbol{\imath}}$, and $\boldsymbol{r}(t) = (t^2 + 1)\,\hat{\boldsymbol{\imath}} + t^4\,\hat{\boldsymbol{\jmath}} + (e^{2t} + t)\,\hat{\boldsymbol{k}}$.

(iii) The speed at $t = 1$ s. is $|v(1)| = |2\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}} + (2e^2 + 1)\hat{\boldsymbol{k}}| = [2^2 + 4^2 + (2e^2 + 1)^2]^{1/2} = \sqrt{4e^4 + 4e^2 + 21}$.

B U Department of Mathematics

Math 102 Calculus II

Summer 2002 First Midterm

1. Find the arc length of the cardioid $r = 5 - 5\cos\theta$ from $\theta = 0$ to $\theta = \pi$.

Solution:

The arc length formula is: $L = \int_a^b \sqrt{f'(\theta)^2 + f(\theta)^2}d\theta$.

The curve is the upper half of the cardioid lying over the x-axis. Also, $f'(\theta) = 5\sin\theta$.

$$\begin{aligned}
L &= \int_0^\pi \sqrt{(5\sin\theta)^2 + (5 - 5\cos\theta)^2}d\theta \\
&= \int_0^\pi \sqrt{25\sin^2\theta + 25 - 50\cos\theta + 25\cos^2\theta d\theta} \\
&= \int_0^\pi \sqrt{50 - 50\cos\theta}d\theta = 5\sqrt{2}\int_0^\pi \sqrt{1-\cos\theta}d\theta \\
&= 5\sqrt{2}\int_0^\pi \sqrt{1 - (1 - 2\sin^2(\theta/2)}d\theta = 5\sqrt{2}\int_0^\pi \sqrt{2}\sin(\theta/2)d\theta \\
&= 10\left(\frac{-\cos(\theta/2)}{1/2}\right)\Bigg|_0^\pi = -20\cos\frac{\theta}{2}\Bigg|_0^\pi = -20\cos\frac{\pi}{2} + 20\cos 0 = 20.
\end{aligned}$$

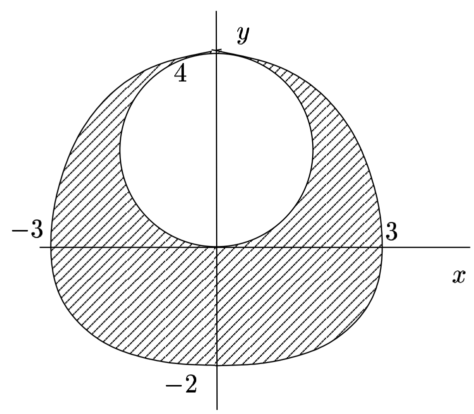
2. Given the curves $r = 3 + \sin\theta$ and $r = 4\sin\theta$,

a) Sketch them on the same set of axes and determine the points of intersection.

b) Find the area of the region which lies inside the first curve but outside the second curve.

Solution:

a) The curve $r = 3 + \sin\theta$ is an egg-shaped figure and $r = 4\sin\theta$ is a circle:



Their common point is determined by $4\sin\theta = 3 + \sin\theta$. Therefore $\sin\theta = 1$, hence $\theta = \pi/2$. The point of intersection is at $r = 4$ and $\theta = \pi/2$.

b) The area inside the egg-shaped figure is :

$$\frac{1}{2}\int_0^{2\pi}(3+\sin\theta)^2 d\theta = \frac{1}{2}\int_0^{2\pi}(9+6\sin\theta+\sin^2\theta)d\theta = \frac{1}{2}\int_0^{2\pi}\left(9+6\sin\theta+\frac{1}{2}-\frac{\cos 2\theta}{2}\right)d\theta$$
$$= \frac{1}{2}\left(9\theta - 6\cos\theta + \frac{1}{2}\theta - \frac{1}{2}\frac{\sin 2\theta}{2}\right)\Big|_0^{2\pi} = \left(\frac{19\theta}{4} - 6\cos\theta - \frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi}$$
$$= \left(\frac{19(2\pi)}{4} - 6\cos 2\pi - \frac{1}{4}\sin 4\pi\right) - \left(\frac{19(0)}{4} - 6\cos 0 - \frac{1}{4}\sin 2(0)\right) = \frac{19\pi}{2}.$$

The area of the circle is $\pi 2^2 = 4\pi$.

Since the circle is inside the egg-shaped figure, the area asked is : $\frac{19\pi}{2} - 4\pi = \frac{11\pi}{2}$.

3. Find the distance from the point $(2, -1, 4)$ to the line whose parametric equations are

$$\begin{array}{rcrcl} x &=& 3 &+& t \\ y &=& -1 &+& 2t \\ z &=& -1 &+& t. \end{array}$$

Solution:

Let P be the point $(2, -1, 4)$ and let P_0 be the point on the line corresponding to $t = 0$. i.e. $P_0 : (3, -1, -1)$. Then the vector $\boldsymbol{PP_0}$ is $< -1, 0, 5 >$, and the direction vector of the line is $\boldsymbol{v} =< 1, 2, 1 >$. The distance asked is $d = \dfrac{\|\boldsymbol{PP_0} \times \boldsymbol{v}\|}{\|\boldsymbol{v}\|}$.

$$\boldsymbol{PP_0} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 0 & 5 \\ 1 & 2 & 1 \end{vmatrix} = -10\boldsymbol{i} + 6\boldsymbol{j} + (-2)\boldsymbol{k} =< -10, 6, -2 > .$$

Then $\|\boldsymbol{PP_0} \times \boldsymbol{v}\| = \sqrt{(-10)^2 + 6^2 + (-2)^2} = \sqrt{140}$, and $\|\boldsymbol{v}\| = \sqrt{1+4+1} = \sqrt{6}$. So,

$$d = \frac{\sqrt{140}}{\sqrt{6}} = \frac{\sqrt{210}}{3}.$$

4. Find the point on the graph of the curve $y = x^2 - 4x + 5$ where the curvature is a maximum; also determine this maximum value.

Solution:

The formula for the curvature κ is : $\kappa(x) = \dfrac{|y''(x)|}{(1 + y'(x)^2)^{\frac{3}{2}}}$. Hence,

$$\kappa(x) = \frac{2}{(1 + (2x-4)^2)^{3/2}} = \frac{2}{(4x^2 - 16x + 17)^{3/2}} = 2(4x^2 - 16x + 17)^{-3/2}.$$

Taking the derivative and equating to zero:

$$\kappa'(x) = 2\left(\frac{-3}{2}\right)(4x^2 - 16x + 17)^{-5/2}(8x - 16) = 0$$

The curvature is a maximum when $x = 2$. The maximum value of the curvature is

$$\kappa(2) = \frac{2}{(4(2^2) - 16(2) + 17)^{3/2}} = 2.$$

B U Department of Mathematics

Math 102 Calculus II

Summer 2003 First Midterm

1. Let $\vec{F} = 2\vec{i} + \vec{j} - 3\vec{k}$ and $\vec{v} = 3\vec{i} - \vec{j}$. Find the vectors $\vec{A}, \vec{B}$ satisfying $\vec{F} = \vec{A} + \vec{B}, \vec{A} \parallel \vec{v}, \vec{B} \perp \vec{v}$.

Solution:

$$\begin{aligned}\vec{A} &= \text{proj}_{\vec{v}}\vec{F} = \frac{\vec{F}\cdot\vec{v}}{||\vec{v}||^2}\vec{v} = \frac{<2,1,-3> \cdot <3,-1,0>}{10} <3,-1,0> \\ &= \frac{6-1}{10} <3,-1,0> = <\frac{3}{2}, \frac{-1}{2}, 0> .\end{aligned}$$

Thus, $\vec{A} = \frac{3}{2}\vec{i} - \frac{1}{2}\vec{j}$.

$\vec{B} = \vec{F} - \text{proj}_{\vec{v}}\vec{F} = <2,1,-3> - <\frac{3}{2}, \frac{-1}{2}, 0> = <\frac{1}{2}, \frac{3}{2}, -3>$.

2. (a) Find the distance between the planes $x + 2y - z = 1$ and $2x + 4y - 2z = 3$.
(b) Find the distance from the point $P = (1,1,1)$ to the line $x = 1 + t, y = 2 - t, z = 2 + 3t$. Justify the reasons underlying your computations.

Solution:

(a) Note that the point $(0,0,-1)$ is on the plane $x + 2y - z = 1$. By the formula of the distance between a point and a plane,

$$d = \frac{|2.0 + 4.0 - 2(-1) - 3|}{\sqrt{4+16+4}} = \frac{1}{\sqrt{24}} = \frac{1}{2\sqrt{6}}.$$

(b) Let us denote the distance between the point P and the line by d. Let us find two points on the line; for $t = 0$, we have the point $A = (1,2,2)$ and for $t = 1$, we have the point $B = (2,1,5)$. Then $\vec{AP} = (1,1,1) - (1,2,2) = <0,-1,-1>$ and $\vec{AB} = (2,1,5) - (1,2,2) = <1,-1,3>$. Since $d = ||\vec{AP}|| \sin\theta$ where θ is the angle between $\vec{AP}$ and $\vec{AB}$,

$$\begin{aligned} d &= \frac{||\vec{AP}||||\vec{AB}|| \sin\theta}{||\vec{AB}||} = \frac{||\vec{AP} \times \vec{AB}||}{||\vec{AB}||} = \frac{|| <0,-1,-1> \times <1,-1,3> ||}{|| <1,-1,3> ||} \\ &= \frac{\left\| \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & -1 \\ 1 & -1 & 3 \end{pmatrix} \right\|}{\sqrt{11}} = \frac{|| <-4,-1,1> ||}{\sqrt{11}} = \frac{3\sqrt{2}}{11}.\end{aligned}$$

3. Does the limit $\displaystyle\lim_{\substack{(x,y)\to(2,2) \\ x+y\neq 4}} \frac{x+y-4}{\sqrt{x+y}-2}$ exist? Evaluate the limit if it exists. Otherwise, explain why it fails to exist.

Solution:

$$\lim_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \frac{x+y-4}{\sqrt{x+y}-2} = \lim_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \frac{(\sqrt{x+y}-2)(\sqrt{x+y}+2)}{\sqrt{x+y}-2} = \lim_{x\to 2} \sqrt{x+y}+2 = 4.$$

4. Suppose $f(x) = \begin{cases} \frac{x+y}{x-y} & \texttt{if } y \neq x \\ 1 & \texttt{if } y = x \end{cases}$.
   (a) Is f continuous at (0,0)? Justify.
   (b) Evaluate $f_x(0,0)$, $f_y(0,0)$ if they exist.

   Solution:

   (a) For $y = mx$ where $m \neq 1$, $\lim_{(x,y)\to(0,0)} \frac{x+y}{x-y} = \lim_{x\to 0} \frac{(m+1)x}{(m-1)x}$. So, this limit depends on m, implying that this function is not continuous at $(0,0)$.

   (b) Since this function is not continuous at $(0,0)$, $f_x(0,0)$ and $f_y(0,0)$ do not exist.

5. The plane $x = 1$ intersects the paraboloid $z = x^2 + 2y^2$ in a parabola. Find the slope of the tangent line in the plane to this parabola at $(1,2,9)$.

   Solution:

   The equation of the parabola is $z = 1 + 2y^2$, $x = 1$. Now, let us find the slope of the tangent line to the parabola at (1,2,9):

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2,9)} = 4y|_{(1,2,9)} = 8.$$

B U Department of Mathematics

Math 102 Calculus II

Summer 2004 First Midterm

1.) Let $\vec{r}(t) = (\sin t - t\cos t)\vec{i} + (\cos t + t\sin t)\vec{j} + \vec{k}$ be a space curve. Find $\vec{T}$ and $\vec{N}$.

Solution:

Now

$$\vec{T}(t) = \frac{\vec{r}'(t)}{||\vec{r}'(t)||} = \frac{(\cos t - (\cos t - t\sin t))\vec{i} + (-\sin t + \sin t + t\cos t)\vec{j} + 0\vec{k}}{||\vec{r}'(t)||}$$

$$= \frac{t\sin t\vec{i} - \sin t\vec{j} + 0\vec{k}}{\sqrt{t^2\sin^2 t + t^2\cos^2 t}} = \sin t\vec{i} + \cos t\vec{j} + 0\vec{k}$$

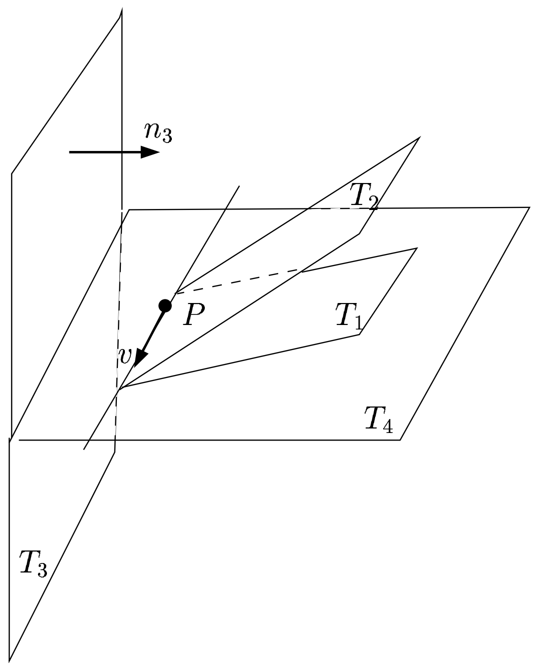
To find $\vec{N}(t)$ we use the equality,

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\left|\left|\vec{T}'(t)\right|\right|}$$

$$\vec{N}(t) = \frac{\cos t\vec{i} - \sin t\vec{j} + 0\vec{k}}{\sqrt{\cos^2 t + \sin^2 t}} = \cos t\vec{i} - \sin t\vec{j} + 0\vec{k}.$$

2.) Find the equation of the plane passing through the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$, and perpendicular to the plane $x + y - 2z = 1$.

Solution:



Let $T_1 : x - z = 1$, $T_2 : y + 2z = 3$ and $T_3 : x + y - 2z = 1$. If we let $z = 0$, then from T_1, we get $x = 1$ and from T_2, we get $y = 3$. The point $P(1, 3, 0)$ will be on the line of intersection of the planes T_1 and T_2. We are asked to find the equation of the plane T_4 in the figure.

Let n_i be the normal vectors to the planes T_i. Then we get:

$$\vec{n_1} = \vec{i} + 0\vec{j} - \vec{k},\ \vec{n_2} = 0\vec{i} + \vec{j} + 2\vec{k} \text{ and } \vec{n_3} = \vec{i} + \vec{j} - 2\vec{k}.$$

The direction vector $\vec{v}$ of the line of intersection will be in the direction perpendicular to both $\vec{n_1}$ and $\vec{n_2}$. So

$$\vec{v} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{vmatrix} = \vec{i} - 2\vec{j} + \vec{k}.$$

The normal vector $\vec{n_4}$ of the plane T_4 will be perpendicular to both $\vec{v}$ and $\vec{n_3}$. So

$$\vec{n_4} = \vec{v} \times \vec{n_3} = \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = 3\vec{i} + 3\vec{j} + 3\vec{k}.$$

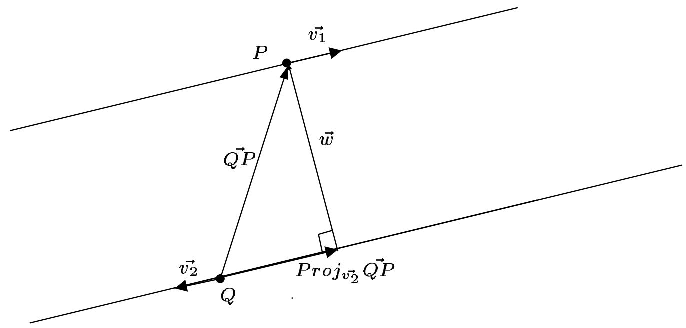
So the equation of the plane through $P(1, 3, 0)$ with the normal vector $\vec{n_4} = 3\vec{i} + 3\vec{j} + 3\vec{k}$ is

$$3(x - 1) + 3(y - 3) + 3(z - 0) = 0$$
$$x + y + z = 4.$$

3.) Find the distance between the lines $L_1: \begin{array}{rcl} x &=& 1+t \\ y &=& 1-t \\ z &=& 2t \end{array}$ and $L_2: \begin{array}{rcl} x &=& 1-2t \\ y &=& 5+2t \\ z &=& -2-4t \end{array}$.

Solution:

Now L_1 has the direction vector $\vec{v_1} = \vec{i} - \vec{j} + \vec{k}$ and L_2 has the direction vector $\vec{v_2} = -2\vec{i} + 2\vec{j} - 4\vec{k}$. That gives $\vec{v_2} = -2\vec{v_1}$, so L_1 and L_2 are parallel lines.



Let $P(1,1,0)$ and $Q(1,5,-2)$ be the points as shown in the figure. If $\vec{w}$ is the vector defined as:

$$\vec{w} = \vec{QP} - Proj_{\vec{v_2}}\vec{QP},$$

then the distance between L_1 and L_2 is $||\vec{w}||$.

Now $\vec{QP} = 0\vec{i} + 4\vec{j} + 2\vec{k}$, and

$$Proj_{\vec{v_2}}\vec{QP} = \left(\frac{\vec{QP}\cdot\vec{v_2}}{\vec{v_2}\cdot\vec{v_2}}\right)\vec{v_2} = \frac{-16}{24}(-2\vec{i} + 2\vec{j} - 4\vec{k}) = \frac{4}{3}\vec{i} - \frac{4}{3}\vec{j} + \frac{8}{3}\vec{k}$$

$$\vec{w} = \vec{QP} - Proj_{\vec{v_2}}\vec{QP} = \frac{-4}{3}\vec{i} - \frac{8}{3}\vec{j} - \frac{2}{3}\vec{k}.$$

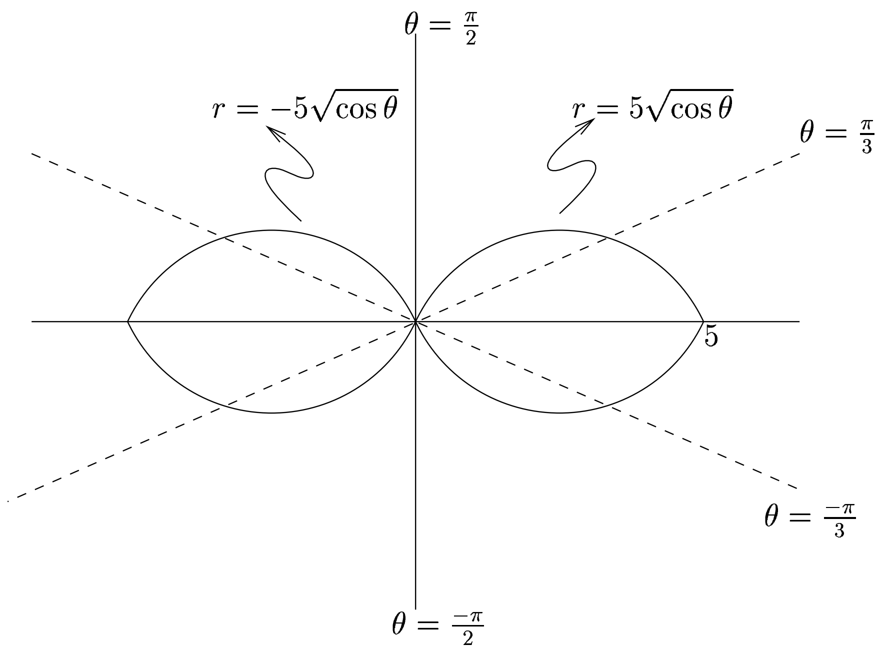
Then the length of the vector $\vec{w} = \sqrt{\frac{16}{9} + \frac{64}{9} + \frac{4}{9}} = \frac{\sqrt{84}}{3} = \frac{2\sqrt{21}}{3}$.

4.) **(a)** Graph the polar curve $r^2 = 25\cos\theta$. Find the domain and the symmetry or symmetries of the curve, if any.

Solution:

Now, realize $r^2 = 25\cos\theta$ can be visualized as two curves $r = 5\sqrt{\cos\theta}$, and $r = -5\sqrt{\cos\theta}$. So $\cos\theta \geq 0$, that is $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. The domain is: $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}]$. Realize that the curve is symmetric with respect to the x-axis, y-axis and the origin.

r	θ
0	$\frac{-\pi}{2}$
$\frac{\pm 5}{\sqrt{2}}$	$\frac{-\pi}{3}$
± 5	0
$\frac{\pm 5}{\sqrt{2}}$	$\frac{\pi}{3}$
0	$\frac{\pi}{2}$



(b) Find the area inside the curve $r^2 = 25\cos\theta$.

Solution:

The area inside one loop will be:

$$\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (5\sqrt{\cos\theta})^2\, d\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 25\cos\theta\, d\theta = \frac{25}{2}\sin\theta|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{25}{2}(1-(-1)) = 25.$$

The total area inside the curve will be $2(25) = 50$.

B U Department of Mathematics

Math 102 Calculus II

Summer 2006 First Midterm

1.) A) Find the equation of the plane through the points $(1,1,-1)$, $(2,0,2)$, $(0,-2,1)$.
B) Find parametric equations of the line perpendicular to this plane at $(4,2,3)$.

Solution:

A) $A(1,1,-1), \qquad B(2,0,2), \qquad C(0,-2,1)$

$\overrightarrow{AB} = (1,-1,3), \qquad \overrightarrow{AC} = (-1,-3,2)$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ -1 & -3 & 2 \end{vmatrix} = 7\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$$

So $\langle 7,-5,-4\rangle$ is normal to the plane. Hence the equation of the plane through the given points is:

$$\begin{aligned} 7(x-1) - 5(y-1) - 4(z+1) &= 0 \\ 7x - 5y - 4z &= 6 \end{aligned}$$

B) $x = 4 + 7, \quad y = 2 - 5t, \quad z = 3 - 4t$

2.) A) Find a unit vector such that the derivative of x^3y^2 along this vector at the point $(2,1)$ is zero. B) Is there a unit vector along which the derivative of x^3y^2 at the point $(2,1)$ is 100? Explain.

Solution:

A) $\nabla f(x,y) = \langle \frac{\partial}{\partial x}(x^3y^2), \frac{\partial}{\partial y}(x^3y^2)\rangle = \langle 3x^2y^2, 2x^3y\rangle$

$\nabla f(2,1) = \langle 12, 16\rangle$

The required unit vector, say $\vec{u} = \langle x, y\rangle$, must satisfy $\vec{u} \cdot \langle 12, 16\rangle = 0$. Let $x = t$, then

$$12t + 16y = 0 \implies y = \frac{-3}{4}t.$$

Also, since $\vec{u}$ is a unit vector we have $\|\vec{u}\| = 1$. So

$$t^2 + \frac{9}{16}t^2 = 1, \qquad t = \pm\frac{4}{5}.$$

Hence we have $\vec{u} = \langle -\frac{4}{5}, \frac{3}{5}\rangle$ or $\vec{u} = \langle \frac{4}{5}, -\frac{3}{5}\rangle$.

B) The derivative along any unit vector $\vec{u}$ at $(2,1)$ is given by

$$\vec{u} \cdot \nabla f(2,1) = \|\vec{u}\|\,\|\nabla f(2,1)\|\cos\theta = 20\cos\theta$$

where θ is the acute angle between $\vec{u}$ and $\nabla f(2,1)$. Clearly $20\cos\theta \leq 20$ so there is no such unit vector.

3.) Find point(s) on the surface $z^2 = xy + 4$ closest to the origin.

Solution:

We want to minimize $f = x^2 + y^2 + z^2$ under the constraint $g = xy - z^2 + 4 = 0$.

$$\nabla f(x, y, z) = \langle 2x, 2y, 2z \rangle, \quad \nabla g(x, y, z) = \langle y, x, -2z \rangle.$$

A point (x_0, y_0, z_0) is an extremum of f only if $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$ for some $\lambda \neq 0$.

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) \implies \begin{aligned} 2x_0 &= \lambda y_0 \\ 2y_0 &= \lambda x_0 \\ 2z_0 &= -2\lambda z_0 \end{aligned}$$

If $z_0 \neq 0$, we have $\lambda = -1$, and so $x_0 = y_0 = 0$ The points on the surface $g(x, y, z) = 0$ satisfying this are $P_1(0, 0, 2)$ and $P_2(0, 0, -2)$.
Otherwise if $z_0 = 0$, we have

$$x_0 = \frac{-4}{y_0}, \quad x_0 = \frac{-8}{\lambda x_0}, \quad x_0^2 = \frac{-8}{\lambda}.$$

Also, $x_0 y_0 = -4$ and $2x_0 = \lambda y_0$ so $x_0^2 = -2\lambda$. It follows that $x_0^4 = 16$ so $x_0 = \pm 2$, and the points on $g(x, y, z) = 0$ satisfying this are $P_3(2, -2, 0)$ and $P_4(-2, 2, 0)$.

Therefore the points closest to the origin on the suface $z^2 = xy + 4$ are P_1 and P_2.

4.) Find the area of the region shared by $r = 2 + 2cos\theta$ and $r = 2 - 2cos\theta$.

Solution:

$$
\begin{aligned}
A &= 4 \cdot \frac{1}{2} \int_0^{\pi/2} r^2 d\theta = 2 \int_0^{\pi/2} (2 + 2\cos\theta)^2 d\theta \\
&= 2 \int_0^{\pi/2} 4 d\theta - 2 \int_0^{\pi/2} 8\cos\theta d\theta + 2 \int_0^{\pi/2} 4\cos^2\theta d\theta
\end{aligned}
$$

We have the identity $\cos^2\theta = \dfrac{\cos 2\theta + 1}{2}$. Hence

$$
\begin{aligned}
A &= 4\pi - 16\sin\theta \Big|_0^{\pi/2} + 2\sin 2\theta \Big|_0^{\pi/2} + 4\theta \Big|_0^{\pi/2} \\
&= 6\pi - 16
\end{aligned}
$$

Boğaziçi University
Department of Mathematics
Math 102 — Calculus II

1	2	3	4	5	Σ
8 points each					40

Date: July 16, 2008	Full Name : KEY
Time: 18:15-19:30	Math 102 Number :
	Student ID :

Summer 2008 — Midterm Examination

IMPORTANT

1. Write your name, surname on top of each page. **2.** The exam consists of 5 questions some of which have more than one part. **3.** Read the questions carefully and write your answers neatly under the corresponding questions. **4.** Show all your work. Correct answers without sufficient explanation might <u>not</u> get full credit. **5.** Calculators are <u>not</u> allowed.

---

1. Find the plane through $P_1(0,0,0)$ and $P_2(1,1,-2)$, parallel to the line of intersection of the planes

$$\underbrace{2x+y-z=3}_{A} \quad \text{and} \quad \underbrace{x+y-z=1}_{B}.$$

The planes A & B have normals

$$n_A = \langle 2,1,-1\rangle \quad \& \quad n_B = \langle 1,1,-1\rangle$$

so that their line of intersection is parallel to

$$v = n_A \times n_B = \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ 1 & 1 & -1 \end{vmatrix} = \langle 0,+1,1\rangle$$

Thus the plane in question has normal

$$n = \overrightarrow{P_1P_2} \times v = \begin{vmatrix} i & j & k \\ 1 & 1 & -2 \\ 0 & 1 & 1 \end{vmatrix} = \langle 3,-1,1\rangle;$$

accordingly its equation is given by

$$\langle x,y,z\rangle \cdot \langle 3,-1,1\rangle = 0$$

or equivalently

$$\boxed{3x-y+z=0}$$

2. The points $P(1,1,4)$ and $Q\left(-\frac{5}{4}, -\frac{5}{4}, \frac{41}{8}\right)$ lie on the paraboloid $z = x^2 + y^2 + 2$.

(a) (2 pts.) Find the equation of the tangent plane to the paraboloid at P.

(b) (2 pts.) Find the equation of the normal line l_P to the paraboloid passing through P.

(c) (2 pts.) Show that the line l_P found above meets the paraboloid again at the point Q.

(d) (2 pts.) Let l_Q be the normal line to the paraboloid passing through Q. If θ is the angle between l_P and l_Q, show that $\sin\theta = \frac{1}{\sqrt{3}}$.

(a) Let $z = f(x,y) = x^2 + y^2 + 2$

Equation of the tangent plane to the paraboloid at P;

$z = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$\Rightarrow z = 4 + 2(x-1) + 2(y-1)$

$\Rightarrow z - 2x - 2y = 0$

(b) The equation of the normal line l_p;

$x = x_0 + f_x(x_0,y_0)t \Rightarrow x = 1+2t$

$y = y_0 + f_y(x_0,y_0)t \Rightarrow y = 1+2t$

$z = f(x_0,y_0) - t \Rightarrow z = 4-t$.

(c) $\frac{41}{8} = \left(\frac{-5}{4}\right)^2 + \left(-5/4\right)^2 + 2$ so Q is on paraboloid

for $t = -9/8$, Q is on l_p.

Thus, l_p meets the paraboloid again at the point Q.

(d) $\sin\theta = \frac{\|n_p \times n_q\|}{\|n_p\|\|n_q\|}$ where n_p and n_q are normal vectors at P and Q

$n_p = \langle 2, 2, -1\rangle$ and $n_q = \langle -5/2, -5/2, -1\rangle$

$n_p \times n_q = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & -1 \\ -5/2 & -5/2 & -1 \end{vmatrix} = \frac{-9}{2}\hat{i} + 9/2\,\hat{j}$

$\Rightarrow \sin\theta = \frac{\frac{9\sqrt{2}}{2}}{3 \cdot \frac{3\sqrt{3}}{\sqrt{2}}} = \frac{1}{\sqrt{3}}$.

3. **Find the unit tangent vector, the unit normal vector**, and the **curvature** for the *plane* curve

$$x = t$$
$$y = -\ln(\cos t)$$

where $-\pi/2 < t < \pi/2$.

Let $r(t) = \langle t, -\ln(\cos t) \rangle$

So, $r'(t) = \langle 1, \frac{\sin t}{\cos t} \rangle = \langle 1, \tan t \rangle$

and $\|r'(t)\| = \sqrt{1+\tan^2 t} = \frac{1}{\cos t}$

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{\langle 1, \tan t \rangle}{1/\cos t} = \langle \cos t, \sin t \rangle$$

$$N(t) = \frac{T'(t)}{\|T'(t)\|} = \frac{\langle -\sin t, \cos t \rangle}{1} = \langle -\sin t, \cos t \rangle$$

$$K(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} \quad \text{where} \quad r'(t) \times r''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & \tan t & 0 \\ 0 & \sec^2 t & 0 \end{vmatrix} = \sec^2 t \, \hat{k}$$

Thus, $K(t) = \dfrac{\sec^2 t}{\left(\frac{1}{\cos t}\right)^3} = \cos t$

4. Suppose that the function $f(x,y)$ satisfies $f(1,1)=1$, $f_x(1,1)=a$ and $f_y(1,1)=b$ for some real numbers a and b. Define $\phi(t)=f(t,f(t,t))$. Find $\phi(1)$ and $\phi'(1)$.

Let $x(t)=t$ & $y(t)=f(t,t)$ so that

$$\phi(t)=f(x(t),y(t)).$$

The chain rule thus yields

$$\phi'(t)=f_x(x(t),y(t))\frac{dx}{dt}+f_y(x(t),y(t))\frac{dy}{dt}. \qquad (1)$$

Clearly, $\frac{dx}{dt}=1$. On the other hand, letting $\tilde{x}(t)=t$ and $\tilde{y}(t)=t$, we get

$$y(t)=f(\tilde{x}(t),\tilde{y}(t))$$

so that by the chain rule

$$\left.\begin{aligned}\frac{dy}{dt}&=f_{\tilde{x}}(\tilde{x}(t),\tilde{y}(t))\frac{d\tilde{x}}{dt}+f_{\tilde{y}}(\tilde{x}(t),\tilde{y}(t))\frac{d\tilde{y}}{dt}\\&=f_x(t,t)+f_y(t,t)\end{aligned}\right\}\ (2)$$

Combining (1) & (2), we get

$$\phi'(t)=f_x(t,f(t,t))+f_y(t,f(t,t))\left[f_x(t,t)+f_y(t,t)\right]$$

and thus

$$\begin{aligned}\phi'(1)&=f_x(1,f(1,1))+f_y(1,f(1,1))\left[f_x(1,1)+f_y(1,1)\right]\\&=f_x(1,1)+f_y(1,1)\left[f_x(1,1)+f_y(1,1)\right]\\&=a+b(a+b)\\&=a+ab+b^2\end{aligned}$$

Note also that

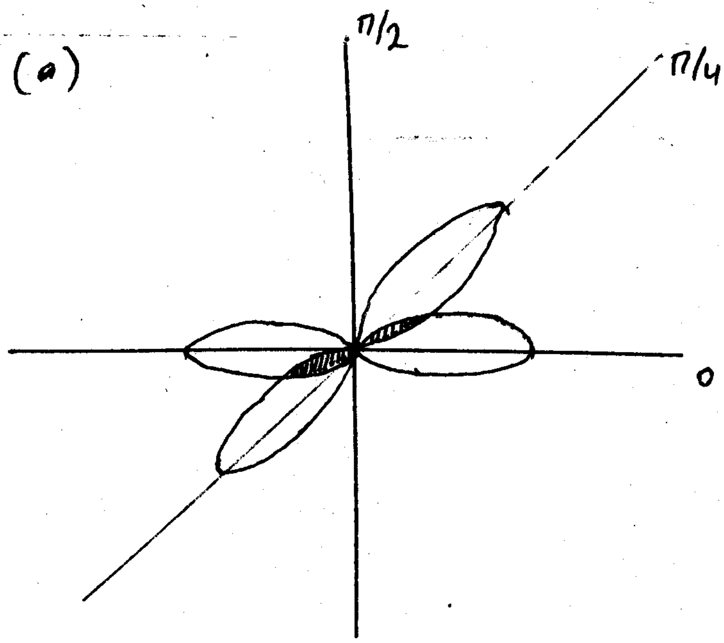
$$\phi(1)=f(1,f(1,1))=f(1,1)=1.$$

5. Given the two lemniscates $r^2 = \cos 2\theta$ and $r^2 = \sin 2\theta$:

(a) (3 pts.) Sketch both curves on the same axes and find all the points of intersection.

(b) (5 pts.) Find the area of the region bounded by and common to both curves.

(a)



intersection pts occur at $\theta = \pi/8$, $\theta = 9\pi/8$ and at the origin.

(b) By Symmetry,

$$\text{Area} = 2 \cdot \left\{ \int_0^{\pi/8} \frac{1}{2} \left(\sqrt{\sin 2\theta}\right)^2 d\theta + \int_{\pi/8}^{\pi/4} \frac{1}{2} \left(\sqrt{\cos 2\theta}\right)^2 d\theta \right\}$$

$$= 2 \cdot \frac{1}{2} \cdot \left\{ \int_0^{\pi/8} \sin 2\theta \, d\theta + \int_{\pi/8}^{\pi/4} \cos 2\theta \, d\theta \right\}$$

$$= \left. \frac{-\cos 2\theta}{2} \right|_{\theta=0}^{\theta=\pi/8} + \left. \frac{\sin 2\theta}{2} \right|_{\theta=\pi/8}^{\theta=\pi/4}$$

$$= \frac{-\sqrt{2}}{4} + \frac{1}{2} + \frac{1}{2} - \frac{\sqrt{2}}{4}$$

$$= 1 - \frac{\sqrt{2}}{2}.$$