B U Department of Mathematics

Math 102 Calculus II

Fall 2000 Final

1. Find the work done by a particle which follows a trajectory $y = 3x$ under a force field $\vec{F}(x,y) = (\cos x \cos y - \sin x \sin y)(\vec{i} + \vec{j})$ from $(0,0)$ to $(\pi/6, \pi/2)$.

Solution:

$W = \int_C \vec{F}.d\vec{r}$, where C is given by $y = 3x$.
$\vec{F}(x,y) = (\cos x \cos y - \sin x \sin y)(\vec{i} + \vec{j}) = M(x,y)\vec{i} + N(x,y)\vec{j}$.
So $M(x,y) = N(x,y) = (\cos x \cos y - \sin x \sin y) = \cos(x+y)$.
So, necessarily $M_y(x,y) = N_x(x,y) \Rightarrow \vec{F}(x,y)$ is conservative.
So there's a potential function ϕ defined as follows:
$\vec{F}(x,y) = \vec{\nabla}\phi(x,y)$, where $\phi_x = \phi_y = \cos(x+y)$. From here,

$$\phi_x = \cos(x+y) \Rightarrow \phi(x,y) = \sin(x+y) + f(y)$$

$$\phi_y = \cos(x+y) \Rightarrow \phi(x,y) = \sin(x+y) + g(x)$$

Equating the two solutions,
$\phi(x,y) = \sin(x+y) + f(y) = \sin(x+y) + g(x) \Rightarrow f(y) = g(x) = c$, a constant.

$$\phi(x,y) = \sin(x+y) + c$$

and since F is conservative,

$$\begin{aligned} W &= \int_C \vec{F}.d\vec{r} = \int_{(x_1,y_1)}^{(x_2,y_2)} \vec{F}(x,y).d\vec{r} = \phi(x_2,y_2) - \phi(x_1,y_1) \\ &= \phi(\pi/6, \pi/2) + c - \phi(0,0) - c = \sin(\tfrac{\pi}{6} + \tfrac{\pi}{2}) - \sin(0+0) \\ &= \sin(\frac{2\pi}{3}) = \frac{\sqrt{3}}{2} \end{aligned}$$

2. Let $\vec{\nabla} f(x(r,\theta), y(r,\theta)) = \dfrac{\partial f}{\partial r}\vec{e_1}(\theta) + \dfrac{1}{r}\dfrac{\partial f}{\partial \theta}\vec{e_2}(\theta)$, where r and θ are polar coordinates. What are $\vec{e_1}(\theta)$ and $\vec{e_2}(\theta)$ in terms of $\vec{i}$ and $\vec{j}$? Are they perpendicular? Give an interpretation for $\vec{e_1}(\theta)$ and $\vec{e_2}(\theta)$. $[Hint: (\arctan x)' = \frac{1}{1+x^2}]$

Solution:

$$\vec{\nabla} f(x,y) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} \,; x = r\cos\theta, y = r\sin\theta \Rightarrow r = \sqrt{x^2+y^2}, \theta = \arctan(\tfrac{y}{x})$$

$$\frac{\partial f}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial f}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial f}{\partial \theta} = \frac{x}{\sqrt{x^2+y^2}}\frac{\partial f}{\partial r} + \frac{(-y/x^2)}{1+(y/x)^2}\frac{\partial f}{\partial \theta} = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}$$

$$\frac{\partial f}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial f}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial f}{\partial \theta} = \frac{y}{\sqrt{x^2+y^2}}\frac{\partial f}{\partial r} + \frac{(1/x)}{1+(y/x)^2}\frac{\partial f}{\partial \theta} = \sin\theta\frac{\partial f}{\partial r} - \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}$$

$$\Rightarrow \vec{\nabla} f(x(r,\theta), y(r,\theta)) = \left[\cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}\right]\vec{i} + \left[\sin\theta\frac{\partial f}{\partial r} - \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}\right]\vec{i}$$

$$\vec{\nabla} f(x(r,\theta), y(r,\theta)) = \frac{\partial f}{\partial r}(\cos\theta\vec{i} + \sin\theta\vec{j}) + \frac{1}{r}\frac{\partial f}{\partial \theta}(-\sin\theta\vec{i} + \cos\theta\vec{j})$$

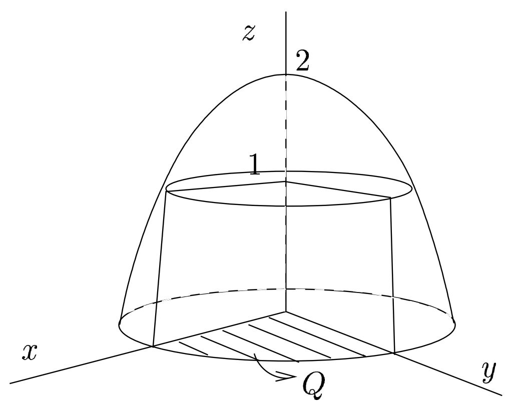
Hence $\vec{e_1}(\theta) = \cos\theta\vec{i} + \sin\theta\vec{j}$ and $\vec{e_2}(\theta) = -\sin\theta\vec{i} + \cos\theta\vec{j}$
$\vec{e_1}(\theta).\vec{e_2}(\theta) = 0 \Rightarrow$They are perpendicular.

These are the basis unit vectors for a system of polar coordinates $\frac{\partial f}{\partial r} \equiv$ rate of change of f with respect to r where θ is kept constant
$\Rightarrow \vec{e_1}(\theta)$ is the radial unit vector along which the change in f occurs with respect to r when θ is kept constant. $\frac{1}{r}\frac{\partial f}{\partial \theta} \equiv$ rate of change of f with respect to θ where r is kept constant
$\Rightarrow \vec{e_2(\theta)}$ is the angular unit vector along which the change in f occurs with respect to θ when r is kept constant.

3. By drawing relevant graphs, write but not evaluate, triple integrals for the volume of the region bounded outside by $z = 2 - x^2 - y^2$ and above by $z = 1$ in the first octant.
(a) in Cartesian coordinates only WITH $dV = dxdydz$. (b) in cylindrical coordinates.

Solution:

(a)

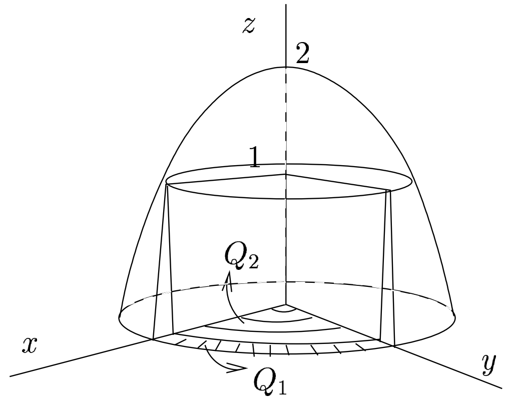


$Q = \{(x,z)|0 \le x \le \sqrt{2-z}, 0 \le z \le 1\}$

$S = \{(x,y,z)|0 \le x \le \sqrt{2-z}, 0 \le y \le \sqrt{2-x^2-z}, 0 \le z \le 1\}$

$$V = \int_o^1 \int_0^{\sqrt{2-z}} \int_0^{\sqrt{2-y^2-z}} dxdydz$$

(b)



The intersection of $z = 1$ and $z = 2 - r^2$ is found by setting $z = 2 - r^2 = 1$, that is, on $z = 1$ plane. When one draws straight lines along z axis, the region S is seen to be the union of two regions S_1 and S_2 along z having different boundaries.
$Q_1 = \{(r,\theta)|0 \le \theta \le \pi/2, 0 \le 1\}$, $Q_2 = \{(r,\theta)|0 \le \theta \le \pi/2, 1 \le r \le \sqrt{2}\}$ and therefore

$$\begin{aligned} S = S_1 \bigcup S_2 &= \{(r,\theta,z)|0 \le \theta \le \pi/2, 0 \le r \le 1, 0 \le z \le 1\} \\ &\bigcup \{(r,\theta,z)|0 \le \theta \le \pi/2, 1 \le r \le \sqrt{2}, 0 \le z \le 2 - r^2\} \end{aligned}$$

so that $V = \displaystyle\int\int\int_{S_1} r dz dr d\theta + \int\int\int_{S_2} r dz dr d\theta$. Finally,

$$V = \int_0^{\pi/2}\int_0^1\int_0^1 r dz dr d\theta + \int_0^{\pi/2}\int_1^{\sqrt{2}}\int_0^{2-r^2} r dz dr d\theta.$$

4. Consider the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{16} = 1$

(a) Find the equation of the plane tangent to the ellipsoid at $P(\frac{2}{\sqrt{3}}, \frac{3}{\sqrt{3}}, \frac{-4}{\sqrt{3}})$ by using the gradient.

(b)Find a normal vector at P without using the gradient. (Do not derive the equation of the tangent plane)

Solution:

(a) A normal $\vec{N}(x,y,z)$to the ellipsoid is:

$$\vec{N}(x,y,z) = \vec{\nabla}(\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16}) = \frac{x}{2}\vec{i} + \frac{2y}{9}\vec{j} + \frac{z}{8}\vec{k}$$

$$\vec{N}(\frac{2}{\sqrt{3}}, \frac{3}{\sqrt{3}}, \frac{-4}{\sqrt{3}}) = \frac{1}{\sqrt{3}}(\vec{i} + \frac{2}{3}\vec{j} - \frac{1}{2}\vec{k}).$$

Hence the tangent plane σ is given by:

$$\sigma : \frac{1}{\sqrt{3}}(\vec{i} + \frac{2}{3}\vec{j} - \frac{1}{2}\vec{k}) \cdot \left[(x - \frac{2}{\sqrt{3}})\vec{i} + (y - \frac{3}{\sqrt{3}})\vec{j} + (z + \frac{4}{\sqrt{3}})\vec{j}\right] = 0$$

$$(x - \frac{2}{\sqrt{3}}) + \frac{2}{3}(y - \frac{3}{\sqrt{3}}) - \frac{1}{2}(z + \frac{4}{\sqrt{3}}) = 0$$

$$\sigma : x + \frac{2}{3}y - \frac{1}{2}z = 2\sqrt{3}.$$

(b)

$$\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}=1 \Rightarrow z=\mp 4\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}$$

Since $P(\frac{2}{\sqrt{3}},\frac{3}{\sqrt{3}},\frac{-4}{\sqrt{3}})$ $z<0$. So one considers the region below the xy plane

$$z=-4\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}$$

So from $\vec{N}_{up}(x,y)=\frac{\partial f}{\partial x}(x,y)\vec{\imath}-\frac{\partial f}{\partial y}(x,y)\vec{\jmath}+\vec{k}=-\vec{N}_{down}(x,y)$

$$\vec{N}_{up}(x,y)=-\frac{x}{\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}}\vec{\imath}-\frac{4y}{9\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}}\vec{\jmath}+\vec{k}$$

$$\Rightarrow \vec{N}_{up}(\frac{2}{\sqrt{3}},\frac{3}{\sqrt{3}})=-2\vec{\imath}-\frac{4}{3}\vec{\jmath}+\vec{k}$$

The normal vector in part a($\times -2\sqrt{3}$) is found.

5. Consider the function $f(x,y)=-\sqrt{2x+1}-\sqrt{8y-3}$.
(a)What are the domain and range of f? Find all critical points.
Can one use the second derivative test in this case?
(b)What is the absolute maximum of the function? By studying infinitesimal translations of it,
verify that it is really a maximum.
(Note:Positivity or negativity of the infinitesimals should be discussed).
Solution:

(a)

$$Domain\{f\}=\{(x,y)|-\frac{1}{2}\leq x,\frac{3}{8}\leq y\}\Rightarrow Range\{f\}=\{f(x,y)|-\infty<f(x,y)\leq 0\}$$

$$\frac{\partial f}{\partial x}=-\frac{1}{\sqrt{2x+1}}\neq 0,\ \frac{\partial f}{\partial x}\text{ is undefined at } x=-\frac{1}{2}$$
$$\frac{\partial f}{\partial y}=-\frac{4}{\sqrt{8y-3}}\neq 0,\ \frac{\partial f}{\partial y}\text{ is undefined at } y=\frac{3}{8}$$

So, $\{(x,y)|x=-\frac{1}{2},y\geq\frac{3}{8}\}$ and $\{(x,y)|x\geq-\frac{1}{2},y=\frac{3}{8}\}$ are the sets of critical points.
Second derivative set cannot be used since the first partial derivatives are undefined.

(b) The value of the function $f(-\frac{1}{2},\frac{3}{8})=0$ is the absolute maximum since $f(x,y)\leq 0$.
Consider translation at $(-\frac{1}{2},\frac{3}{8})$ as $x=-\frac{1}{2}+k$, $y=\frac{3}{8}+h$. Then,

$$f(-\frac{1}{2}+k,\frac{3}{8}+h)=-\sqrt{2(-\frac{1}{2}+k)+1}-\sqrt{8(\frac{3}{8}+h)-3}=-\sqrt{2k}-\sqrt{8h}$$

But since $x\geq-\frac{1}{2}$ and $y\geq\frac{3}{8}$,
$-\frac{1}{2}+k\geq-\frac{1}{2}$ and $\frac{3}{8}+h\geq\frac{3}{8}\Rightarrow h,\ k\geq 0$.
So when $k\geq 0$ and $h\geq 0$,$f(-\frac{1}{2}+k,\frac{3}{8}+h)$ is defined.

$f(-\frac{1}{2}+k,\frac{3}{8}+h)-f(-\frac{1}{2},\frac{3}{8})=-\sqrt{2k}-\sqrt{8h}\le 0$
Which means that every translation from $(-\frac{1}{2},\frac{3}{8})$ gives rise to lower values of f $\Rightarrow$ $f(-\frac{1}{2},\frac{3}{8})=0$ is really a maximum for f.

6. Find the distance from the point $P(1,-1,2)$ to the surface $z=2-x-2y$
   (a) By geometrical considerations (without using Lagrange's method)
   (b) Construct explicitly, but do NOT solve, the Lagrange equations to find the distance. (State clearly the constraint, the function to be extremized etc.)
   Solution:

Take an arbitrary point on the plane, e.g. $Q(0,0,2)$ Then, the distance, say d, is given by: $d=\dfrac{|\vec{N}.\vec{PQ}|}{\vec{N}}$,where $\vec{N}$ is the normal vector for the plane.
$\vec{N}=\vec{i}+2\vec{j}+\vec{k}$, $\vec{PQ}=(0-1)\vec{i}+(0+1)\vec{j}+(2-2)\vec{k}=-\vec{i}+\vec{j}$
$d=\dfrac{|\vec{N}.\vec{PQ}|}{\vec{N}}=\dfrac{1}{\sqrt{6}}.$

(b) Let an arbitrary point on the plan be given by (x,y,z). Then the distance between any point
of the plane and P is given by:
$d(x,y,z)=\sqrt{(x-1)^2+(y+1)^2+(z-2)^2}$ which is the function to be extremized.
Define also $g(x,y,z)=x+2y-z-2=0$ which is the constraint.
The Lagrange equations are given by:

$$\vec{\nabla}d(x,y,z)=\lambda\vec{\nabla}g(x,y,z)$$
$$\vec{\nabla}(\sqrt{(x-1)^2+(y+1)^2+(z-2)^2})=\lambda\vec{\nabla}(x+2y-z-2)$$

$$\frac{x-1}{d}\vec{i}+\frac{y+1}{d}\vec{j}+\frac{z-2}{d}\vec{k}=\lambda(\vec{i}+2\vec{j}+\vec{k})\Rightarrow$$

Lagrange equations:

$$\frac{(x-1)}{\sqrt{(x-1)^2+(y+1)^2+(z-2)^2}}=\lambda$$

$$\frac{(y+1)}{\sqrt{(x-1)^2+(y+1)^2+(z-2)^2}}=2\lambda$$
$$\frac{(z-2)}{\sqrt{(x-1)^2+(y+1)^2+(z-2)^2}}=\lambda$$

With the constraint:

$$x+2y+z-2=0.$$

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Math 102 Calculus II

Fall 2001 Final

1. Show that the lines:

$$\begin{aligned} L_1 &: x+1=4t, y-3=t, z-1=0 \\ L_2 &: x+13=12t, y-1=6t, z-2=3t \end{aligned}$$

intersect and find an equation of the plane they determine.

Solution:

The lines intersect if and only if the following equations can be solved simultaneously for t_1 and t_2:

$$\begin{aligned} 4t_1-1 &= 12t_2-13 \\ t_1+3 &= 6t_2+1 \\ z &= 3t_2+2=1 \end{aligned}$$

Hence $t_2=-1/3, t_1=-4$. The lines intersect at $(-17,-1,1)$.
The vectors $\boldsymbol{v}_1=\langle 4,1,0\rangle$ and $\boldsymbol{v}_2=\langle 4,2,1\rangle$ are parallel to L_1 and L_2 respectively.
So, $\boldsymbol{v}_1\times\boldsymbol{v}_2=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 4 & 1 & 0 \\ 4 & 2 & 1 \end{vmatrix}=\boldsymbol{i}-4\boldsymbol{j}+4\boldsymbol{k}$ is normal to the desired plane. The equation of the plane is

$$1\cdot(x+17)-4(y+1)+4(z-1)=0 \Rightarrow x-4y+4z=-9.$$

2. Find the point on the plane $2x+3y+z-14=0$ nearest to the origin.

Solution:

First way: To minimize $f(x,y,z)=x^2+y^2+z^2$ subject to the constraint $g(x,y,z)=2x+3y+z-14=0$ the following equality must be satisfied:

$$\begin{aligned} \nabla f &= \lambda\nabla g \\ 2x\boldsymbol{i}+2y\boldsymbol{j}+2z\boldsymbol{k} &= \lambda(2\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}) \end{aligned}$$

Hence $\lambda=x, y=\frac{3x}{2}, z=\frac{x}{2}$. Since $2x+3y+z=14\Rightarrow 2x+\frac{9x}{2}+\frac{x}{2}=14$, we get $x=2, y=3, z=1$. The point nearest to the origin is $(2,3,1)$.
Second way: For any point (x,y,z), the distance from the point to the origin is $r=\sqrt{x^2+y^2+z^2}$. Since $z=14-2x-3y$ on the plane, we get $r(x,y)=\sqrt{x^2+y^2+(14-2x-3y)^2}$. Now the absolute minimum of $f(x,y)=r^2=x^2+y^2+(14-2x-3y)^2$: $f_x=10x+12y-56=0$ and $f_y=12x+20y-84=0$ implies $x=2, y=3, z=1$. Hence the point on the plane closest to the origin is $(2,3,1)$.

3. Use a double integral in polar coordinates to find the area of the region outside $r = 2\cos\theta$ and inside the cardioid $r = 1+\cos\theta$. Sketch the region.

Solution:

$$\begin{aligned}
A &= \iint_R dA = 2\left(\int_0^{\pi/2}\int_{2\cos\theta}^{1+\cos\theta} r\,dr\,d\theta + \int_{\pi/2}^{\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta\right)\\
&= \int_0^{\pi/2} r^2\Big|_{2\cos\theta}^{1+\cos\theta} d\theta + \int_{\pi/2}^{\pi} r^2\Big|_0^{1+\cos\theta} d\theta\\
&= \int_0^{\pi/2} ((1+\cos\theta)^2 - 4\cos^2\theta)d\theta + \int_{\pi/2}^{\pi}(1+\cos\theta)^2 d\theta\\
&= (\theta + 2\sin\theta - \frac{3}{2}\theta - \frac{3}{4}\sin 2\theta)\Big|_0^{\pi/2} + (\theta + 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta)\Big|_{\pi/2}^{\pi}\\
&= -\pi/4 + 2 + 3/2\pi - 3/4\pi - 2 = \pi/2
\end{aligned}$$

4. Use a double integral to find the volume under the plane $x+y+z=0$ and over the region R bounded by the x-axis, the line $x=1$ and the parabola $y=x^2$. Sketch the region R.

Solution:

$$\begin{aligned}
V &= \iint_R (x+y)dA = \int_0^1\int_0^{x^2}(x+y)dydx\\
&= \int_0^1 (xy + y^2/2)\Big|_0^{x^2} dx = \int_0^1 (x^3 + x^4/2)dx\\
&= (x^4/4 + x^5/10)\Big|_0^1 = 7/20
\end{aligned}$$

or equivalently

$$\begin{aligned}
V &= \int_0^1\int_{\sqrt{y}}^1 (x+y)dxdy\\
&= \int_0^1 (x^{2/2+xy})\Big|_{\sqrt{y}}^1 dy = \int_0^1(-y^{3/2} + y/2 + 1/2)dy\\
&= (-2/5y^{5/2} + y^2/4 + y/2)\Big|_0^1 = 7/20
\end{aligned}$$

5. Let G be the solid in the first octant bounded by the sphere $x^2+y^2+z^2=4$ and the coordinate planes. Evaluate $\iint_G xyzdV$

(a) using cylindrical coordinates;

(b) using spherical coordinates.

Solution:

$$\begin{aligned}
\int_{\theta=0}^{\pi/2}\int_{r=0}^{2}\int_{z=0}^{\sqrt{4-r^2}} r^3 z\sin\theta\cos\theta dzdrd\theta &= \int_0^{\pi/2}\int_0^2 \frac{1}{2}(4r^3 - r^5)\sin\theta\cos\theta drd\theta\\
&= \frac{8}{5}\int_0^{\pi/2}\sin\theta\cos\theta d\theta = \frac{4}{5}.
\end{aligned}$$

(b)

$$\begin{aligned}
\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{2}\rho^5\sin^3\phi\cos\phi\sin\theta\cos\theta d\rho d\phi d\theta &= \int_0^{\pi/2}\int_0^{\pi/2}\frac{32}{5}\sin^3\phi\cos\phi\sin\theta\cos\theta d\phi d\theta\\
&= \frac{8}{5}\int_0^{\pi/2}\sin\theta\cos\theta d\theta = \frac{4}{5}.
\end{aligned}$$

6. **a)** Let $\phi = \phi(x, y, z)$. Show that $curl(\nabla\phi) = 0$, assuming all derivatives involved exist and are continuous.

Solution:

$$\nabla\phi = \frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k}$$

$$\begin{aligned}
\mathrm{curl}(\nabla\phi) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} \\
&= (\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y})\boldsymbol{i} + (\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z})\boldsymbol{j} + (\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x})\boldsymbol{k} = 0
\end{aligned}$$

assuming equality of mixed second partial derivatives.

b) Let $P = (0, 1), Q = (2, -1), R = (2, 5), S = (4, 1)$ and C be the curve consisting of line segments from P to Q, Q to R and R to S. Evaluate $\int_C (x - y)dx + xydy$.

Solution:

$$\begin{aligned}
C_1 &: r_1(t) = 2t\boldsymbol{i} + (1 - 2t)\boldsymbol{j} \\
C_2 &: r_2(t) = 2\boldsymbol{i} + (6t - 1)\boldsymbol{j} \\
C_3 &: r_3(t) = (2 + 2t)\boldsymbol{i} + (5 - 4t)\boldsymbol{j}
\end{aligned}$$

$$\begin{aligned}
\int_C (x - y)dx + xydy &= \int_{C_1} + \int_{C_2} + \int_{C_3} \\
&= \int_0^1 (8t^2 + 4t - 2)dt + \int_0^1 (72t - 12)dt + \int_0^1 (32t^2 + 4t - 46)dt \\
&= \int_0^1 (40t^2 + 80t - 60)dt = (40\frac{t^3}{3} + 80\frac{t^2}{2} - 60t)\Big|_0^1 = -\frac{20}{3}.
\end{aligned}$$

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Math 102 Calculus II

Fall 2002 Final

1. **(a)** Find the directional derivative of the function $f(x,y,z) = x^3 - xy^2 - z$ at the point $P(1,1,0)$ in the direction of the vector $\boldsymbol{v} = 2\boldsymbol{i} - 3\boldsymbol{j} + 6\boldsymbol{k}$.

   (b) In which direction does f change most rapidly at P and what is the rate of change of f in this direction?

Solution:

(a)

$$\left.\begin{array}{rcl} f_x &=& 3x^2 - y^2 \\ f_y &=& -2xy \\ f_z &=& -1 \end{array}\right\} \Rightarrow (\boldsymbol{\nabla} f)_P = 2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}.$$

A unit vector in the direction of $\boldsymbol{v}$ is

$$\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} = \frac{2\boldsymbol{i} - 3\boldsymbol{j} + 6\boldsymbol{k}}{\sqrt{4+9+36}} = \frac{1}{7}(2\boldsymbol{i} - 3\boldsymbol{j} + 6\boldsymbol{k}).$$

Hence, $\dfrac{df}{ds} = (\boldsymbol{\nabla} f)_P \cdot \boldsymbol{u} = \dfrac{1}{7}(4+6-6) = \dfrac{4}{7}$.

(b) f changes most rapidly along the direction of $(\boldsymbol{\nabla} f)_P = 2\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}$. The maximum rate of change is $\|(\boldsymbol{\nabla} f)_P\| = \sqrt{4+4+1} = 3$.

2. The ellipsoid $x^2 + 2y + 3z^2 = 66$ has two tangent planes parallel to the plane $x + y + z = 1$. Find the equations of planes and the coordinates of their points of tangency.

Solution:

Let $w = x^2 + 2y^2 + 3z^2 - 66$. A normal vector at an arbitrary point is $\boldsymbol{\nabla} w = \langle w_x, w_y, w_z \rangle$.

$$\left.\begin{array}{rcl} w_x &=& 2x \\ w_y &=& 4y \\ w_z &=& 6z \end{array}\right\} \Rightarrow \begin{array}{l} \boldsymbol{N} = x_0\boldsymbol{i} + 2y_0\boldsymbol{j} + 3z_0\boldsymbol{k} \\ \text{is normal to the ellipsiod} \\ \text{at the point } P_0(x_0, y_0, z_0). \end{array}$$

The normal vector of the given plane is $\boldsymbol{v} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$. $\boldsymbol{N}$ must be parallel to $\boldsymbol{v}$ which implies that $\boldsymbol{N} = t\boldsymbol{v}$ for some real number t. Hence:

$$\left.\begin{array}{rcl} x_0 &=& t \\ 2y_0 &=& t \\ 3z_0 &=& t \end{array}\right\} \Rightarrow x_0 = 2y_0 = 3z_0.$$

So let $x_0 = 6s$, $y_0 = 3s$ and $z_0 = 2s$ for some parameter s. Since P_0 is on the ellipse,

$$6s)^2 + 2(3s)^2 + 3(2s)^2 = 66 \Rightarrow (36+18+12)s^2 = 66 \Rightarrow s = \pm 1.$$

The points are $(6, 3, 2)$ and $(-6, -3, -2)$.

The corresponding tangent plane will be of the form $x+y+z = D$. Substituting the coordinates of the points, we find

$$x + y + z = 11 \quad \text{and} \quad x + y + z = -11.$$

3. Given the triple integral

$$\int_0^3 \int_{-\sqrt{9-x^2}}^0 \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} e^{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} dz dy dx$$

(a) Sketch and describe the region of integration.

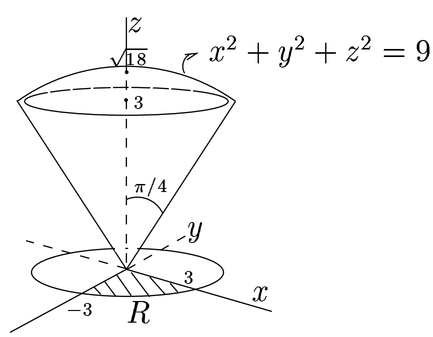
(b) Express this integral using spherical coordinates.

(c) Express this integral using cylindrical coordinates.

(d) Evaluate one of the multiple integrals found in parts **(b)** and **(c)**.

Solution:

(a) Intersection of the hemisphere $z = \sqrt{18 - x^2 - y^2}$ and the cone $z = \sqrt{x^2 + y^2}$ is the curve $x^2 + y^2 = 9$ on the plane $z = 3$. The region is the part of the solid enclosed between the hemisphere and the cone which projects onto R.



(b) Spherical coordinates are given by: $x = \rho \cos\theta \sin\varphi$, $y = \rho \sin\theta \sin\varphi$ and $z = \rho \cos\varphi$. Then the given integral has the representation:

$$\int_{-\pi/2}^{0} \int_0^{\pi/4} \int_0^{\sqrt{18}} \rho^2 \sin\varphi e^{\rho^3} d\rho d\varphi d\theta.$$

(c) Cylindrical coordinates are $x = r\cos\theta$, $y = r\sin\theta$ and $z = z$. In these coordinates the given integral has the form:

$$\int_{-\pi/2}^{0} \int_0^3 \int_r^{\sqrt{18-r^2}} re^{\left(r^2+z^2\right)^{\frac{3}{2}}} dz dr d\theta.$$

The limits for θ can also be taken as $\displaystyle\int_{3\pi/2}^{2\pi}$.

(d) Evaluating **(b)**:

$$\frac{1}{3}\frac{\pi}{2}\int_0^{\pi/4}\int_0^{\sqrt{18}} 3\rho^2 \sin\varphi e^{\rho^3}\, d\rho d\varphi = \frac{\pi}{6}\int_0^{\pi/4} \sin\varphi e^{\rho^3}\Big|_0^{3\sqrt{2}}\, d\varphi$$
$$= \frac{\pi}{6}\left(e^{54\sqrt{2}}-1\right)\int_0^{\pi/4} \sin\varphi d\varphi = \frac{\pi}{6}\left(e^{54\sqrt{2}}-1\right)\cos\varphi\Big|_0^{\pi/4}$$
$$= \frac{\pi}{6}\left(e^{54\sqrt{2}}-1\right)\left(1-\frac{1}{\sqrt{2}}\right).$$

4. Calculate the outward flux of the vector field $\boldsymbol{F}(x,y) = x\boldsymbol{i} + y^2\boldsymbol{j}$ across the square bounded by the lines $x = \mp 1$ and $y = \mp 1$.

Solution:

$$\begin{aligned}\oint_C \boldsymbol{F}\cdot\boldsymbol{n}ds &= \iint_R \boldsymbol{\nabla}\cdot\boldsymbol{F}dA \quad \text{(using the vector form of Green's theorem)}\\ &= \int_{-1}^{1}\int_{-1}^{1}(1+2y)dxdy = \int_{-1}^{1}(1+2y)2dy\\ &= 2\left(y+y^2\right)\Big|_{-1}^{1} = 2(1+1-(-1+1)) = 4\end{aligned}$$

5. Let $\boldsymbol{F} = (1+2xy^2)\boldsymbol{i} + 2x^2y\boldsymbol{j}$.
(a) Determine whether $\boldsymbol{F}$ is a conservative vector field or not; if it is, find a potential function for $\boldsymbol{F}$.
(b) Evaluate the work done by the force $\boldsymbol{F}$ from $(1,2)$ to $(2,5)$ along the parabola $y = x^2+1$.

Solution:

(a) Set $P = 1+2xy^2$ and $Q = 2x^2y$. The field $\boldsymbol{F}$ is conservative if $P_y = Q_x$. Indeed: $P_y = 4xy = Q_x$. So, the given vector field is conservative.
There is, as a consequence, a potential function ϕ which satisfies $\boldsymbol{\nabla}\phi = \boldsymbol{F}$. Let us now find it:

$$\phi_x = 1+2xy^2 \quad \text{and} \quad \phi_y = 2x^2y.$$

Integrating one these say the first one we get $\phi = x + x^2y^2 + a(y)$. Now using the second condition we have $\phi_y = 2x^2y + a'(y) = 2x^2y \Rightarrow a(y) = \text{constant}$. We may always choose $a(y) = \text{constant} = 0$. Hence a potential function is:

$$\phi(x,y) = x^2y^2 + x.$$

(b) Let C denote the curve along $y = x^2+1$ from $(1,2)$ to $(2,5)$. Then the work done by $\boldsymbol{F}$ is:

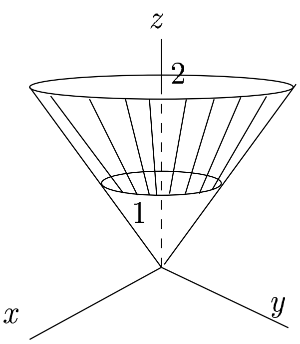
$$W = \int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_C \boldsymbol{\nabla}\phi\cdot d\boldsymbol{r}.$$

This then equals, by the fundamental theorem of line integrals:

$$\begin{aligned}W &= \phi(\text{initial point}) - \phi(\text{end point})\\ &= \phi(2,5) - \phi(1,2) = 102 - 5 = 97.\end{aligned}$$

6. Evaluate the surface integral $\iint_S (\ln z)dS$ where S is the part of the cone $z = \sqrt{x^2+y^2}$ between the planes $z = 1$ and $z = 2$. Sketch the region of integration.

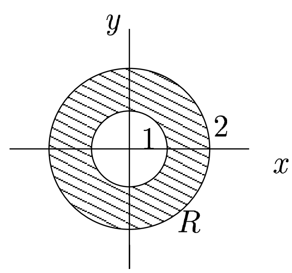
Solution:



We first compute the necessary partial derivatives $z^2 = x^2 + y^2 \Rightarrow 2z\dfrac{\partial z}{\partial x} = 2x \Rightarrow \dfrac{\partial z}{\partial x} = \dfrac{x}{z}$. Similarly, $\dfrac{\partial z}{\partial y} = \dfrac{y}{z}$. Hence,

$$\begin{aligned} dS &= \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \sqrt{1 + \frac{x^2+y^2}{z^2}}\, dA \\ &= \sqrt{\frac{x^2+y^2+z^2}{z^2}}\, dA = \sqrt{\frac{2(x^2+y^2)}{x^2+y^2}}\, dA = \sqrt{2}dA \end{aligned}$$

Then, $I = \iint_S (\ln z)dS = \sqrt{2}\iint_R \ln\left(x^2+y^2\right)^{1/2} dydx.$



We change to polar coordinates to obtain the double integral

$$I = \sqrt{2}\int_0^{2\pi}\int_1^2 r\ln r dr d\theta,$$

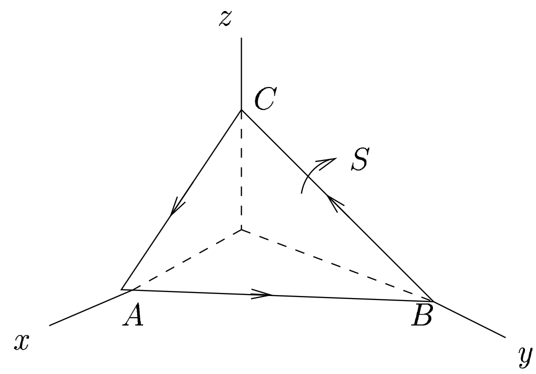
and then integrate by parts by using:

$$\begin{aligned} u &= \ln r & dv &= r dr \\ du &= \frac{dr}{r} & v &= \frac{r^2}{2} \end{aligned}$$

Hence,

$$\begin{aligned} I &= 2\pi\sqrt{2}\int_1^2 r\ln r dr = 2\pi\sqrt{2}\left(\frac{r^2}{2}\ln r\Big|_1^2 - \int_1^2 \frac{1}{2} r dr\right) \\ &= 2\pi\sqrt{2}\left(2\ln 2 - \frac{r^2}{4}\Big|_1^2\right) = 2\pi\sqrt{2}\left(2\ln 2 - \frac{3}{4}\right). \end{aligned}$$

7. Evaluate the integral $\oint_{\mathbf{C}} \boldsymbol{G} \cdot d\boldsymbol{R}$ where C is the triangle with vertices $A(1,0,0)$, $B(0,2,0)$ and $C(0,0,1)$ oriented as shown, and $\boldsymbol{G}$ is the vector field $\boldsymbol{G}(x,y,z) = y^2\boldsymbol{i} + z\boldsymbol{j} + \boldsymbol{k}$.



Solution:

Using the Stokes' Theorem:

$$\oint_C \boldsymbol{G} \cdot d\boldsymbol{R} = \iint_S (\boldsymbol{\nabla} \times \boldsymbol{G}) \cdot \boldsymbol{n}\, dS$$

where

$$\boldsymbol{\nabla} \times \boldsymbol{G} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & z & 1 \end{vmatrix}$$

Let S be the plane through the given points, with normal

$$\boldsymbol{AB} \times \boldsymbol{BC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 2 & 0 \\ 0 & -2 & 1 \end{vmatrix} = 2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}.$$

The unit normal, oriented correctly, is

$$\boldsymbol{n} = \frac{2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}}{\sqrt{9}} = \frac{1}{3}(2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}).$$

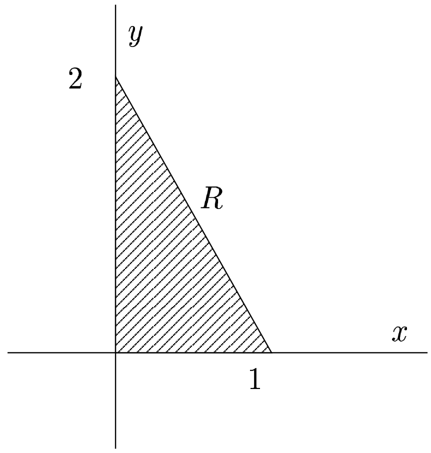
(it will have a positive $z-$component).

$$(\boldsymbol{\nabla} \times \boldsymbol{G}) \cdot \boldsymbol{n} = \frac{1}{3}(-2 - 4y).$$

The equation of the plane S is $2(x-1) + y + 2z = 0$. Hence:

$$2 + 2\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -1 \quad \text{and} \quad 1 + 2\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{1}{2}$$

Finally, $dS = \sqrt{1 + \left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2}\, dA = \sqrt{1 + 1 + \dfrac{1}{4}}\, dA = \dfrac{3}{2}\, dA.$



So,

$$\begin{aligned}
\int_S (\boldsymbol{\nabla} \times \boldsymbol{G}) \cdot \boldsymbol{n}\, dS &= \frac{1}{3} \cdot \frac{3}{2} \iint_R (-2 - 4y) dA = -\frac{1}{2} \cdot 2 \int_0^1 \int_0^{2-2x} (1 + 2y)\, dy dx \\
&= -\int_0^1 (y + y^2)\big|_0^{2-2x}\, dx = -\int_0^1 \left[(2 - 2x + (2 - 2x)^2)\right] dx \\
&= \int_0^1 \left[-4x^2 + 10x - 6\right] dx = \left[-\frac{4}{3}x^3 + 5x^2 - 6x\right]_0^1 \\
&= -\frac{4}{3} + 5 - 6 = -\frac{7}{3}.
\end{aligned}$$

B U Department of Mathematics

Math 102 Calculus II

Fall 2003 Final

1. Show that the lines
$L_1 : x = -2 + t, y = 3 + 2t, z = 4 - t,\ L_2 : x = 3 - t, y = 4 - 2t, z = t$
(a) are parallel.
(b) Find an equation of the plane they determine.

Solution:

(a) $\boldsymbol{v_1} =< 1, 2, -1 >$ is parallel to L_1, $\boldsymbol{v_2} =< -1, -2, 1 >$ is parallel to L_2 and $\boldsymbol{v_1} = -\boldsymbol{v_2}$ so, $L_1 \parallel L_2$.
(b) Let $t = 0$ to find the points $\boldsymbol{P_1}(-2, 3, 4)$, $\boldsymbol{P_2}(3, 4, 0)$ that lie, respectively, on the given lines. $\boldsymbol{P_1P_2} =< 5, 1, -4 >$ and
$\boldsymbol{v_1} \times \boldsymbol{P_1P_2} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & -1 \\ 5 & 1 & -4 \end{vmatrix} = -7\boldsymbol{i} - \boldsymbol{j} - 9\boldsymbol{k}$ is normal to the required plane. $-7(x - 3) - (y - 4) - 9z = 0$
$-7x + 21 - y + 4 - 9z = 0$
$-7x - y - 9z = -25$ or $7x + y + 9z = 25$

2. Find an equation for the plane tangent to the surface $x^2 + 4y^2 + z^2 = 12$ at the point $(2, 1, 2)$.

Solution:

Let $f(x, y, z) = x^2 + 4y^2 + z^2$
$\nabla f(x, y, z) = 2x\boldsymbol{i} + 8y\boldsymbol{j} + 2z\boldsymbol{k}$
$\nabla f(2, 1, 2) = 4\boldsymbol{i} + 8\boldsymbol{j} + 4\boldsymbol{k} = 4(\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$
Use $\boldsymbol{n} = \boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ as the normal vector for the plane. The equation of the tangent plane is
$(x - 2) + 2(y - 1) + (z - 2) = 0$
$x - 2 + 2y - 2 + z - 2 = 0$
$x + 2y + z = 6$

3. Find the work done on a particle by the force field $\boldsymbol{F}(x, y) = (x - y)\boldsymbol{i} + xy\boldsymbol{j}$ in moving a particle counterclockwise around the ellipse $9x^2 + 4y^2 = 36$ from $(2, 0)$ to $(-2, 0)$.

Solution:

$9x^2 + 4y^2 = 36 \Rightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1$. Parametric equations of the specified part of the ellipse is $x = 2\cos t$, $y = 3\sin t$, $0 \leq t \leq \pi$.
$\boldsymbol{r}(t) = 2\cos t\boldsymbol{i} + 3\sin t\boldsymbol{j}$
$\boldsymbol{r}'(t) = -2\sin t\boldsymbol{i} + 3\cos t\boldsymbol{j}$
$\boldsymbol{F} \cdot \boldsymbol{r}(t) = [(2\cos t - 3\sin t)\boldsymbol{i} + (6\cos t\sin t)\boldsymbol{j}][-2\sin t\boldsymbol{i} + 3\cos t\boldsymbol{j}] = -4\cos t\sin t + 6\sin^2 t + 18\cos^2 t\sin t$
$W = \int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^\pi (-4\cos t\sin t + 3 - 3\cos 2t + 18\cos^2 t\sin t)dt$
$= (-2\sin^2 t + 3t - \frac{3}{2}\sin 2t - 6\cos^3 t)|_0^\pi = 3\pi + 12$

4. State the volume enclosed by $x^2 + y^2 + z^2 = a^2$ by triple integrals using
(a) cylindrical coordinates,
(b) spherical coordinates,
(c) evaluate ONE of the triple integrals found in (a) and (b).

Solution:

(a) $V = 2\int_0^{2\pi}\int_0^a\int_0^{\sqrt{a^2-r^2}} rdzdrd\theta$
$= 2\int_0^{2\pi}\int_0^a (rz)|_0^{\sqrt{a^2-r^2}} drd\theta$
$= 2\int_0^{2\pi}\int_0^a r\sqrt{a^2-r^2}drd\theta$
$= \frac{-2}{2}2\int_0^{2\pi}\frac{2}{3}\sqrt{(a^2-r^2)^3}|_0^a d\theta$
$= \frac{2}{3}a^3\int_0^{2\pi} d\theta$
$= \frac{2}{3}a^3\theta|_0^{2\pi} = \frac{4\pi a^3}{3}$.
(b)$V = \int_0^{2\pi}\int_0^{\pi}\int_0^a \rho^2 \sin\phi d\rho d\phi d\theta$
$= \int_0^{2\pi}\int_0^{\pi} \sin\phi \left(\frac{\rho^3}{3}\right)\Big|_0^a d\phi d\theta$
$= \frac{a^3}{3}\int_0^{2\pi}\int_0^{\pi} \sin\phi d\phi d\theta$
$= \frac{a^3}{3}\int_0^{2\pi}\int_0^{\pi}(-\cos\phi)|_0^{\pi} d\phi d\theta$
$= \frac{2a^3}{3}\int_0^{2\pi} d\theta = \frac{2a^3}{3}\int_0^{2\pi}\theta|_0^{2\pi}$
$= \frac{4\pi a^3}{3}$.

5. Let $\boldsymbol{F}(x,y) = (ye^{xy}-1)\boldsymbol{i} + xe^{xy}\boldsymbol{j}$. (a) Determine whether or not the vector field $\boldsymbol{F}$ is conservative.
(b) If $\boldsymbol{F}$ is conservative, find a potential function for $\boldsymbol{F}$.

Solution:

(a) Let $f(x,y) = ye^{xy} - 1$ and $g(x,y) = xe^{xy}$,
$\frac{\partial f}{\partial y} = e^{xy} + xye^{xy}$, $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ and $\boldsymbol{F}(x,y)$ is conservative.
(b) $\frac{\partial \phi}{\partial x} = ye^{xy} - 1 \Rightarrow \phi(x,y) = \int(ye^{xy}-1)dx = y\frac{1}{y}e^{xy} - x + k(y) = e^{xy} - x + k(y)$.
$\frac{\partial \phi}{\partial y} = xe^{xy} + k'(y) = g(x,y) = xe^{xy} \Rightarrow k'(y) = 0 \Rightarrow k(y) = c$, (constant) $\Rightarrow \phi(x,y) = e^{xy} - x + c$.

6. Let C be the unit circle oriented counterclockwise. Evaluate the line integral $\oint_C ydx + xdy$ using Green's Theorem and check the result by evaluating it directly.

Solution:

Let R denote a domain in $\mathbb{R}^2$ with boundary the curve C. Green's Theorem states that

$$\oint_C f(x,y)dx + g(x,y)dy = \iint_R \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}dA.$$

But $\displaystyle\int\int_R (1-1)dA = 0$ because for $f(x,y) = y$, $g(x,y) = x$ and $\frac{\partial g}{\partial x} = 1$, $\frac{\partial f}{\partial y} = 1$, $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 1 - 1 = 0$.
For the line integral, let $x = \cos t$, $y = \sin t$,
$\oint ydx + xdy = \int_0^{2\pi} \sin t(-\sin t)dx + \cos t \cdot \cos tdt$

$\int_0^{2\pi}(-\sin^2 t+\cos^2 t)dt = -\int_0^{2\pi}\frac{1-\cos 2t}{2}dt + \int_0^{2\pi}\frac{1+\cos 2t}{2}dt = -(\frac{t}{2}-\frac{1}{4}\sin 2t)|_0^{2\pi} + (\frac{t}{2}+\frac{\sin 2t}{4})|_0^{2\pi} = -\pi+\pi = 0.$

B U Department of Mathematics

Math 102 Calculus II

Fall 2004 Final

1.) Find and classify all the critical points of

$$F(x,y) = \frac{1}{3}x^3 - x + xy^2 \,.$$

Solution:

First order partial derivatives are

$$F_x = x^2 - 1 + y^2 \,, \qquad F_y = 2xy$$

Equating these to zero we see that there are 4 critical points: $(0,1)$, $(0,-1)$, $(1,0)$, $(-1,0)$.

Next, we calculate

$$D(x,y) = F_{xx}F_{yy} - F_{xy}^2 = 4x^2 - 4y^2$$

Since $D(0,1) = D(0,-1) = -4 < 0$, $(0,1)$ and $(0,-1)$ are saddle points.

We have $D(1,0) = D(-1,0) = 4 > 0$ and $F_{xx}(1,0) = 2 > 0$, $F_{xx}(-1,0) = -2 < 0$, therefore $(1,0)$ is a local minimum and $(-1,0)$ is a local maximum.

2.) a) At the point (1,2) the directional derivative of $F(x,y)$ is $2\sqrt{2}$ toward $P_1(2,3)$ and -3 toward $P_2(1,0)$. Find the directional derivative of $F(x,y)$ at $(1,2)$ toward $P_3(4,6)$.

Solution:

The unit vector at (1,2) toward $P_1(2,3)$ is $\vec{u} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$. The directional derivative of $F(x,y)$ at $(1,2)$ toward $P_1(2,3)$ is

$$D_{\vec{u}}F(1,2) = \vec{\nabla}F(1,2).\vec{u} = \frac{F_x(1,2) + F_y(1,2)}{\sqrt{2}} = 2\sqrt{2}\,,$$

which implies $F_x(1,2) + F_y(1,2) = 4$.

Now, the unit vector from (1,2) toward $P_2(1,0)$ is $\vec{v} = -\mathbf{j}$.

$$D_{\vec{v}}F(1,2) = \vec{\nabla}F(1,2).\vec{v} = -F_y(1,2) = -3\,,$$

Since $F_x(1,2) + F_y(1,2) = 4$, we get $F_y(1,2) = 3$ and $F_x(1,2) = 1$ which implies $\vec{\nabla}F(1,2) = (\mathbf{i} + 3\mathbf{j})$.

The unit vector from (1,2) toward $P_3(4,6)$ is $\vec{w} = (3\mathbf{i} + 4\mathbf{j})/5$. The directional derivative is

$$D_{\vec{w}}F(1,2) = \vec{\nabla}F(1,2).\vec{w} = 3$$

b) A ball is put on the surface $z = 2e^{-x-4y}$ at the point $(0,0,2)$. Along which direction in the xy-plane will it roll?

Solution:

The ball will roll in the direction in which the surface is falling away at the fastest rate, i.e., $-\vec{\nabla}z$.

$$z_x = -2e^{-x-4y}, \qquad z_y = -8e^{-x-4y}$$

Using these we find

$$\vec{\nabla}z(0,0) = -2\mathbf{i} - 8\mathbf{j}$$

Therefore, the ball will roll in the direction $-\vec{\nabla}z(0,0) = \mathbf{2i} + 8\mathbf{j}$.

3.) Find the plane that is tangent to the surface $x^2 - y^2 + 3z = 0$. This plane contains the point $(0,0,1)$ and it is parallel to the line

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}.$$

Solution:

Since the plane we are looking for contains the point (0,0,1) its equation is of the form

$$A(x-0) + B(y-0) + C(z-1) = 0,$$

where A, B, C are real numbers and a normal of this plane is $\vec{n_1} =< A, B, C >$. Let this plane be tangent to the surface $x^2 - y^2 + 3z = 0$ at the point (m, n, p). At this point the normal of the surface is $\vec{n_2} =< 2m, -2n, 3 >$ and it is parallel to n_1, i.e., $\vec{n_1} = k\vec{n_2}$ for some real number k which can be chosen to be 1 without loss of generality. Therefore, we have $A = 2m, B = -2n$ and $C = 3$.

The line given above is parallel to the vector $< 2, 1, -2 >$ and its dot product with the plane normal $< A, B, C >$ should be zero:

$$< 2, 1, -2 > . < 2m, -2n, 3 >= 0,$$

which implies $2m - n = 3$. Since the point (m, n, p) is contained in the surface and the plane we also have

$$m^2 - n^2 + 3p = 0\,, \qquad\qquad 2m^2 - 2n^2 + 3(p-1) = 0$$

Solving these three equations simultaneously we find that $p = -1, m = 2, n = 1$. Thus, the equation of the plane is

$$4x - 2y + 3(z-1) = 0$$

4.) Find the surface area of the sphere $x^2 + y^2 + z^2 = 4$ above $z = 1$.

Solution:

We have $z = \pm\sqrt{4 - x^2 - y^2}$. Since we want to find the area above the line $z = 1$ we need to have z positive and therefore we define

$$F(x, y) = \sqrt{4 - x^2 - y^2}$$

whise first order partial derivatives are

$$F_x = -\frac{x}{\sqrt{4 - x^2 - y^2}}\,, \qquad\qquad F_y = -\frac{y}{\sqrt{4 - x^2 - y^2}}$$

Now, the surface area is

$$A = \int\int_R \sqrt{F_x^2 + F_y^2 + 1}\, dA = \int\int_R \frac{2}{\sqrt{4 - x^2 - y^2}} dA\,,$$

where R is the region on the xy-plane defined by $(x^2 + y^2) \leq \sqrt{3}$. Converting this double integral into polar coordinates

$$A = \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{2r}{\sqrt{4 - r^2}} dr d\theta$$

from which we find $A = 4\pi$.

5.) a) Using spherical coordinates express the volume of the solid region that is inside the sphere $x^2 + y^2 + z^2 = 4$ and outside the cylinder $x^2 + y^2 = 1$ in the first octant. (Do not evaluate the integral.)

Solution:

In spherical coordinates the cylinder $x^2 + y^2 = 1$ becomes $\rho = (1/\sin\phi)$ and the sphere $x^2 + y^2 + z^2 = 4$ is $\rho = 2$. Solving these two equations simultaneously we find that these two solid regions intersect at $\phi = \pi/6$ and $\phi = 5\pi/6$. Since the volume is in the first octant the limits for the angle ϕ are $\pi/6 \leq \phi \leq \pi/2$. The angle θ should satisfy $0 \leq \theta \leq \pi/2$. For ρ we have $1/\sin\phi \leq \rho \leq 2$.

Therefore the volume is

$$V = \int\int\int_R dV = \int_0^{\pi/2} \int_{\pi/6}^{\pi/2} \int_{1/\sin\phi}^{2} \rho^2 \sin\phi d\rho d\phi d\theta$$

5.) b) Express the volume of the region R bounded by the parabolic cylinder $x = y^2$ and the planes $z = 0$ and $x + z = 1$ as a triple integral in the orders $dzdxdy$ and $dydzdx$. (Do not evaluate the integrals.)

Solution:

The projection of the region on the xy-plane is $x = y^2$, and $0 \le x \le 1$. So, the volume is

$$V = \int_{-1}^{1} \int_{y^2}^{1} \int_{0}^{1-x} dzdxdy$$

The projection of the region on the zx-plane is $x + z = 1$ and $0 \le x \le 1$ which implies

$$V = \int_{0}^{1} \int_{0}^{1-x} \int_{-\sqrt{x}}^{\sqrt{x}} dydzdx$$

5.) c) Express the following triple integral in rectangular coordinates in the order $dzdydx$.(Do not evaluate the integral.)

$$I = \int_{0}^{\pi/2} \int_{1}^{\sqrt{3}} \int_{1}^{\sqrt{4-r^2}} z^2 r^3 \sin\theta \cos\theta dzdrd\theta$$

Solution:

$$I = \int_{0}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{3-x^2}} \int_{1}^{\sqrt{4-x^2-y^2}} xyz^2 dzdydx + \int_{1}^{\sqrt{3}} \int_{0}^{\sqrt{3-x^2}} \int_{1}^{\sqrt{4-x^2-y^2}} xyz^2 dzdydx$$

6.) Let R be the region in xyz-space defined by the inequalities $1 \le x \le 2,\ \ 0 \le xy \le 2,\ \ 0 \le z \le 1$. Evaluate the integral below by first applying the transformation $u = x, \quad v = xy, \quad w = 3z$.

$$\iiint_R (x^2y + 3xyz)dxdydz\,.$$

Solution:

The inverse transformation is $x = u$, $y = v/u$ and $z = w/3$. The region is mapped into $1 \le u \le 2, \quad 0 \le v \le 2, \quad 0 \le w \le 3$, and calculating the Jacobian we find $J = 1/(3u)$ which is positive. Therefore the integral becomes,

$$\begin{aligned} I &= \int_{1}^{2} \int_{0}^{3} \int_{0}^{2} (uv + vw)\frac{1}{3u} dvdwdu \\ &= \int_{1}^{2} \int_{0}^{3} (2u + 2w)\frac{1}{3u} dwdu \\ &= \int_{1}^{2} (6u + 9)\frac{1}{3u} du \\ &= 2 + 3\ln 2 \end{aligned}$$

7.) Let R be the region between the line $y = x$ and the parabola $y = x^2$ and let C be its boundary swept out counterclockwise. Find the work done by the force field $\vec{F} = (-y)\mathbf{i} + (xy)\mathbf{j}$ along C

a) Without using Green's theorem.

Solution:

The work done by $\vec{F}$ is

$$W = \int_C \vec{F}.d\vec{r} = \int_{C_1} (xydy - ydx) + \int_{C_2} (xydy - ydx)\,,$$

where C_1 is the curve defined by $y = x^2$ and C_2 is the line $y = x$. We can parametrize C_1 as $x = t$ and $y = t^2$ with $t \,\epsilon\, [0, 1]$. C_2 can be parametrized as $x = y = t$ with $t \,\epsilon\, [1, 0]$. Using these parametrizations the work integral becomes

$$W = \int_0^1 (2t^4 - t^2)dt + \int_1^0 (t^2 - t)dt = \frac{7}{30}$$

b) Using Green's theorem.

Solution:

Let $\vec{F} = P\mathbf{i} + Q\mathbf{j}$, where $P = -y$ and $Q = xy$. Using Green's theorem we get

$$W = \int_0^1 \int_{x^2}^{x} (Q_x - P_y)dydx = \int_0^1 \int_{x^2}^{x} (y + 1)dydx = \frac{7}{30}$$

8.) Show that the following vector field is conservative.

$$\vec{F} = (2xy + z^2)\mathbf{i} + (x^2)\mathbf{j} + (2xz + \cos z)\mathbf{k}$$

a) without finding a potential for $\vec{F}$.

Solution:

Let $\vec{F} = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$, where $P(x, y, z) = (2xy + z^2)$, $Q(x, y, z) = x^2$ and $R(x, y, z) = (2xz + \cos z)$. $\vec{F}$ is conservative when all of the following conditions are satisfied:

$$P_y = Q_x\,,\; P_z = R_x\,,\; Q_z = R_y$$

Since we have $P_y = Q_x = 2x$, $P_z = R_x = 2z$, $Q_z = R_y = 0$, $\vec{F}$ is conservative

b) by finding a potential for $\vec{F}$.

Solution:

Let $\vec{F} = \vec{\nabla}\phi(x, y, z) = \phi_x\mathbf{i} + \phi_y\mathbf{j} + \phi_z\mathbf{k}$ where $\phi(x, y, z)$ is a scalar function. Then, we have

$$
\begin{aligned}
\phi_x &= 2xy + z^2 \\
\phi_y &= x^2 \\
\phi_z &= 2xz + \cos z
\end{aligned}
$$

Integrating the first equation we find $\phi = x^2y+z^2x+G(y,z)$ where $G(y,z)$ is an arbitrary function of y and z. Now using this in the second equation we obtain $G_y(y,z) = 0$ which means that G is only function of z. Finally, using the third equation we see that $G_z = \cos z$ which gives $G = \sin z+\phi_0$ where ϕ_0 is an arbitrary real constant. Therefore, the potential is

$$\phi = x^2y + z^2x + \sin z + \phi_0$$

B U Department of Mathematics

Math 102 Calculus II

Fall 2005 Final

1. **(a)** Find an equation for the plane through $A(0,0,1)$, $B(2,0,0)$, and $C(0,3,0)$.
   (b) Find the cosine of the angle between this plane and xy-plane.
   (c) Find the area of the triangle ABC.

Solution:

(a) We find a vector normal to the plane and use it with one of the points (it doesn't matter which) to write an equation for the plane.

$$\begin{aligned} \overrightarrow{AB} &= <2-0, 0-0, 0-1> &= <2,0,-1> \\ \overrightarrow{AC} &= <0-0, 3-0, 0-1> &= <0,3,-1> \end{aligned}$$

The cross product

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3\vec{i} + 2\vec{j} + 6\vec{k}$$

is normal to the plane. By using components of this vector and coordinates of the point (0,0,1) we can write the equation of the plane as

$$\begin{aligned} 3(x-0) + 2(y-0) + 6(z-1) &= 0 \\ 3x + 3y + 6z &= 6 \end{aligned}$$

(b) The angle between two intersecting planes is defined to be the (acute) angle determined by their normal vectors.
The vectors $\vec{n_1} =< 0,0,1 >$ and $\vec{n_2} =< 3,2,6 >$ are normal vectors for xy-plane and the plane $3x + 2y + 6z = 6$, respectively. To find the cosine of the angle between these planes, let's take the dot product of these vectors:

$$\begin{aligned} \vec{n_1} \cdot \vec{n_2} &= ||\vec{n_1}|| \cdot ||\vec{n_2}|| \cdot \cos\theta \\ < 0,0,1 > \cdot < 3,2,6 > &= 1 \cdot \sqrt{9+4+36} \cdot \cos\theta \\ &\Rightarrow \cos\theta = \frac{6}{\sqrt{49}} = \frac{6}{7} \end{aligned}$$

(c) The area A of the triangle is half the area of the parallelogram determined by the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. In the (a) part, we calculated,

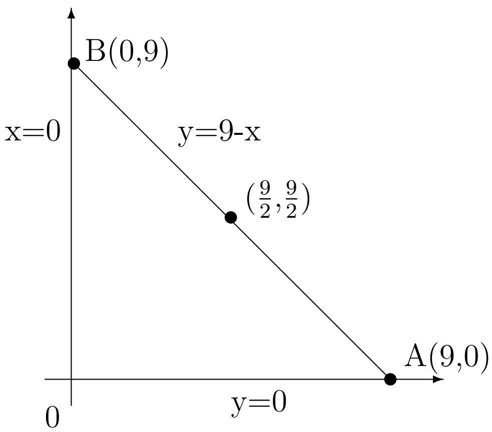
$$\begin{aligned} \overrightarrow{AB} \times \overrightarrow{AC} &= 3\vec{i} + 2\vec{j} + 6\vec{k} \\ \text{and } ||\overrightarrow{AB} \times \overrightarrow{AC}|| &= \sqrt{9+4+36} = 7 \\ \text{and consequently } A &= \frac{1}{2}||\overrightarrow{AB} \times \overrightarrow{AC}|| = \frac{7}{2} \end{aligned}$$

2. Find the absolute maximum and minimum values of

$$f(x,y) = 2 + 2x + 2y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0,\ y = 0,\ y = 9 - x$.

Solution:



Since f is differentiable, the only places where f can take these values are points inside the triangle where $f_x = f_y = 0$ and points on the boundary.

Interior points : For these we have $f_x = 2 - 2x = 0, f_y = 2 - 2y = 0$ which gives the single point $(x,y) = (1,1)$. The value of f there is $f(1,1) = 4$

Boundary points : We take the triangle one side at a time:

1. On the segment OA, y =0. The function $f(x,y) = f(x,0) = 2 + 2x - x^2$ may now be regarded as a function of x defined on the closed interval $0 \le x \le 9$. Its extreme values may occur at endpoints

$$\begin{array}{lll} x = 0 & \text{where} & f(0,0) = 2 \\ x = 9 & \text{where} & f(9,0) = -61 \end{array}$$

and at the interior points where $f'(x,0) = 2 - 2x = 0$. The only interior point where $f'(x,0) = 0$ is $x = 1$ where $f(x,y) = f(1,0) = 3$.

2. On the segment OB, x =0 and $f(x,y) = f(0,y) = 2 + 2y - y^2$ may be regarded as a function of y defined on the closed interval $0 \le y \le 9$. Its extreme values may occur at endpoints

$$\begin{array}{lll} y = 0 & \text{where} & f(0,0) = 2 \\ y = 9 & \text{where} & f(0,9) = -61 \end{array}$$

and at the interior points where $f'(0,y) = 2 - 2y = 0$. The only interior point where $f'(0,y) = 0$ is $y = 1$ where $f(0,1) = 3$.

3. On the segment AB, We have already looked at the values of f at the endpoints of AB, so we need only look at the interiorpoints of AB. With $y = 9 - x$, we have

$$f(x,y) = f(x, 9-x) = 2 + 2x + 2(9-x) - x^2 - (9-x)^2 = -61 + 18x - 2x^2$$

Setting $f'(x, 9-x) = 18 - 4x = 0$ gives $x = \dfrac{9}{2}$

$$y = 9 - \frac{9}{2} = \frac{9}{2} \text{ and } f(x,y) = f\left(\frac{9}{2}, \frac{9}{2}\right) = -\frac{41}{2}$$

Finally, we list all candidates: $4, 2, -61, 3, -\dfrac{41}{2}$.

The maximum is 4, which f takes at $(1,1)$.
The minimum is -61, which f takes at $(0,9)$ and $(9,0)$.

3. Evaluate the following integrals:

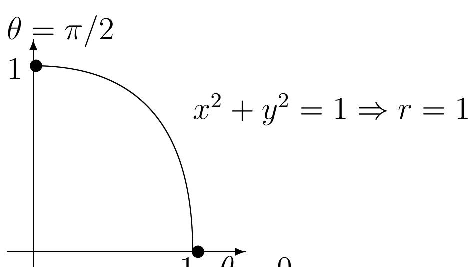
(1) $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$

(2) $\displaystyle\iint_R e^{x^2+y^2}\,dy\,dx$

where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$.

Solution:

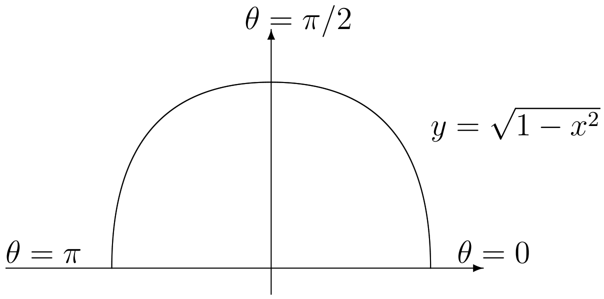
(1) It is easier to evaluate this double integral in polar coordinates. In order to determine limits of integration let's sketch the region of integration.



Substituting $x = r\cos\theta, y = r\sin\theta$, and replacing $dx\,dy$ by $r\,dr\,d\theta$, we get

$$\begin{aligned}
\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx &= \int_0^{\pi/2} \int_0^1 r^2\, r\,dr\,d\theta \\
&= \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^1 d\theta \\
&= \int_0^{\pi/2} \frac{1}{4}\,d\theta = \frac{\pi}{8}
\end{aligned}$$

(2) In Cartesian coordinates, the integral in question is a non-elementary integral and there is no direct way to integrate $e^{x^2+y^2}$ with respect to x or y. Again we will apply polar coordinates.



The region R is substituting $x = r\cos\theta, y = r\sin\theta$ and replacing $dy\,dx$ by $r\,dr\,d\theta$ enables us to evaluate the integral as

$$\begin{aligned}
\iint_R e^{x^2+y^2}\,dy\,dx &= \int_0^{\pi}\int_0^1 e^{r^2}\,r\,dr\,d\theta \\
&= \int_0^{pi}\left[\frac{e^{r^2}}{2}\right]_0^1 d\theta \\
&= \int_0^{\pi}\frac{1}{2}(e-1)d\theta = \frac{\pi}{2}(e-1)
\end{aligned}$$

4. Evaluate

$$\int_0^3\int_0^4\int_{y/2}^{(y/2)+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dx\,dy\,dz$$

by applying the transformation $u = \dfrac{(2x-y)}{2},\ v = \dfrac{y}{2},\ w = \dfrac{z}{3}$

Solution:

Let's call the region of integration in xyz-space as D and corresponding uvw-region as G. We need to find the boundaries of the region in uvw-region and the Jacobian of the transformation in order to evaluate this integral. To find them, we first solve equations $u = \dfrac{(2x-y)}{2},\ v = \dfrac{y}{2},\ w = \dfrac{z}{3}$ for x,y and z in terms of u,v and w. Routine algebra gives:

$$\begin{aligned}
x &= u+v \\
y &= 2v \\
z &= 3w
\end{aligned}$$

We then find the boundaries of G by substituting these expressions into the equations for the boundaries of D.

xyz-equations for the boundary of D	Corresponding uvw-equations for the boundary of G	Simplified uvw-equations
$x = \dfrac{y}{2}$	$u + v = \dfrac{2v}{2} = v$	$u = 0$
$x = \dfrac{y}{2} + 1$	$u + v = \left(\dfrac{2v}{2}\right) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$
$z = 0$	$3w = 0$	$w = 0$
$z = 3$	$3w = 3$	$w = 1$

The Jacobian of the transformation is

$$\hat{J}(u, v, w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6$$

$$\begin{aligned}
\int_0^3 \int_0^4 \int_{y/2}^{(y/2)+1} \left(\frac{2x - y}{2} + \frac{z}{3} \right) dx\, dy\, dz &= \int_0^1 \int_0^2 \int_0^1 (u + w) |\hat{J}(u, v, w)| du\, dv\, dw \\
&= \int_0^1 \int_0^2 \int_0^1 (u + w) 6\, du\, dv\, dw \\
&= 6 \int_0^1 \int_0^2 \left[\frac{u^2}{2} + uw \right]_0^1 dv\, dw \\
&= 6 \int_0^1 \int_0^2 \left(\frac{1}{2} + w \right) dv\, dw \\
&= 6 \int_0^1 \left(\frac{v}{2} + vw \right)_0^2 dw = 6 \int_0^1 (1 + 2w) dw \\
&= 6 \Big[w + w^2 \Big]_0^1 = 6 \cdot 2 = 12
\end{aligned}$$

B U Department of Mathematics

Math 102 Calculus II

Fall 2005 Make-up Exam

1. Find the area of the region that lies inside the circle $r = 1$ and outside of the cardioid $r = 1 - \cos\theta$.

2. Calculate

(a) $I = \int_0^{\pi/2} \int_0^{\cos x} x^2 dy\, dx$

(b) Reverse the order of the integral.

B U Department of Mathematics

Math 102 Calculus II

Date: January 12, 2007 Time: 9:00-12:00	Full Name : Student ID :

Fall 2006 Final Exam

IMPORTANT Show all your work. Correct answers without sufficient explanation may <u>not</u> get full credit.

Q1	Q2	Q3	Q4	Q5	Q6	Q7	TOTAL
10 pts	10+10 pts	10 pts	6 pts	8 pts	6 pts	10 pts	60 pts

---

1. Use *Lagrange multipliers* method to find all relative extrema of x^2y^2 on the ellipse $4x^2+y^2=8$.

Solution:

Let $f(x,y)=x^2y^2$ and $g(x,y)=4x^2+y^2=8$. We solve $\nabla f=\lambda\nabla g$ for x,y and λ:

$$2xy^2=\lambda\,8x; \quad 2x^2y=\lambda\,2y.$$

First equation gives either $x=0$ or $y^2=4\lambda$. Second gives $y=0$ or $x^2=\lambda$. Either $(x,y)=(0,0)$ which is not on the ellipse or $\lambda=1$, $x=\pm1$, $y=\pm2$. Hence we obtain four points: $(-1,-2)$,$(-1,2)$, $(1,-2)$ and $(1,2)$. To show that these points are relative extrema (not saddle points), one can look at the level curves of f, which are hyperbola. At the four points we found, these curves stay at the same side of the ellipse.

2. Let Ω be the solid bounded by the *half* cone $z=-\sqrt{x^2+y^2}$ and the circular paraboloid $z=2-x^2-y^2$ and suppose $f:\Omega\to\mathbb{R}$. Express $\displaystyle\iiint_\Omega f\,dV$ as an iterated triple integral
(a) in rectangular coordinates in the order $dz\,dy\,dx$;
(b) in spherical coordinates in the order $d\rho\,d\phi\,d\theta$;
(c) [**Bonus.**] in cylindrical coordinates in the order $dr\,dz\,d\theta$.

Solution:

The half cone and the paraboloid intersect at points with x,y satisfying $-\sqrt{x^2+y^2}=2-x^2-y^2$. Set $u=\sqrt{x^2+y^2}$ to get $-u=2-u^2$ so that either $u=2$ or $u=-1$. Hence the two surfaces intersect at the circle $z=-2$, $x^2+y^2=4$. (Not at $z=1$! See the figure.)

(a) $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{-\sqrt{x^2+y^2}}^{2-x^2-y^2} f\,dz\,dy\,dx.$

(b) Express ρ on the paraboloid as a function of θ and ϕ:

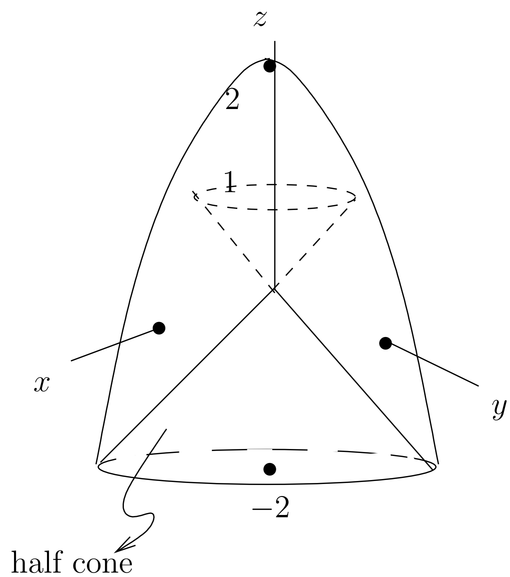
$$z=2-x^2-y^2\Rightarrow\rho\cos\phi=2-\rho^2\sin^2\phi\Rightarrow\rho(\phi)=\frac{-\cos\phi+\sqrt{1+7\sin^2\phi}}{2\sin^2\phi}$$

Hence, the iterated integral is

$$\int_0^{2\pi}\int_0^{\frac{3\pi}{4}}\int_0^{\rho(\phi)} f\,\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

where $\rho(\phi)$ is as above.

(c) $\displaystyle\int_0^{2\pi}\int_{-2}^{0}\int_{-z}^{\sqrt{2-z}} f\,r\,dr\,dz\,d\theta + \int_0^{2\pi}\int_0^{2}\int_0^{\sqrt{2-z}} f\,r\,dr\,dz\,d\theta.$

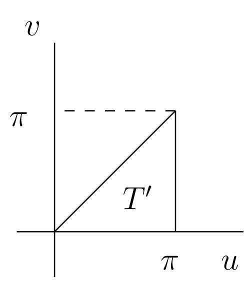


3. Use the transformation $u = \frac{1}{2}(x+y)$, $v = \frac{1}{2}(x-y)$ to find

$$\iint_T \sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)dA$$

over the triangular region T with vertices $(0,0)$, $(2\pi,0)$, (π,π).

Solution:



The region T is mapped to the triangle T' in the u-v world as shown in the figure above. Observing that $x = u+v$ and $y = u-v$, the Jacobian $J = \dfrac{\partial(x,y)}{\partial(u,v)} =$

$\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$. Hence,

$$\begin{aligned} \int\int_T &= \int_0^{\pi}\int_0^u \sin u \cos v |J|\, dv du \\ &= 2\int_0^{\pi} \sin u \big(\sin v\big)_0^u du \\ &= 2\int_0^{\pi} \sin^2 u\, du \\ &= \big(u - \frac{1}{2}\sin 2u\big)_0^{\pi} = \pi \end{aligned}$$

4. Consider the vector field $\mathbf{F}(x,y) = e^x \cos y\,\mathbf{i} - e^x \sin y\,\mathbf{j}$.
(a) Is $\mathbf{F}$ conservative on $\mathbb{R}^2$?
If yes, find a potential ϕ for $\mathbf{F}$; check that ϕ is really a potential for $\mathbf{F}$.
If no, explain the reason in detail, using related theorems.
(b) Find curl $\mathbf{F}$.

Solution:

(a) $\mathbf{F}$ is conservative on $\mathbb{R}^2$. The potential for $\mathbf{F}$ is $\phi(x,y) = e^x \cos y$.
(b) curl $\mathbf{F}$ =curl grad $\phi = 0$.

5. Find the work done by the force field

$$\mathbf{F}(x,y) = 2xy\,\mathbf{i} + (x^2 + \cos y)\,\mathbf{j}$$

along the curve $C : \mathbf{r}(t) = t\,\mathbf{i} + t\cos\frac{t}{3}\mathbf{j}$, $0 \le t \le \frac{\pi}{2}$.

Solution:

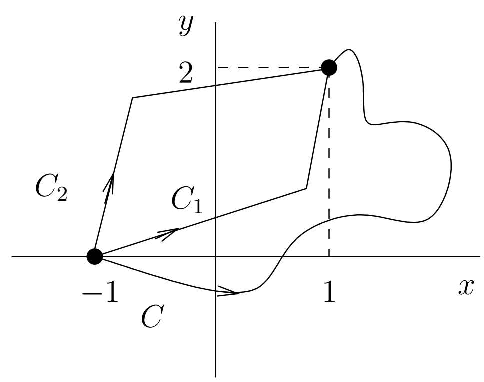
Note that $\mathbf{F}$ is conservative with a potential $\phi = x^2y + \sin y$. Hence the work done is

$$W = \phi(\frac{\pi}{2}, \frac{\pi}{2}\cos\frac{\pi}{6}) - \phi(0,0) = \phi(\frac{\pi}{2}, \frac{\sqrt{3}\pi}{4}) - \phi(0,0) = \frac{3\pi^3}{64} + \sin\frac{\sqrt{3}\pi}{4}.$$

6. Consider the vector field

$$\mathbf{F}(x,y) = \frac{1-y}{x^2 + (y-1)^2}\,\mathbf{i} + \frac{x}{x^2+(y-1)^2}\,\mathbf{j}$$

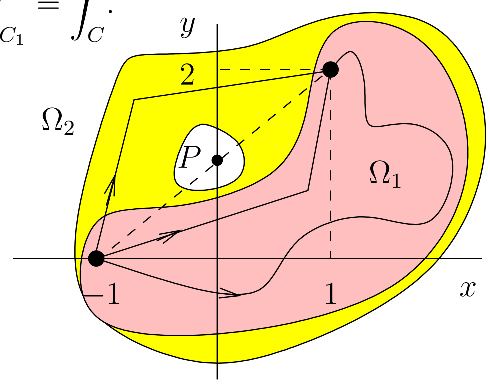
and the curves C, C_1 and C_2.



(a) Is it always true that $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r}$? Why?

Solution:

Let $\mathbf{F} = \langle f, g \rangle$. Observe that $f_y = g_x$. By theorem, $\mathbf{F}$ is conservative on any *simply connected* domain. Since $\mathbf{F}$ is not defined at $P = (0, 1)$, if a domain contains P, it must be deleted. One can find a simply-connected domain, as Ω_1 in the figure below, which contains C and C_1 and does not contain P. On this domain $\mathbf{F}$ is conservative by theorem, therefore $\int_{C_1} = \int_C$.



(b) Is it always true that $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r}$? Why?

Solution:

Not necessarily. Since every domain containing C and C_2 contains P as well, we have no chance to find a simply connected domain containing the curves but not P. Therefore we cannot apply the theorem. Hence we cannot be sure about the equality. Indeed, the equality is false in general.

7. Let C be the boundary of the region enclosed between $y = x^2$ and $y = 2x$. Assuming that C is oriented counterclockwise, use *Green's theorem* to evaluate $\oint_C (6xy - y^2)dy$.

Solution:

The region Ω is simply connected with a connected, smooth, simple boundary. Set $\mathbf{F} = 0\,\mathbf{i} + (6xy - y^2)\,\mathbf{j} = \langle f, g \rangle$. Then

$$\begin{aligned}
\oint_C (6xy - y^2)dy &= \oint_C \mathbf{F} \cdot d\mathbf{r} \\
&= \int\!\!\int_\Omega \frac{\partial g}{\partial x} dA \qquad (Green's) \\
&= \int_0^2 \int_{x^2}^{2x} 6y\, dy\, dx \\
&= 3 \int_0^2 y^2 \Big|_{x^2}^{2x} dx \\
&= 3 \left(\frac{4x^3}{3} - \frac{x^5}{5} \right)_0^2 \\
&= \frac{64}{5}
\end{aligned}$$

B U Department of Mathematics

Math 102 Calculus II

Fall 1999 Final

1. Evaluate the triple integral:

$$\iiint_Q (2-z)dV$$

where Q is the solid region bounded by the cone $z = 2 - \sqrt{x^2+y^2}$ and the xy-plane.

Solution:

In cylindrical coordinates (r,θ,z),
$Q = \{(r,\theta,z)|\ 0 \le \theta \le 2\pi,\ \ 0 \le r \le 2,\ \ 0 \le z \le 2-r\}$

$$\begin{aligned}
\iiint_Q (2-z)dV &= \int_0^{2\pi}\int_0^{2}\int_0^{2-r} (2-z)r\ dz\ dr\ d\theta \\
&= \int_0^{2\pi}\int_0^{2} (2r - \frac{1}{2}r^3)dr\ d\theta \\
&= 2\pi\left[r^2 - \frac{1}{8}r^4\right]_0^2 = 4\pi
\end{aligned}$$

2. **(a)** Find the angle between the planes $x = 7$ and $x + y + \sqrt{2}\,z = -3$.
(b) Find a vector $\vec{\boldsymbol{v}}$ that is tangent to the curve of intersection of the plane $z = 2x + 3y$ and the sphere $x^2 + y^2 + z^2 = 6$ at the point $(2,-1,1)$.

Solution:

(a) This is the angle between the normals $\vec{\boldsymbol{N_1}} = \hat{\boldsymbol{\imath}}$ and $\vec{\boldsymbol{N_2}} = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \sqrt{2}\,\hat{\boldsymbol{k}}$. So,

$$\begin{aligned}
\cos\theta &= \frac{\vec{\boldsymbol{N_1}}\cdot\vec{\boldsymbol{N_2}}}{|\vec{\boldsymbol{N_1}}||\vec{\boldsymbol{N_2}}|} = \frac{1}{2} \\
\Rightarrow \theta &= \frac{\pi}{3}
\end{aligned}$$

(b) The curve of intersection is the circle

$$5x^2 + 10y^2 + 12xy = 6, \qquad z = 2x + 3y$$

Let

$$\begin{aligned}
\vec{\boldsymbol{r}} &= x\hat{\boldsymbol{\imath}} + y(x)\hat{\boldsymbol{\jmath}} + z(x)\hat{\boldsymbol{k}} \\
\vec{\boldsymbol{v}} &= c\vec{\boldsymbol{r'}},\ c \in \mathbb{R}
\end{aligned}$$

One finds $z' = 2 + 3y'$, and $10x + 12y + (20y + 12x)y' = 0$.
$\Rightarrow$ At $(2,-1,1)$,

$$\begin{aligned}
y' &= -2, \\
z' &= -4
\end{aligned}$$

$\Rightarrow \vec{\boldsymbol{v}} = c(\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} - 4\hat{\boldsymbol{k}})$.

3. **(a)** Find the directional derivative of $f(x, y, z) = x^2 + y^2 + z^2$ in the direction of the upward normal vector to the surface $z = 3x^2 + 2y^2$ at the point $(2, 3, 30)$.

Solution:

Upward $\qquad \vec{\boldsymbol{N}} = \hat{\boldsymbol{k}} - 6x\hat{\boldsymbol{\imath}} - 4y\hat{\boldsymbol{\jmath}}.$

At $(2, 3, 30)$

$$\begin{aligned} \vec{\boldsymbol{N}} &= \hat{\boldsymbol{k}} - 12\hat{\boldsymbol{\imath}} - 12\hat{\boldsymbol{\jmath}}, \qquad |\vec{\boldsymbol{N}}| = 17 \\ \hat{\boldsymbol{u}} &= \frac{1}{17}(\hat{\boldsymbol{k}} - 12\hat{\boldsymbol{\imath}} - 12\hat{\boldsymbol{\jmath}}) \end{aligned}$$

$$\begin{aligned} \vec{\nabla}\boldsymbol{f}(2,3,30) &= 4\hat{\boldsymbol{\imath}} + 6\hat{\boldsymbol{\jmath}} + 60\hat{\boldsymbol{k}} \\ D_{\hat{\boldsymbol{u}}} f &= \hat{\boldsymbol{u}} \cdot \vec{\nabla}\boldsymbol{f}(2,3,30) \\ &= \frac{1}{17}(60 - 48 - 72) \\ &= \frac{-60}{17} \end{aligned}$$

(b) Let $\rho = \sqrt{x^2 + y^2 + z^2}$ and $f = f(\rho)$. Determine the most general $f(\rho)$ which satisfies $f_{xx} + f_{yy} + f_{zz} = 0$.

Solution:

$$\begin{aligned} f_x &= f'\rho_x, \quad f_{xx} = f''\rho_x^2 + f'\rho_{xx}, \quad f' \equiv \frac{df}{d\rho} \\ \rho_x &= \frac{1}{\rho}\,x, \quad \rho_{xx} = \frac{1}{\rho} - \frac{1}{\rho^3}x^2 \end{aligned}$$

Since ρ is symmetric in x,y and z

$$\begin{aligned} f_{xx} + f_{yy} + f_{zz} &= f''(\rho_x^2 + \rho_y^2 + \rho_z^2) + f'(\rho_{xx} + \rho_{yy} + \rho_{zz}) \\ \rho_{zz} + \rho_{yy} + \rho_{zz} &= \frac{3}{\rho} - \frac{1}{\rho^3}\rho^2 = \frac{2}{\rho} \\ \rho_x^2 + \rho_y^2 + \rho_z^2 &= 1 \qquad \text{Thus} \\ f_{xx} + f_{yy} + f_{zz} &= f'' + \frac{2}{\rho}f' = 0 \\ \Rightarrow \frac{f''}{f'} = (\ln f')' &= -\frac{2}{\rho} \\ \Rightarrow f' &= \frac{c}{\rho^2} \\ \Rightarrow f(\rho) &= \frac{A}{\rho} + B, \qquad A, B \in \mathbb{R} \end{aligned}$$

4. **(a)** Find the surface area of the part of the hyperbolic paraboloid $z = 10 + xy$ that lies inside the cylinder $x^2 + y^2 = 9$.

Solution:

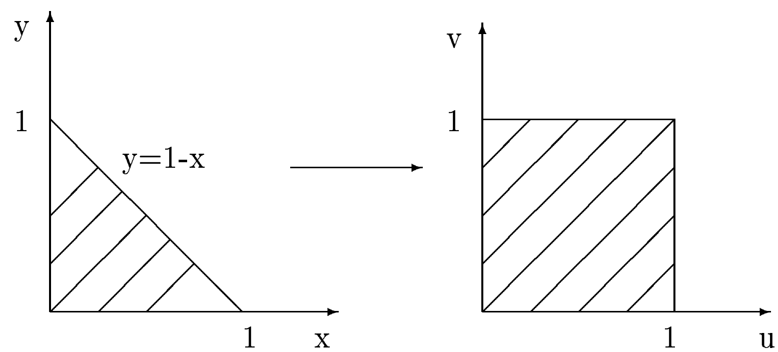
$$
\begin{aligned}
A(s) &= \iint_Q \sqrt{1+x^2+y^2}\, dA \qquad \text{where} \\
Q &= \{(x,y)|x^2+y^2 \le 9\}. \qquad \text{Using polar coordinates,} \\
A(s) &= \int_0^{2\pi}\int_0^3 \sqrt{1+r^2}\; r\; dr\; d\theta \\
&= \frac{2\pi}{2}\cdot \left.\frac{(1+r^2)^{\frac{3}{2}}}{3/2}\right|_0^3 \\
&= \frac{2\pi}{3}(10^{3/2}-1).
\end{aligned}
$$

(b) By using the transformations $x+y=u$, $y=uv$, evaluate the double integral

$$\int_0^1\int_0^{1-x} e^{\frac{y}{x+y}}\; dy\; dx.$$

Solution:

$$
\begin{aligned}
&x = u - y = u - uv, \\
&y = uv, \\
&\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} 1-v & -u \\ v & u \end{bmatrix} = (1-v)u-(-u)v = u.
\end{aligned}
$$



$$
\begin{aligned}
\int_0^1\int_0^{1-x} e^{\frac{y}{x+y}}\; dy\; dx &= \int_0^1\int_0^1 e^v|u|\; du\; dv \\
&= \frac{1}{2}(e-1).
\end{aligned}
$$

5. **(a)** Evaluate the line integral $\int_C \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{r}}$, where $\vec{\boldsymbol{F}} = 2xz^3\ \hat{\boldsymbol{\imath}} + 3x^2z^2\ \hat{\boldsymbol{k}}$ and C is the curve $x=\cos t, y=\sin t, z=\sin 2t, 0\le t\le 2\pi$.

Solution:

$$\vec{\boldsymbol{F}} = \vec{\nabla}(x^2z^3), \text{ conservative.}$$

C is closed: $\qquad \vec{\boldsymbol{r}}(0) = \vec{\boldsymbol{r}}(2\pi)$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = 0$$

(b) Let $\vec{F} = \vec{a} \times \vec{r}$, where $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ is a constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Find $\vec{\nabla} \cdot \vec{F}$ and $\vec{\nabla} \times \vec{F}$. Is $\vec{F}$ a conservative vector field?

Solution:

$$\begin{aligned}
\vec{F} &= det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{bmatrix} \\
&= (za_2 - ya_3)\hat{i} + (xa_3 - za_1)\hat{j} + (ya_1 - xa_2)\hat{k} \\
&\Rightarrow \vec{\nabla} \cdot \vec{F} = 0
\end{aligned}$$

$$\begin{aligned}
\vec{\nabla} \times \vec{F} &= det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (za_2 - ya_3) & (xa_3 - za_1) & (ya_1 - xa_2) \end{bmatrix} \\
&= \left[\frac{\partial}{\partial y}(ya_1 - xa_2) - \frac{\partial}{\partial z}(xa_3 - za_1)\right]\hat{i} \\
&+ \left[\frac{\partial}{\partial z}(za_2 - ya_3) - \frac{\partial}{\partial x}(ya_1 - xa_2)\right]\hat{j} \\
&+ \left[\frac{\partial}{\partial x}(xa_3 - za_1) - \frac{\partial}{\partial y}(za_2 - ya_3)\right]\hat{k} \\
&= 2\vec{a}
\end{aligned}$$

Thus $\vec{\nabla} \times \vec{F} = \vec{\nabla} \times (\vec{a} \times \vec{r}) = 2\vec{a}$ and since $\vec{\nabla} \times \vec{F} \neq 0$, $\vec{F}$ is <u>not</u> conservative.

6. Let $f(x, y)$ and $g(x, y)$ be two functions that are continuously differentiable on an open set containing the circular disk Q whose boundary is the circle $x^2 + y^2 = 1$. Two vector fields $\vec{F}$ and $\vec{G}$ are defined as

$$\vec{F} = f\hat{i} + g\hat{j}, \qquad \vec{G} = (g_x - g_y)\hat{i} + (f_x - f_y)\hat{j}.$$

Find the value of the double integral

$$\iint_Q \vec{F} \cdot \vec{G}\, dx\, dy,$$

if it is known that on the boundary of Q we have $f(x, y) = y$ and $g(x, y) = 1$.

Solution:

$$\vec{F} \cdot \vec{G} = f_x g - f_y g + f g_x - f g_y = \frac{\partial}{\partial x}(fg) - \frac{\partial}{\partial y}(fg).$$

By Green's Theorem we know:

$$\int\limits_C M\ dx + N\ dy = \iint\limits_Q (N_x - M_y)\ dA$$

Identify $\qquad M = N = fg$. Then

$$\iint\limits_Q \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{G}}\ dx\ dy = \int\limits_C (fg)(dx + dy),$$

where C is the circle $x^2 + y^2 = 1$. Using

$$\begin{aligned} &x(t) = \cos t,\ y = \sin t, \qquad\qquad t \in [0, 2\pi], \\ &f = y,\ g = 1 \\ &\int\limits_C (fg)(dx + dy) = \int_0^{2\pi} \sin t(\cos t - \sin t) dt = -\pi \end{aligned}$$

Therefore,

$$\iint\limits_Q \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{G}}\ dx\ dy = -\pi.$$

B U Department of Mathematics

Math 102 Calculus II

Spring 2000 Final

1. Consider the curve $r = \theta$ in polar coordinates. Compute the curvature at $\theta = \pi$.

Solution:

$$\left.\begin{array}{r} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \Rightarrow \begin{array}{l} x = \theta\cos\theta \\ y = \theta\sin\theta \end{array}$$

$$\kappa = \frac{|\ddot{x}\dot{y} - \dot{x}\ddot{y}|}{\left(\dot{x}^2 + \dot{y}^2\right)^{3/2}}$$

$$\begin{array}{lll} \dot{x} = \cos\theta + \theta\left(-\sin\theta\right) & \Rightarrow & \dot{x}\left(\pi\right) = -1. \\ \ddot{x} = -\sin\theta + \left(-\sin\theta\right) + \theta\left(-\cos\theta\right) & \Rightarrow & \ddot{x}\left(\pi\right) = \pi. \\ \dot{y} = \sin\theta + \theta\left(\cos\theta\right) & \Rightarrow & \dot{y}\left(\pi\right) = -\pi. \\ \ddot{y} = \cos\theta + \cos\theta + \theta\left(-\sin\theta\right) & \Rightarrow & \ddot{y}\left(\pi\right) = -2. \end{array}$$

$$\Rightarrow \kappa = \frac{|\pi\left(-\pi\right) - \left(-1\right)\left(-2\right)|}{\left(\left(-1\right)^2 + \left(-\pi\right)^2\right)^{3/2}} = \frac{2 + \pi^2}{\left(1 + \pi^2\right)^{3/2}}\ .$$

2. Compute $\nabla f(r)$ where $r = \sqrt{x^2 + y^2 + z^2}$ and interpret your result geometrically.

Solution:

By chain rule , $\nabla f\left(r\right) = f'\left(r\right)\dfrac{\partial r}{\partial x}\hat{\boldsymbol{i}} + f'\left(r\right)\dfrac{\partial r}{\partial y}\hat{\boldsymbol{j}} + f'\left(r\right)\dfrac{\partial r}{\partial z}\hat{\boldsymbol{k}}$

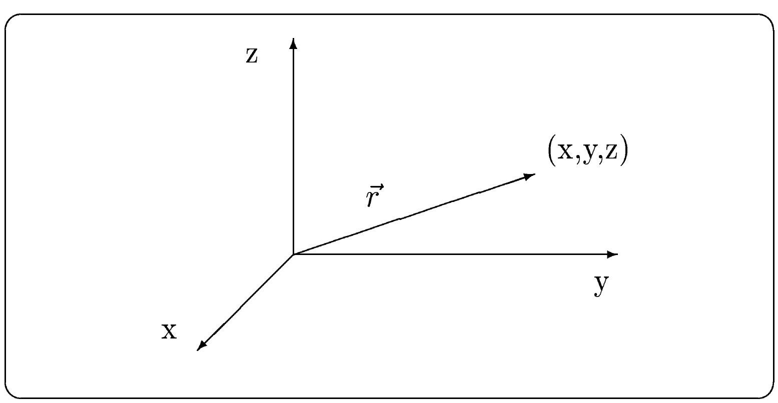
$$= f'\left(r\right)\left(\frac{2x}{2\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{i}} + \frac{2y}{2\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{j}} + \frac{2z}{2\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{k}}\right)$$

$$\frac{f'\left(r\right)}{r}\left(x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}\right) = \frac{f'\left(r\right)}{r}\cdot\vec{\boldsymbol{r}}$$

Here the term $\dfrac{f'\left(r\right)}{r}$ is a scalar and the second term $\vec{\boldsymbol{r}}$ is a vector.

$\Rightarrow \nabla f\left(r\right)$ is parallel to the vector $\vec{\boldsymbol{r}}$.

$\Leftrightarrow \nabla f$ flows outwards away from the origin.



$\Leftrightarrow f$ is constant on all circles $r = \mathit{constant}$ and ∇f is $\perp$ to these circles.

3. Is there a function $z = f(x,y)$ with continuous second partial derivatives such that $\dfrac{\partial f}{\partial x} = 2y - e^x \sin y$, $\dfrac{\partial f}{\partial y} = 1 + e^x \cos y$? If yes, find such a function. If no, justify your answer.

Solution:

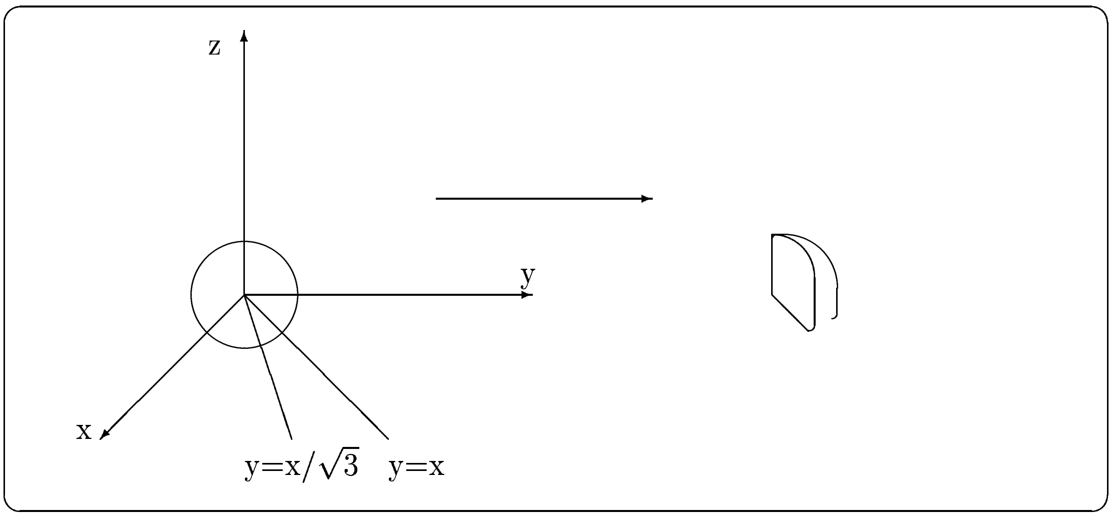
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2y - e^x \sin y) = 2 - e^x \cos y = f_{yx}$$
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(1 + e^x \cos y) = e^x \cos y = f_{xy}$$

The equation $f_{xy} = f_{yx} \Leftrightarrow 2 - e^x \cos y = e^x \cos y \Leftrightarrow 1 = e^x \cos y$ is NOT satisfied on any region Ω.

Therefore there is no such function on any region Ω.

4. Evaluate $\iiint\limits_{\Omega} (x^2+y^2+z^2)dV$, where Ω is the region in the first quadrant $(x,y,z \geq 0)$ bounded by the sphere $x^2+y^2+z^2 = 1$ and the planes $y = x$, $y = \dfrac{1}{\sqrt{3}}\,x$.

Solution:



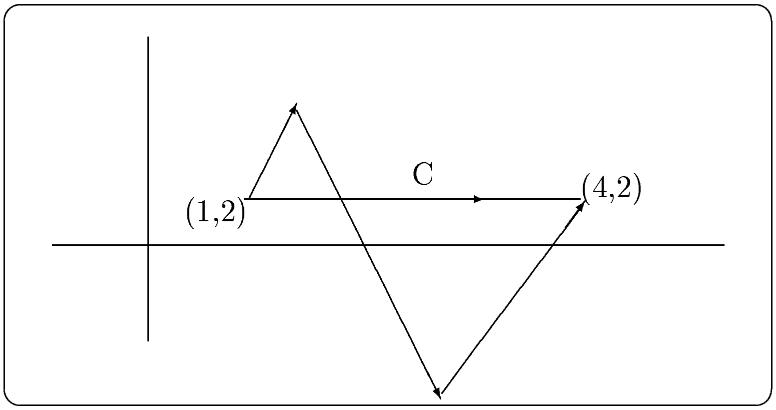
$$0 \leq \rho \leq 1 \quad \tfrac{\pi}{6} \leq \theta \leq \tfrac{\pi}{4} \quad 0 \leq \phi \leq \tfrac{\pi}{2}$$

$$\iiint\limits_{\Omega} (x^2+y^2+z^2)dV = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^1 \rho^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \left(\frac{\pi}{4} - \frac{\pi}{6}\left([-\cos\phi]_0^{\frac{\pi}{2}}\right) \cdot \left(\frac{1}{5}\right) = \frac{\pi}{60}.$$

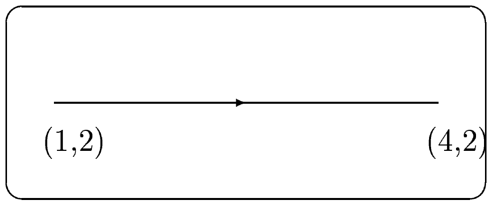
5. Let $P = (1,2)$, $Q = (2,4)$, $R = (3,-7)$, S=(4,2) and C the curve consisting from line segments from P to Q, Q to R and R to S. Evaluate

$$\int\limits_C (6xy^2 - y^3)dx + (6x^2y - 3xy^2)dy$$

Solution:



$P_y = (6xy^2 - y^3)_y = 12xy - 3y^2$
$Q_x = (6x^2y - 3xy^2)_x = 12xy - 3y^2$
Therefore $P_y = Q_x$ on $\mathbb{R}^2$(= simply connected) $\Rightarrow$ line integral is path independent.
We therefore replace C by the line segment $\begin{cases} x = t & 1 \leq t \leq 4 \\ y = 2 & \end{cases}$



$$\int_1^4 (6t(2)^2 - (2)^3)dt = \left[\frac{24t^2}{2} - 8t\right]_1^4 = 156.$$

6. Let C be a closed and simple ($\equiv$ without self intersections) curve. Show that

$$\int_C (-y^3)dx + (x^3)dy$$

is a positive number.

Solution:

If $g(x,y) > 0$ for all $(x,y) \in \Omega$, then $\iint_\Omega g(x,y)dA > C$
This follows from the definition of the double integral as a limit of Riemann Sums.
Now, by Green's Theorem,
$\int_C (-y^3)dx + (x^3)dy = \iint_\Omega (3x^2 + 3y^2)dA > 0$ because $x^2 + y^2 > 0$ on Ω.

B U Department of Mathematics

Math 102 Calculus II

Spring 2001 Final

1. Find an equation for the plane that contains the lines $x = y = 2z - 2$ and $x/2 = y = 1 - z$.

Solution:

The lines intersect at the point $A = (0, 0, 1)$, so there is such a plane.
The first line has the parametric representation $\langle x, y, z\rangle = \langle 2t - 2, 2t - 2, t\rangle$, so its direction vector is $\boldsymbol{u} = \langle 2, 2, 1\rangle$.
The second line has the parametric representation $\langle x, y, z\rangle = \langle 2t, t, 1 - t\rangle$, so its direction vector is $\boldsymbol{v} = \langle 2, 1, -1\rangle$.
The vector $\boldsymbol{n} = \boldsymbol{u} \times \boldsymbol{v}$ is perpendicular to both direction vectors, hence it is normal to the plane that contains the lines.

$$\boldsymbol{n} = \boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 2 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} 2 & 1 \\ 2 & -1 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} \boldsymbol{k} = -3\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}.$$

For any two points A and B on the plane, the vector $\boldsymbol{BA}$ is perpendicular to the normal. We have $A = (0, 0, 1)$ on the plane. Letting $B = (x, y, z)$ be an arbitrary point on the plane, this gives us the equation of the plane:

$$\begin{gathered} \boldsymbol{n} \cdot \boldsymbol{BA} = 0 \\ (-3\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}) \cdot (x\boldsymbol{i} + y\boldsymbol{j} + (z - 1)\boldsymbol{k}) = 0 \\ -3x + 4y - 2(z - 1) = 0 \\ -3x + 4y - 2z + 2 = 0. \end{gathered}$$

2. The temperature distribution on a metal plate is $T(x, y) = 200 - (x^2 + 4y^2)$. Let P be a heat seeking particle in the plane (heat-seeking means that the particle moves in the direction where the increase in the temperature is maximum).

(a) Suppose that we place the particle at $(2, 0)$. In which direction will the particle move?

(b) Find the path followed by this particle.

Solution:

The direction of maximum increase of $T(x, y)$ is given by the gradient of $T(x, y)$:

$$\boldsymbol{\nabla} T = \frac{\partial T}{\partial x}\boldsymbol{i} + \frac{\partial T}{\partial y}\boldsymbol{j} = -2x\boldsymbol{i} - 8y\boldsymbol{j}.$$

(a) At the given point the direction is $\boldsymbol{\nabla} T(2, 0) = -4\boldsymbol{i}$.
(b) Note that when the particle is on the $x-$axis, the direction of movement $-2x\boldsymbol{i}$ is along the x-axis towards the origin because the temperature T is increasing as particle comes closer to the origin. So the particle cannot leave the $x-$axis, it moves from the starting point to the origin, where $T(x, y)$ is maximum, along a straight line.

3. Find $\displaystyle\iint_Q \frac{\sin x}{x}\, dA$, where Q is the triangle with vertices $(0,0)$, $(2,0)$ and $(2,2)$.

Solution:

We can evaluate this integral if we do the first integration with respect to variable y.

$$\begin{aligned}\iint_Q \frac{\sin x}{x}\, dA &= \int_{x=0}^{2}\int_{y=0}^{x} \frac{\sin x}{x}\, dydx = \int_{x=0}^{2}\left[\frac{\sin x}{x} y\right]_{y=0}^{x} dx \\ &= \int_{x=0}^{2} \sin x\, dx = \left[-\cos x\right]_{x=0}^{2} = -\cos 2 + \cos 0 = 1 - \cos 2.\end{aligned}$$

4. Find the volume of the solid bounded by the cone $z^2 = x^2 + y^2$ and the cylinder $x^2 + y^2 = 9$.

Solution:

Both the cone and the cylinder have rotational symmetry around the $z-$axis, so we should better work in cylindrical coordinates (r, θ, z). In cylindrical coordinates, the cylinder has the equation $r = 3$, and the cone has the equation $z^2 = r^2$. The equation $z = r$ gives us the upper half of the cone, and $z = -r$ gives the lower half. Hence the volume is:

$$\begin{aligned}\int_{\theta=0}^{2\pi}\int_{r=0}^{3}\int_{z=-r}^{r} dV &= \int_{0}^{2\pi}\int_{0}^{3}\int_{-r}^{r} r\, dzdrd\theta = \int_{0}^{2\pi}\int_{0}^{3}\Big[rz\Big]_{z=-r}^{r} dr\, d\theta \\ &= \int_{0}^{2\pi}\int_{0}^{3} 2r^2\, drd\theta = \int_{0}^{2\pi}\left[\frac{2}{3}r^3\right]_{r=0}^{3} d\theta = \int_{\theta=0}^{2\pi} 18\, d\theta = 36\pi.\end{aligned}$$

5. Consider the unit sphere $x^2 + y^2 + z^2 = 1$ and the points $A = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, $B = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ on that sphere. Find the coordinates of the point C on the sphere such that the triangle ABC has maximum area.

Solution:

Let the third point be $C = (x, y, x)$. Then construct:

$$\begin{aligned}\boldsymbol{u} = \boldsymbol{BA} &= \left\langle 0, 0, \frac{2}{\sqrt{3}}\right\rangle \\ \boldsymbol{v} = \boldsymbol{CA} &= \left\langle x - \frac{1}{\sqrt{3}}, y - \frac{1}{\sqrt{3}}, z + \frac{1}{\sqrt{3}}\right\rangle\end{aligned}$$

are two sides of the triangle ABC.
The length of the cross product of $\boldsymbol{u}$ and $\boldsymbol{v}$ is equal to $\|\boldsymbol{u}\|\|\boldsymbol{v}\| \sin\alpha$, where α is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$. This is equal to the area of the parallelogram with sides $\boldsymbol{u}$

and $\boldsymbol{v}$, or twice the area of our triangle. So, we want to maximize $\|\boldsymbol{u}\times\boldsymbol{v}\|$.

$$\begin{aligned}
\boldsymbol{u}\times\boldsymbol{v} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & \frac{2}{\sqrt{3}} \\ x-\frac{1}{\sqrt{3}} & y-\frac{1}{\sqrt{3}} & z+\frac{1}{\sqrt{3}} \end{vmatrix} \\
&= \begin{vmatrix} 0 & \frac{2}{\sqrt{3}} \\ y-\frac{1}{\sqrt{3}} & z+\frac{1}{\sqrt{3}} \end{vmatrix}\boldsymbol{i} - \begin{vmatrix} 0 & \frac{2}{\sqrt{3}} \\ x-\frac{1}{\sqrt{3}} & z+\frac{1}{\sqrt{3}} \end{vmatrix}\boldsymbol{j} + \begin{vmatrix} 0 & 0 \\ x-\frac{1}{\sqrt{3}} & y-\frac{1}{\sqrt{3}} \end{vmatrix}\boldsymbol{k} \\
&= -\frac{2}{\sqrt{3}}\left(y-\frac{1}{\sqrt{3}}\right)\boldsymbol{i} + \frac{2}{\sqrt{3}}\left(x-\frac{1}{\sqrt{3}}\right)\boldsymbol{j} \\
\|\boldsymbol{u}\times\boldsymbol{v}\|^2 &= \frac{4}{3}\left(y-\frac{1}{\sqrt{3}}\right)^2 + \frac{4}{3}\left(x-\frac{1}{\sqrt{3}}\right)^2.
\end{aligned}$$

We want to maximize the square root of the last expression, but this is the same as maximizing the expression itself. So we want to maximize

$$H(x,y,z) = \left(y-\frac{1}{\sqrt{3}}\right)^2 + \left(x-\frac{1}{\sqrt{3}}\right)^2$$

subject to the constraint $S(x,y,z) = x^2+y^2+z^2-1 = 0$. We can solve this problem using the method of Lagrange multipliers:

$$\boldsymbol{\nabla} H(x,y,z) = \lambda \boldsymbol{\nabla} S(x,y,z),$$

that is equivalent to:

$$2\left(x-\frac{1}{\sqrt{3}}\right)\boldsymbol{i} + 2\left(y-\frac{1}{\sqrt{3}}\right)\boldsymbol{j} = \lambda(2x\boldsymbol{i}+2y\boldsymbol{j}+2z\boldsymbol{k})$$

or componentwise:

$$\begin{aligned}
\lambda x &= x-\frac{1}{\sqrt{3}} \\
\lambda y &= y-\frac{1}{\sqrt{3}} \\
\lambda z &= 0
\end{aligned}$$

together with the constraint $x^2+y^2+z^2-1=0$.

The first two equations give us $x=y$. By the third equation, we have $\lambda=0$, or $z=0$.

If $\lambda=0$, first two equations give us $x=y=1/\sqrt{3}$. We substitute these in $x^2+y^2+z^2-1=0$ and solve for z to get $z=\pm 1/\sqrt{3}$. The points we get are A and B. At these points we have $H(x,y,z)=0$, i.e. there is no triangle at all. Clearly H has its local (global) minimum at these critical points. So they are not what we want.

Then we must have $z=0$. This gives us $x^2+y^2=1$. Since we also have $x=y$, we have $x=y=\pm 1/\sqrt{2}$. We get the critical points:

$$\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right) \quad \text{and} \quad \left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right).$$

By evaluating H at these points, we see that H achieves its global maximum at the second one.

6. Find the work done by the force field $\boldsymbol{F}$ in moving a particle along the path $y = 1/x$ from $y \to +\infty$ to $x \to +\infty$, if the field has a potential $\phi(x,y) = \dfrac{ax+by}{x+y}$, $a, b \in \mathbb{R}$ and $\boldsymbol{F}(1,1) = \dfrac{\pi}{4}\boldsymbol{i} - \dfrac{\pi}{4}\boldsymbol{j}$.

Solution:

In a force field defined by a potential, the work done depends only on the initial and final points, and is equal to the difference of potentials at these points. Since our initial and final points are at infinity, we have to compute the values of the potential there by computing appropriate limits.
At the initial point, the potential is

$$\lim_{y \to +\infty} \phi(1/y, y) = \lim_{y \to +\infty} \frac{a/y + by}{1/y + y} = b.$$

At the final point, the potential is

$$\lim_{x \to +\infty} \phi(x, 1/x) = \lim_{x \to +\infty} \frac{ax + b/x}{x + 1/x} = a.$$

Therefore, the work done is equal to $a - b$.

B U Department of Mathematics

Math 102 Calculus II

Spring 2002 Final

1. Find parametric equations for the line that is orthogonal at the origin to the plane $3x+y+z=0$.

Solution:

Let L be the line. $P(0,0,0)$ is on L. The vector $3\vec{i}+\vec{j}+\vec{k}$ is a normal vector for the plane $3x+y+z=0$, so, is a direction vector for L.

$$\text{Thus, } L: \begin{cases} x = 3t \\ y = t \\ z = t\ ,\ t \in \mathbb{R}. \end{cases}$$

2. Show that the ellipsoid $9x^2+4y^2+9z^2=36$ and the sphere $x^2+y^2+z^2-10z+16=0$ have a common tangent plane at the point $(0,0,2)$. Also find the equation of this tangent plane.

Solution:

$P(0,0,2)$ satisfies the equations of both surfaces.
Let $w = 9x^2+4y^2+9z^2-36$. Then, $\nabla w|_P = 18x\vec{i}+8y\vec{j}+18z\vec{k}\ |_P = 36\vec{k}$
$\Rightarrow\ N_1 = \vec{k}$ is normal to the ellipsoid at P.
Let $u = x^2+y^2+z^2-10z+16$. Then, $\nabla u|_P = 2x\vec{i}+2y\vec{j}+(2z-10)\vec{k}\ |_P = -6\vec{k}$
$\Rightarrow\ N_2 = \vec{k}$ is normal to the sphere at P.
The plane through P and normal to $\vec{k}$ is tangent to both surfaces at P. The equation of this tangent plane is $0(x-0)+0(y-0)+1(z-2)=0$ i.e. $z=2$.

3. Suppose $f(x, y)$ is a differentiable function satisfying the partial differential equation $3f_x + 5f_y = 0$. Let $u = \alpha x + y$ and $v = \alpha x - y$. Find the value(s) of α for which $f_u = 0$.

Solution:

By the chain rule

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = f_u \cdot \alpha + f_v \cdot \alpha$$

and

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = f_u \cdot 1 + f_v \cdot (-1)$$

Substituting into $3f_x + 5f_y = 0$, we get $(3\alpha + 5)f_u + (3\alpha - 5)f_v = 0$. To have $f_u = 0$, we require $(3\alpha - 5)f_v = 0$ and hence $\alpha = \frac{5}{3}$.

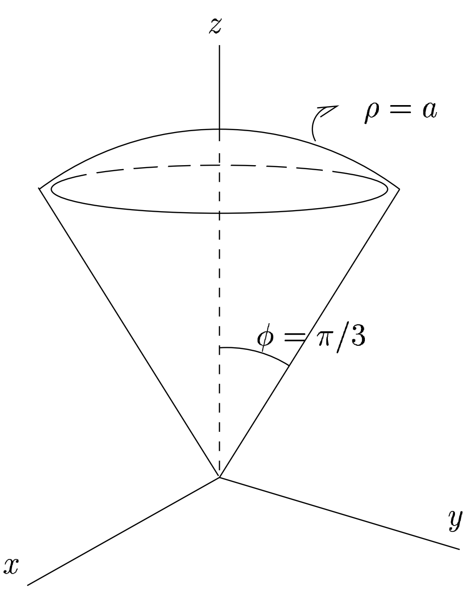
4. Let G be the region bounded above by the sphere $\rho = a, a > 0$ and below by the cone $\phi = \dfrac{\pi}{3}$, where ρ and ϕ are spherical coordinates. Express the triple integral

$$\iiint_G (x^2 + y^2)\,dV.$$

a) using spherical coordinates, and sketch the region G.

Solution:

(a)



$x^2 + y^2 = r^2 = (\rho \sin\phi)^2; \quad dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$

Hence, $I = \iiint_G \rho^2 \sin^2\phi \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^a \rho^4 \sin^3\phi \, d\rho \, d\phi \, d\theta.$

b) using cylindrical coordinates.

Solution:

$\phi = \dfrac{\pi}{3} \;\Rightarrow\; \tan\phi = \sqrt{3} = \dfrac{r}{z} \;\Rightarrow\; z = \dfrac{r}{\sqrt{3}}$ is the equation of the cone.
$\rho = a \;\Rightarrow\; x^2 + y^2 + z^2 = a^2 \;\Rightarrow\; r^2 + z^2 = a^2 \;\Rightarrow\; z^2 = a^2 - r^2$ is the equation of the sphere.

$r = \rho \sin\phi = a \sin\dfrac{\pi}{3} = \dfrac{a\sqrt{3}}{2}$

The surfaces intersect in the circle with radius $r = \dfrac{a\sqrt{3}}{2}$.

Hence, $I = \displaystyle\iiint_G r^2 r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^{\frac{a\sqrt{3}}{2}} \int_{\frac{r}{\sqrt{3}}}^{\sqrt{a^2-r^2}} r^3\, dz\, dr\, d\theta.$

c) Evaluate *one* of the multiple integrals found in parts (a) and (b).

Solution:

From the integral found in part (a),

$$\begin{aligned}
I &= 2\pi \int_0^{\frac{\pi}{3}} \left(\frac{\rho^5}{5} \sin^3\phi\right)\Bigg|_{\rho=0}^{\rho=a} d\phi \\
&= \frac{2\pi a^5}{5} \int_0^{\frac{\pi}{3}} \sin^3\phi\, d\phi \\
&= \frac{2\pi a^5}{5} \int_0^{\frac{\pi}{3}} (1 - \cos^2\phi)\sin\phi\, d\phi \\
&= \frac{2\pi a^5}{5} \left(-\cos\phi + \frac{\cos^3\phi}{3}\right)\Bigg|_0^{\frac{\pi}{3}} \\
&= \frac{2\pi a^5}{5} \left\{-\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{8} - \left(-1 + \frac{1}{3}\right)\right\} \\
&= \frac{2\pi a^5}{5} \left(\frac{1}{2} + \frac{1}{24} + \frac{2}{3}\right) = \frac{2\pi a^5}{5} \cdot \frac{29}{24}.
\end{aligned}$$

5. Find the work done by the force field $\vec{F}(x,y) = \frac{\pi}{2} e^{2y} \cos \frac{\pi x}{2} \vec{i} + 2e^{2y} \sin \frac{\pi x}{2} \vec{j}$ along the path given by the semicircle $x^2 + y^2 = 1$, $y \geq 0$ from point $A(1,0)$ to point $B(-1,0)$.

Solution:

Let $M = \frac{\pi}{2} e^{2y} \cos \frac{\pi x}{2}$ and $N = 2e^{2y} \sin \frac{\pi x}{2}$.

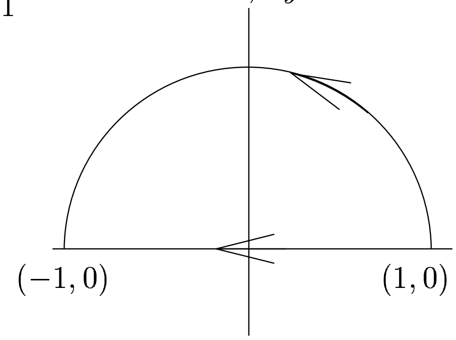
Method I: Calculate the first partial derivatives M_y and N_x:

$M_y = \frac{\pi}{2} \cdot 2e^{2y} \cos \frac{\pi x}{2} = \pi e^{2y} \cos \frac{\pi x}{2}$ and $N_x = 2e^{2y} \cdot \frac{\pi}{2} \cos \frac{\pi x}{2} = \pi e^{2y} \cos \frac{\pi x}{2}$

$M_y = N_x$ everywhere $\Rightarrow$ $\vec{F}$ is conservative. So, work is path-independent.

Choose an easier path from A to B.

$C' : \begin{cases} x = -t \\ y = 0 \,, \ -1 \leq t \leq 1 \end{cases} \quad \Rightarrow \quad dx = -dt, \ dy = 0.$



Then,

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{R} = \int_{C'} \mathbf{F} \cdot d\mathbf{R} \quad \text{(by path-independence)} \\ &= \int_{-1}^{1} \frac{\pi}{2}\, e^0 \cos \frac{\pi(-t)}{2}\, (-dt) \\ &= -\frac{\pi}{2} \int_{-1}^{1} \cos\left(-\frac{\pi}{2} t\right) dt \\ &= -\frac{\pi}{2} \cdot \left. \frac{\sin(-\frac{\pi}{2} t)}{-\frac{\pi}{2}} \right|_{-1}^{1} = -1 - 1 = -2. \end{aligned}$$

Method II: Find the potential energy function w.

$\frac{\partial w}{\partial x} = M = \frac{\pi}{2} e^{2y} \cos \frac{\pi x}{2}$. Then,

$$\begin{aligned} w(x,y) &= \frac{\pi}{2} \int e^{2y} \cos \frac{\pi x}{2} dx \\ &= \frac{\pi}{2}\, e^{2y} \frac{\sin \frac{\pi x}{2}}{\frac{\pi}{2}} + \psi(y) \\ &= e^{2y} \sin \frac{\pi x}{2} + \psi(y) \end{aligned}$$

$\Rightarrow \frac{\partial w}{\partial y} = 2e^{2y} \sin \frac{\pi x}{2} + \psi' = N = 2e^{2y} \sin \frac{\pi x}{2}$

$\Rightarrow \psi' = 0 \Rightarrow \psi = c$

Thus, $w(x,y) = e^{2y} \sin \frac{\pi x}{2} + c$.

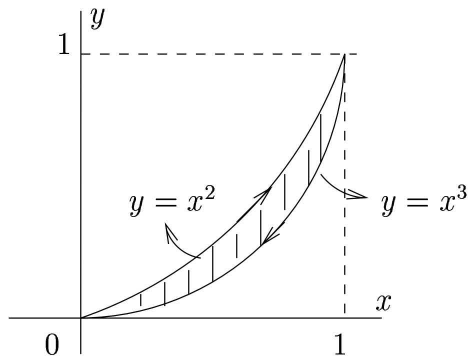
Hence, $W = w(B) - w(A) = \sin\left(-\frac{\pi}{2}\right) - \sin \frac{\pi}{2} = -1 - 1 = -2.$

6. A particle moves along the path C determined by the curves $y = x^2$ and $y = x^3$ where $0 \le x \le 1$. It starts its motion from the origin along $y = x^2$ and then it comes back to the origin along $y = x^3$ under a force field $\vec{F}(x, y) = (x^2 + y)\vec{\imath} + (x^2 - y)\vec{\jmath}$.

(a) Determine the total work done on the particle by the force field $\vec{F}$, by directly evaluating the proper line integrals.

Solution:

Let $M = x^2 + y$ and $N = x^2 - y$.



$$C_1 : \begin{cases} x = t \\ y = t^2 \text{ (from } t = 0 \text{ to } t = 1) \end{cases} \Rightarrow dx = dt,\ dy = 2tdt$$

$$C_2 : \begin{cases} x = t \\ y = t^3 \text{ (from } t = 1 \text{ to } t = 0) \end{cases} \Rightarrow dx = dt,\ dy = 3t^2dt$$

$W = \int_{C_1} \mathbf{F} \cdot d\mathbf{R} + \int_{C_2} \mathbf{F} \cdot d\mathbf{R}$ where $\mathbf{F} \cdot d\mathbf{R} = Mdx + Ndy = (x^2 + y)dx + (x^2 - y)dy$.

Then,

$$\begin{aligned} W &= \int_0^1 (t^2 + t^2)dt + (t^2 - t^2)2tdt + \int_1^0 (t^2 + t^3)dt + (t^2 - t^3)3t^2dt \\ &= \int_0^1 (t^2 - t^3 - 3t^4 + 3t^5)dt = \left(\frac{t^3}{3} - \frac{t^4}{4} - \frac{3t^5}{5} + \frac{t^6}{2}\right)\Bigg|_0^1 = \frac{1}{3} - \frac{1}{4} - \frac{3}{5} + \frac{1}{2} = -\frac{1}{60}. \end{aligned}$$

(b) Determine the total work done on the particle by the force field $\vec{F}$, by using Green's theorem.

Solution:

By Green's theorem, $W = \oint_C \mathbf{F} \cdot d\mathbf{R} = \iint_R -(N_x - M_y)\,dA$ (because C is oriented clockwise). Since $N_x = 2x$, $M_y = 1$,

$$\begin{aligned} W &= \iint_R (1 - 2x)\,dy\,dx = \int_0^1 \int_{x^3}^{x^2} (1 - 2x)\,dy\,dx \\ &= \int_0^1 (1 - 2x)y\,\Big|_{x^3}^{x^2}\,dx = \int_0^1 (1 - 2x)(x^2 - x^3)\,dx \\ &= \int_0^1 (x^2 - x^3 - 2x^3 + 2x^4)\,dx = \left(\frac{x^3}{3} - \frac{x^4}{4} - \frac{x^4}{2} + \frac{2x^5}{5}\right)\Bigg|_0^1 \\ &= \frac{1}{3} - \frac{1}{4} - \frac{1}{2} + \frac{2}{5} = -\frac{1}{60}. \end{aligned}$$

B U Department of Mathematics

Math 102 Calculus II

Spring 2003 Final

1. Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{u}$ and $\boldsymbol{v}$ be unit vectors in 3-space lying on the same plane. Assume that $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}$ and that $\boldsymbol{u}$ is perpendicular to $\boldsymbol{v}$. Show that $(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{u}\times\boldsymbol{v}) = \pm 1$. Explain what the $+$ and $-$ signs geometrically mean.

Solution:

Given is $||\boldsymbol{a}|| = ||\boldsymbol{b}|| = ||\boldsymbol{u}|| = ||\boldsymbol{v}|| = 1$ and $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{u}, \boldsymbol{v} \in \mathcal{P}$ where $\boldsymbol{n}$ is a normal vector for $\mathcal{P}$.
$\boldsymbol{a}\times\boldsymbol{b} \perp \mathcal{P} \Rightarrow \boldsymbol{a}\times\boldsymbol{b} /\!/ \boldsymbol{n}$ and $||\boldsymbol{a}\times\boldsymbol{b}|| = ||\boldsymbol{a}||\,||\boldsymbol{b}|| \sin\theta = 1$ as $\theta = \pi/2$ ($\boldsymbol{a}\perp\boldsymbol{b}$).
$\boldsymbol{u}\times\boldsymbol{v} \perp \mathcal{P} \Rightarrow \boldsymbol{u}\times\boldsymbol{v} /\!/ \boldsymbol{n}$ and $||\boldsymbol{u}\times\boldsymbol{v}|| = ||\boldsymbol{u}||\,||\boldsymbol{v}|| \sin\phi = 1$ as $\phi = \pi/2$ ($\boldsymbol{u}\perp\boldsymbol{v}$).
We conclude that both $\boldsymbol{a}\times\boldsymbol{b}$ and $\boldsymbol{u}\times\boldsymbol{v}$ are parallel unit vectors, so the angle between them is either 0 or π. Then:

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{u}\times\boldsymbol{v}) = \underbrace{||\boldsymbol{a}\times\boldsymbol{b}||}_{1}\ \underbrace{||\boldsymbol{u}\times\boldsymbol{v}||}_{1}\underbrace{\cos\alpha}_{\pm 1} = \pm 1.$$

If $+1$ then $\boldsymbol{a}\times\boldsymbol{b}$ and $\boldsymbol{u}\times\boldsymbol{v}$ have the same direction, if -1 then they have opposite directions.

2. Using the chain rule compute $\left.\dfrac{\partial w}{\partial u}\right|_{u=0,v=1}$ if $w = \sin xy + \sin y$, $x = u^2 + v^2$ and $y = uv$.

Solution:

$\dfrac{\partial w}{\partial u} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial u} = (y\cos xy + \sin y)2u + (x\cos xy + x\cos y)v$. At $u = 0, v = 1$ we have $x = 1, y = 0$. Substituting all these values:

$$\left.\frac{\partial w}{\partial u}\right|_{u=0,v=1,x=1,y=0} = 2$$

3. Find the points on the sphere $x^2 + y^2 + z^2 = 36$ that are closest to and farthest from the point $(1, 2, 2)$.

Solution:

We use Lagrange multipliers to solve this problem. The distance function is $d = \sqrt{(x-1)^2 + (y-2)^2 + (z-2)^2}$, hence enough to extremize its square. We find the maximum and minimum of $f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-2)^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 - 36 = 0$.
For a nonzero λ we set:

$$\boldsymbol{\nabla} f = \lambda \boldsymbol{\nabla} g \Rightarrow \langle 2(x-1), 2(y-2), 2(z-2)\rangle = \lambda\langle 2x, 2y, 2z\rangle.$$

Equating each corresponding slot:

$$(1-\lambda)x = 1,\ (1-\lambda)y = 2,\ (1-\lambda)z = 2.$$

Note that if $\lambda = 1$ then these three equations become inconsistent. So let $\lambda \neq 1$. Then we get:

$$x = \frac{1}{1-\lambda},\ y = \frac{2}{1-\lambda},\ z = \frac{2}{1-\lambda}.$$

The point we are seeking is on the sphere meaning that this triple must satisfy $g(x, y, z) = 0$:

$$\frac{1}{(1-\lambda)^2} + \frac{4}{(1-\lambda)^2} + \frac{4}{(1-\lambda)^2} = 36.$$

This is equivalent to saying $\lambda = 3/2$ or $\lambda = 1/2$. Now we compute f at each of these points induced by different values of λ, i.e. $\lambda = 3/2 \Leftrightarrow x = -2, y = -4, z = -4$ and $\lambda = 1/2 \Leftrightarrow x = 2, y = 4, z = 4$. Then:

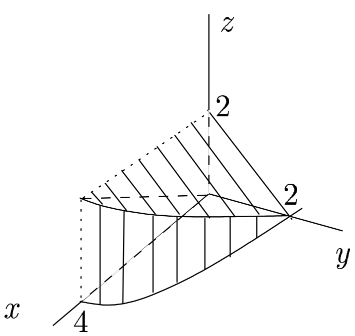
$$f(-2, -4, -4) = 81, \quad \text{and} \quad f(2, 4, 4) = 9.$$

We conclude that the closest point is $(2, 4, 4)$ with the distance 3 and the farthest point is $(-2, -4, -4)$ with distance 9.

4. Let R be the solid region in the first octant bounded by the coordinate planes, the plane $y + z = 2$ and the cylinder $x = 4 - y^2$. Express the volume of R as an iterated triple integral:

(a) in $dzdxdy$ order (do not evaluate).

Solution:



The solid is bounded by the coordinate planes which are $x = 0, y = 0, z = 0$. Hence easily:

$$V(R) = \int_0^2 \int_0^{4-y^2} \int_0^{2-y} dzdxdy.$$

(b) in $dydxdz$ order (do not evaluate).

Solution:

Along the y-axis (slices parallel to xz-plane) there are two possible boundaries. So this is the sum two triple integrals:

$$V(R) = \iint_{D_1} \int_0^{2-z} dydA + \iint_{D_2} \int_0^{\sqrt{4-x}} dydA,$$

where $D_1 = \{(x, z)|0 \le x \le 4 - (2-z)^2,\ 0 \le z \le 2\}$ and $D_2 = \{(x, z)|4 - (2-z)^2 \le x \le 4,\ 0 \le z \le 2\}$. Hence:

$$V(R) = \int_0^2 \int_0^{4-(2-z)^2} \int_0^{2-z} dydxdz + \int_0^2 \int_{4-(2-z)^2}^4 \int_0^{\sqrt{4-x}} dydxdz.$$

(c) Find the volume of R by evaluating one of the above triple integrals.

Solution:

First integral is obviously easier to evaluate:

$$\begin{aligned} V(R) &= \int_0^2\int_0^{4-y^2}\int_0^{2-y} dzdxdy = \int_0^2\int_0^{4-y^2}(2-y)dxdy = \int_0^2 (2-y)(4-y^2)dy \\ &= \left[8y - \frac{2}{3}y^3 - 2y^2 + \frac{y^4}{2}\right]_0^2 = 12 - \frac{16}{3} = \frac{20}{3}. \end{aligned}$$

5. Evaluate $\displaystyle\int_0^3\int_0^4\int_{y/2}^{1+y/2}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dxdydz$ by using the transformation $u=(2x-y)/2$, $v=y/2$ and $w=z/3$ and integrating over an appropriate region in the uvw-space.

Solution:

We easily find the back transformation to be $x=u+v$, $y=2v$, $z=3w$. Now we have to find new boundaries or the transformed region via the principle boundaries go to boundaries:

upper boundaries	lower boundaries
$x=1+y/2 \Rightarrow u+v=1+v \Rightarrow u=1$	$x=y/2 \Rightarrow u+v=v \Rightarrow u=0$
$y=4 \Rightarrow 2v=4 \Rightarrow v=2$	$y=0 \Rightarrow v=0$
$z=3 \Rightarrow 3w=3 \Rightarrow w=1$	$z=0 \Rightarrow w=0,$

which shows that the new region is a rectangular box (i.e. all boundaries are constant).
To transform the integral we need also the Jacobian of the transformation which is:

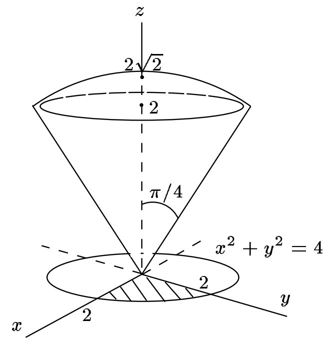
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

The integral transforms as follows:

$$\begin{aligned} \int_0^3\int_0^4\int_{y/2}^{1+y/2}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dxdydz &= \int_0^1\int_0^2\int_0^1 (u+w)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw \\ &= 6\int_0^1\int_0^2 (w+1/2)dvdw = 12\int_0^1 (w+1/2)dw \\ &= 12\left[\frac{w}{2}+\frac{w^2}{2}\right]_0^1 = 12. \end{aligned}$$

6. Evaluate $\displaystyle\int_0^2\int_0^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} z^2dzdxdy.$

Solution:



The boundaries suggest that it is convenient to use spherical coordinates: $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$ where $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$.
First remark is $\underbrace{\sqrt{x^2+y^2}}_{\text{cone}} \le z \le \underbrace{\sqrt{8-x^2-y^2}}_{\text{sphere}}$. Transform these into spherical coordinates:

$$\begin{aligned} z &= \sqrt{x^2+y^2} \Leftrightarrow \rho\cos\phi = \rho\sin\phi \Rightarrow \phi = \tfrac{\pi}{4} \\ z &= \sqrt{8-x^2-y^2} \Leftrightarrow \rho\cos\phi = \sqrt{8-\rho\sin^2\phi} \Rightarrow \rho = 2\sqrt{2}. \end{aligned}$$

Intersection of the sphere and the cone: $x^2+y^2 = 8-x^2-y^2 \Rightarrow x^2+y^2 = 4$ on the xy-plane. This means:

$$0 \le x \le \sqrt{4-y^2} \quad \text{and} \quad 0 \le y \le 2$$

which is the first quadrant of the circle with radius 2 and center at the origin on the xy-plane. Hence in spherical coordinates: $0 \le \theta \le \pi/2$. The radius of this circle is nothing but the projection of ρ onto xy-plane with the angle $\phi = \pi/4$. Thus, new boundaries are:

$$0 \le \theta \le \pi/2, \quad 0 \le \rho \le 2\sqrt{2}, \quad 0 \le \phi \le \pi/4.$$

Now rewriting and evaluating the triple integral:

$$\begin{aligned} \int_0^2\int_0^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} z^2 dzdxdy &= \int_0^{\pi/2}\int_0^{\pi/4}\int_0^{2\sqrt{2}} \rho^2\cos^2\phi\rho^2\sin\phi\, d\rho d\phi d\theta \\ &= \frac{2^{15/2}}{5}\int_0^{\pi/2}\int_0^{\pi/4}\cos^2\phi\sin\phi\, d\phi d\theta \\ &= \frac{2^{15/2}}{5}\left[\frac{1}{3} - \frac{1}{3\cdot 2^{3/2}}\right]\int_0^{\pi/2} d\theta \\ &= \frac{2^5}{15}(2\sqrt{2}-1)\pi = \frac{32}{15}(2\sqrt{2}-1)\pi. \end{aligned}$$

7. Consider the vector field $\boldsymbol{F} = 2xy^3\boldsymbol{i} + (1+3x^2y^2)\boldsymbol{j}$ defined in the xy-plane.

(a) Find a potential $\phi(x,y)$ for $\boldsymbol{F}$, i.e. such that $\boldsymbol{F} = \boldsymbol{\nabla}\phi$.

Solution:

First note that $\boldsymbol{\nabla}\phi = \langle \phi_x, \phi_y \rangle$. Equating this to $\boldsymbol{F}$ we obtain:

$$\langle 2xy^3, 1 + 3x^2y^2 \rangle = \langle \phi_x, \phi_y \rangle$$

and this yields two differential equations to solve:

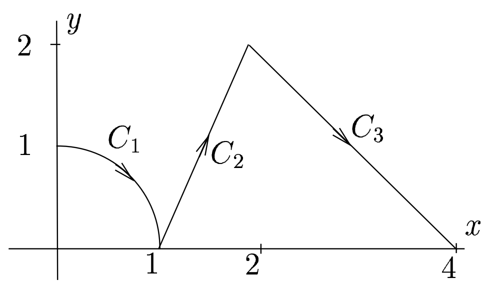
$$\phi_x = 2xy^3 \quad \text{and} \quad \phi_y = 1 + 3x^2y^2.$$

Integrating the first one: $\phi = x^2y^3 + a(y)$. From here we can compute the derivative with respect to y: $\phi_y = 3x^2y^2 + a'(y)$. But this must naturally coincide with $\phi_y = 1 + 3x^2y^2$ which entails $a'(y) = 1$ or $a(y) = y + c$, where $c \in \mathbb{R}$. Here the arbitrary constant c may be chosen as 0. The potential consequently becomes:

$$\phi = x^2y^3 + y.$$

(b) Find the work done by the force field $\boldsymbol{F}$ on a particle that moves along the curve C in the direction of increasing x, if C is formed by $y = \sqrt{1 - x^2}$ for $0 \le x \le 1$, $y = 2x - 2$ for $1 \le x \le 2$ and $y = -x + 4$ for $2 \le x \le 4$.

Solution:



Let us first label the three curves forming C by C_1, C_2 and C_3 respectively. Then the work done is:

$$W = \int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{C_1} \cdots + \int_{C_2} \cdots + \int_{C_3} \cdots.$$

But in part **(a)** we showed that $\boldsymbol{F}$ is conserved, namely work done by $\boldsymbol{F}$ is path independent. Hence we can apply the fundamental theorem for line integrals:

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C \boldsymbol{\nabla}\phi \cdot d\boldsymbol{r} = \phi(\text{end point}) - \phi(\text{initial point}).$$

Now we compute these points: $x = 4 \Rightarrow y = 0$ meaning $(4, 0)$ is the end point, and $x = 0 \Rightarrow y = 1$ meaning that $(0, 1)$ is the initial point of the path. Then:

$$W = \phi(4, 0) - \phi(0, 1) = 0 - 1 = -1.$$

8. **(a)** For which region D is the double integral $\displaystyle\iint_D \left(1 - \frac{x^2}{4} - y^2\right) dA$ maximum? (Hint: Think of the volume interpretation of a double integral.)

Solution:

$\iint_D f(x,y)dA$ represents a volume if $f(x,y) \geq 0$ in D. When $f(x,y)$ starts taking negative values the value of the double integral starts decreasing. Hence, to maximize the volume we must choose D as large as possible so that $f(x,y)$ remains nonnegative. After this general argumentation we come back to the question. Our $f \geq 0$ if and only if $\frac{x^2}{4} + y^2 \leq 1$. Thus, we choose D to be the elliptic region described by:

$$D = \{(x,y) \in \mathbb{R}^2 \,|\, \frac{x^2}{4} + y^2 \leq 1\},$$

so that the given integral is maximum.

(b) Find the simple (non-self-intersecting), positively oriented (counter-clockwise) closed curve C such that $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ is maximum if $\boldsymbol{F} = \left(\frac{x^2 y}{4} + \frac{y^3}{3}\right)\boldsymbol{i} + x\boldsymbol{j}$.

Solution:

Since C is a simple curve, the region bounded by it is simply connected. Then Green's theorem is applicable. We set $P = \frac{x^2 y}{4} + \frac{y^3}{3}$ and $Q = x$. Then:

$$\begin{aligned}
\int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_C \underbrace{\left(\frac{x^2 y}{4} + \frac{y^3}{3}\right)}_{P} dx + \underbrace{x}_{Q} dy \\
&= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \quad \text{(Green's thm.)} \\
&= \iint_D \left(1 - \frac{x^2}{4} - y^2\right) dA.
\end{aligned}$$

This is the integral maximized in part **(a)**. If D is the region found above then the required simple curve is the ellipse: $C : \frac{x^2}{4} + y^2 = 1$.

B U Department of Mathematics

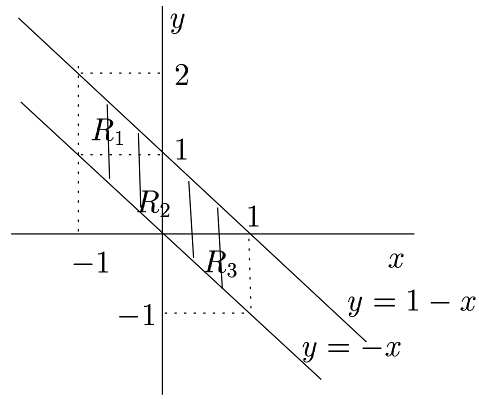
Math 102 Calculus II

Spring 2004 Final

1. Change the order of integration in the following integrals.

(a) $\displaystyle\int_{-1}^{1}\int_{-x}^{1-x} f(x,y)\,dy\,dx.$

Solution:



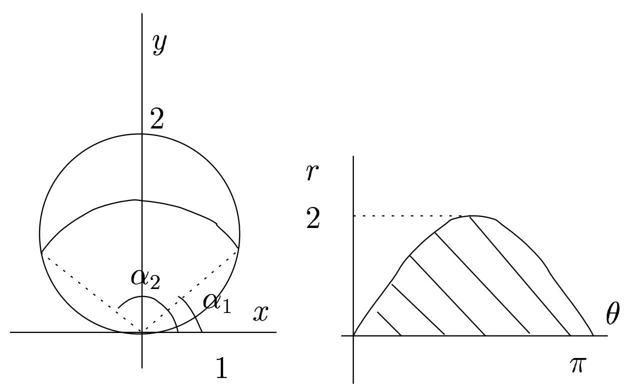
From the sketch we see that to consider $dxdy$ order of integration we divide the region into three parts.

$$R_1:\int_1^2\int_{-1}^{1-y} f(x,y)dxdy;\quad R_2:\int_0^1\int_{-y}^{1-y} f(x,y)dxdy;\quad R_3:\int_{-1}^{0}\int_{-y}^{1} f(x,y)dxdy.$$

$$\Rightarrow \int_{-1}^{1}\int_{-x}^{1-x} f(x,y)dydx = \int_1^2\int_{-1}^{1-y} f(x,y)dxdy+\int_0^1\int_{-y}^{1-y} f(x,y)dxdy+\int_{-1}^{0}\int_{-y}^{1} f(x,y)dxdy.$$

(b) $\displaystyle\int_{0}^{\pi}\int_{0}^{2\sin\theta} dr\,d\theta.$

Solution:



One can solve the problem either in xy-plane or in θr-plane. Let us consider the curve $r = 2\sin\theta$ in xy-plane. Using the relations $r^2 = x^2 + y^2, y = r\sin\theta$, we get $x^2 + (y-1)^2 = 1$. From this equation and the given limits of integration, we have a region as shown above. Now, $2\sin\alpha_1 = r$ and $2\sin\alpha_2 = r$. As we see $\alpha_2 > \pi/2 > \alpha_1$ and $\alpha_2 = \pi - \alpha_1$. Therefore, $\alpha_1 = \sin^{-1}(r/2)$ and $\alpha_2 = \pi - \sin^{-1}(r/2)$. (Note that $\sin^{-1}$ is defined for $-\pi/2 \leq \theta \leq \pi/2$). It follows that

$$\int_0^{\pi}\int_0^{2\sin\theta} dr d\theta = \int_0^2 \int_{\sin^{-1}(r/2)}^{\pi-\sin^{-1}(r/2)} d\theta dr.$$

2. Analyze the behavior of $f(x,y) = x^5y + xy^5 + xy$ at its critical points.

Solution:

The first partial derivatives are

$$\frac{\partial z}{\partial x} = 5x^4y + y^5 + y = y(5x^4 + y^4 + 1)$$

and

$$\frac{\partial z}{\partial y} = x(5y^4 + x^4 + 1).$$

The terms $5x^4 + y^4 + 1$ and $5y^4 + x^4 + 1$ are always greater than or equal to 1, and so it follows that the only critical point is $(0,0)$.
The second partial derivatives are

$$\frac{\partial^2 z}{\partial x^2} = 20x^3y\ ,\ \frac{\partial^2 z}{\partial y^2} = 20xy^3$$

and

$$\frac{\partial^2 z}{\partial x \partial y} = 5x^4 + 5y^4 + 1.$$

Thus at $(0,0)$, $D = -1$ and so $(0,0)$ is a saddle point.

3. Find the maximum and minimum values of the function $f(x,y) = x^2 + y^2 - x - y + 1$ in the disk D defined by $x^2 + y^2 \leq 1$.

Solution:

(i)To find the critical points we set $\partial f/\partial x = \partial f/\partial y = 0$. Thus, $2x - 1 = 0$,$2y - 1 = 0$, and hence $(x,y) = (\frac{1}{2}, \frac{1}{2})$ is the only critical point in the open disk $U = \{(x,y) : x^2 + y^2 < 1\}$.

(ii)The boundary ∂U can be parametrized by $\mathbf{c}(t) = (\sin t, \cos t)$,$0 \leq t \leq 2\pi$. Thus

$$f(\mathbf{c}(t)) = \sin^2 t + \cos^2 t - \sin t - \cos t + 1 = 2 - \sin t - \cos t = g(t).$$

To find the maximum and minimum of f on ∂U it suffices to locate the maximum and minimum of g. Now $g'(t) = 0$ only when

$$\sin t = \cos t \text{ ,that is, } t = \frac{\pi}{4}, \frac{5\pi}{4}\ .$$

Thus the candidates for the maximum and minimum for f and ∂U are the points $\boldsymbol{c}(\frac{\pi}{4})$, $\boldsymbol{c}(\frac{5\pi}{4})$ and the endpoints $\boldsymbol{c}(0) = \boldsymbol{c}(2\pi)$.

(iii)The values of f at the critical points are: $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$ from step (i) and, from step (ii),

$$f(\boldsymbol{c}(\frac{\pi}{4})) = f(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = 2 - \sqrt{2},$$

$$f(\boldsymbol{c}(\frac{5\pi}{4})) = f(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) = 2 + \sqrt{2},$$

and

$$f(\boldsymbol{c}(0)) = f(\boldsymbol{c}(2\pi)) = f(0,1) = 1.$$

(iv) Comparing all the values $\frac{1}{2}, 2 - \sqrt{2}, 2 + \sqrt{2}, 1$, it is clear that the absolute minimum occurs at $(\frac{1}{2}, \frac{1}{2})$ and the absolute maximum occurs at $(-\sqrt{2}/2, -\sqrt{2}/2)$.

4. Let C be an oriented simple curve connecting $(1,1,1)$ and $(1,2,4)$. Evaluate $\int_C 2xyz\,dx + x^2z\,dy + x^2y\,dz$. Justify your evaluation.

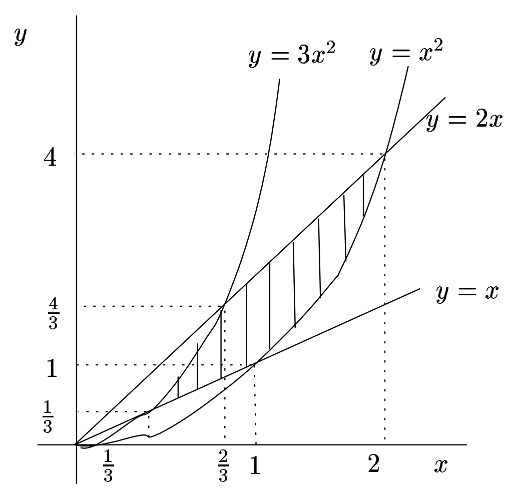
Solution:

Observe that $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k} = 2xyz\vec{i} + x^2z\vec{j} + x^2y\vec{k}$ is conservative with potential $\phi(x,y,z) = x^2yz$, i.e. $\partial\phi/\partial x = F_1$, $\partial\phi/\partial y = F_2$ and $\partial\phi/\partial z = F_3$. By Fundamental Theorem of Work Integrals, it follows that

$$\int_C 2xyz\,dx + x^2z\,dy + x^2y\,dz = \phi(1,2,4) - \phi(1,1,1) = 1 \cdot 2 \cdot 4 - 1 \cdot 1 \cdot 1 = 7$$

5. Let R be the region the the first quadrant of xy-plane bounded by the curves $y = x^2$, $y = 3x^2$, $y = x$ and $y = 2x$. Evaluate $\iint_R x^2\,dx\,dy$. Hint: You might try an appropriate change of variables.

Solution:



Consider the transformation

$$T : \mathbb{R}^2 \to \mathbb{R}^2, \qquad (u, v) \to (x = \frac{v}{u}, y = \frac{v^2}{u})$$

so that $v = \frac{y}{x}$ and $u = \frac{y}{x^2}$. With this transformation the rectangle enclosed by the lines $u = 1$, $u = 3$, $v = 1$, $v = 2$ is mapped onto the shaded region in xy-plane. The corresponding Jacobian is:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{v}{u^2} & \frac{1}{u} \\ -\frac{v^2}{u^2} & \frac{2v}{u} \end{vmatrix} = -2\frac{v^2}{u^3} + \frac{v^2}{u^3} = -\frac{v^2}{u^3}.$$

Note that there is a minus sign since the transformation T reverses orientation. Then,

$$\iint_R x^2\, dx\, dy = \int_1^3 \int_1^2 \frac{v^2}{u^2} \left| -\frac{v^2}{u^3} \right| dv du = -\left. \frac{1}{4} u^{-4} \right]_1^3 \cdot \left. \frac{1}{5} v^5 \right]_1^2 = \frac{124}{81}.$$

6. Let C be the boundary of the square with corners $(1,1)$, $(2,1)$, $(2,2)$, $(1,2)$, oriented counter-clockwise. Compute

$$\int_C (x^2 + y)\, dx + (2x + y^2)\, dy.$$

Solution:

Since (a) the enclosed square S is simply connected, (b) its boundary is a simple and closed curve and (c) $f = x^2 + y$ and $g = 2x + y^2$ are differentiable over S, one can use Green's Theorem to change this line integral into a double integral over S:

$$\int_C = \int_1^2 \int_1^2 \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dy dx = \int_1^2 \int_1^2 (2-1) dy dx = 1.$$

7. Let $\vec{A}$, $\vec{B}$, $\vec{C}$ be the vectors in 3-space. Assume that $\vec{A}$ is orthogonal to neither $\vec{B}$ nor $\vec{C}$.
(a) What is the angle between $\vec{A} \times (\vec{B} \times \vec{C})$ and $\text{proj}_{\vec{B}\times\vec{C}} \vec{A}$?

Solution:

Either (i) $\vec{A}$ is normal to $\vec{B} \times \vec{C}$ such that $\text{proj}_{\vec{B}\times\vec{C}} \vec{A}$ is the zero vector and **the angle is undefined**;

or (ii) the nonzero projection $\text{proj}_{\vec{B}\times\vec{C}} \vec{A} = \frac{\vec{A} \cdot (\vec{B} \times \vec{C})}{||\vec{B} \times \vec{C}||^2} \vec{B} \times \vec{C}$ is parallel to $\vec{B} \times \vec{C}$, hence orthogonal to $\vec{A} \times (\vec{B} \times \vec{C})$ by the definition of cross product. In this case, **the angle sought is $\pi/2$ radians**.

(b) If $\vec{A} \neq \vec{0}$, then does $\vec{A} \times \vec{B} = \vec{A} \times \vec{C}$ imply $\vec{B} = \vec{C}$? Justify your answer.

Solution:

No, for example, let $\vec{A} = \vec{i}$, $\vec{B} = \vec{i} + \vec{j}$ and $\vec{C} = 2\vec{i} + \vec{j}$. Then $\vec{A}$ is not orthogonal to $\vec{B}$, nor to $\vec{C}$, and $\vec{A} \times \vec{B} = \vec{i} \times \vec{i} + \vec{i} \times \vec{j} = \vec{0} + \vec{k} = \vec{k} = 2\vec{0} + \vec{k} = \vec{i} \times 2\vec{i} + \vec{i} \times \vec{j} = \vec{A} \times \vec{C}$, but $\vec{B} = \vec{i} + \vec{j} \neq 2\vec{i} + \vec{j} = \vec{C}$.

8. Evaluate $\displaystyle\lim_{(x,y)\to(2,1)} \frac{\sin^{-1}(xy-2)}{\tan^{-1}(3xy-6)}$.

Solution:

Let $a = xy - 2$. Then $(x, y) \to (2, 1)$ implies $a \to 0$. So

$$\lim_{(x,y)\to(2,1)} \frac{\sin^{-1}(xy-2)}{\tan^{-1}(3xy-6)} = \lim_{a\to 0} \frac{\sin^{-1}(a)}{\tan^{-1}(3a)}$$

which leads $\frac{0}{0}$ indeterminacy. After application of L'Hôpital's rule we get:

$$\lim_{a\to 0} \frac{\sin^{-1}(a)}{\tan^{-1}(3a)} = \lim_{a\to 0} \frac{1+9a^2}{3\sqrt{1-a^2}} = 1/3.$$

9. **Bonus question.** (a) Let $\vec{F} = 4x\vec{\imath} + 4y\vec{\jmath} + 2\vec{k}$. Let S be the surface that is the bottom of the paraboloid $z = x^2 + y^2$ with $z \le 1$. Find the outward flux away from the z-axis of $\vec{F}$ through S.
(b) Let Ω be the solid inside $z = x^2 + y^2$ and below $z = 1$. Let $\vec{F}$ be as above. Evaluate $\iiint_\Omega \vec{\nabla} \cdot \vec{F}\, dV$.
Are the results in part (a) and (b) equal? Why? [You may answer this question without solving (a) and (b).]

Solution:

(a) The surface S is parametrized as $x = u$, $y = v$, $z = u^2 + v^2$. A normal to S is at $(u, v, u^2 + v^2)$ is $\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = -2u\vec{\imath} - 2v\vec{\jmath} + \vec{k}$. This vector points *towards* the z-axis, not away from the z-axis. A normal vector *away from* the z-axis is $2u\vec{\imath} + 2v\vec{\jmath} - \vec{k}$. The flux is given by the surface integral of $\vec{F}$, namely it is equal to

$$\begin{aligned}
\textstyle\iint_S \vec{F} \cdot \vec{n}\, dS &= \iint_{u^2+v^2\le 1} (4u\vec{\imath} + 4v\vec{\jmath} + 2\vec{k}) \cdot (2u\vec{\imath} + 2v\vec{\jmath} - \vec{k})\, dA \\
&= \iint_{u^2+v^2\le 1} (8u^2 + 8v^2 - 2)\, dA \\
&= \int_0^{2\pi} \int_0^1 (8r^2 - 2) r\, dr\, d\theta \\
&= 2\pi[2r^4 - r^2]_0^1 = 2\pi.
\end{aligned}$$

(b) We have $\vec{\nabla} \cdot \vec{F} = (\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}) \cdot (4x\vec{\imath} + 4y\vec{\jmath} + 2\vec{k}) = \frac{\partial}{\partial x}4x + \frac{\partial}{\partial y}4y + \frac{\partial}{\partial z}2 = 4 + 4 + 0 = 8$, so

$$\begin{aligned}
\textstyle\iiint_\Omega \vec{\nabla} \cdot \vec{F}\, dV &= \iiint_\Omega 8\, dV = 8 \iiint_\Omega dV \\
&= 8 \iint_{x^2+y^2\le 1} \int_{x^2+y^2}^1 dz\, dA \\
&= 8 \iint_{x^2+y^2\le 1} [1 - (x^2 + y^2)]\, dA \\
&= 8 \int_0^{2\pi} \int_0^1 [1 - r^2]\, r\, dr\, d\theta \\
&= 8 \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 d\theta \\
&= 8 \cdot 2\pi \cdot \frac{1}{4} = 4\pi.
\end{aligned}$$

(c) The results in (a) and (b) are not equal. In fact, by the divergence theorem, the integral over Ω is equal to the surface integral of $\vec{F}$ over the whole surface enclosing Ω, and this surface consists of S *and* the disk $D = \{(x, y, z) : x^2 + y^2 \leq 1,\ z = 1\}$. Since

$$\iiint_\Omega \vec{\nabla} \cdot \vec{F}\, dV = \iint_S \vec{F} \cdot \vec{n}\, dS + \iint_D \vec{F} \cdot \vec{n}\, dS$$

and since $\vec{F}$ has a positive component in the direction of $\vec{k}$, which is $\vec{n}$ on D, the surface integral over D is positive, not zero, and the results cannot be equal. In more detail, we have

$$\begin{aligned}
\textstyle\iint_D \vec{F} \cdot \vec{n}\, dS &= \iint_{u^2+u^2\leq 1} (4u\vec{\imath} + 4v\vec{\jmath} + 2\vec{k}) \cdot \vec{k}\, dA \\
&= 2 \iint_{u^2+u^2\leq 1} dA \\
&= 2(\text{area of the unit disk}) = 2 \cdot \pi 1^2 = 2\pi \neq 0.
\end{aligned}$$

B U Department of Mathematics

Math 102 Calculus II

Date: June 2, 2005 Time: 15:00-17:00	Full Name :
	Math 102 Number :
	Student ID :

Spring 2005 Final Exam

IMPORTANT

1. Write your name, surname on top of each page. **2.** The exam consists of 8 questions some of which have more than one part. **3.** Read the questions carefully and write your answers neatly under the corresponding questions. **4.** Show all your work. Correct answers without sufficient explanation might not get full credit. **5.** Calculators are not allowed.

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	TOTAL
25 pts	25 pts	25 pts	25 pts	25 pts	25 pts	25 pts	25 pts	200 pts

---

1.) Find the point(s) on the surface $xy - z^2 = 1$ closest to the origin.

2.) Evaluate $\int_{(1,1,2)}^{(3,5,0)} yz\ dx\ +\ xz\ dy\ +\ xy\ dz.$

3.) Let **F** **(x,y,z)** be a vector field with continuous partial derivatives. Show that

div (curl F) $=$ **0**.

4.) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} 5\sqrt{x^2+y^2}\; dy dx$.

5.) Let L_1, L_2 be the lines whose parametric equations are
$L_1 : x = 1 + 7t,\ y = 3 + t,\ z = 5 - 3t$
$L_2 : x = 4 - t,\ y = 6,\ z = 7 + 2t.$
Find the distance between the two lines.

6.) Let a curve in the plane be parametrized by $x = 2t,\ y = t^2$.
a) Find the curvature function of the curve.
b) Find the tangential and normal components of the acceleration of a particle moving on this curve.

7.) Evaluate the line integral $\oint_C y^2 \, dx + x^2 \, dy$, where C is the square with vertices $(0,0), (1,0), (1,1)$ and $(0,1)$ oriented counterclockwise, using Green's Theorem and then check your answer by evaluating it directly.

8.) Find the volume of the solid enclosed between the surfaces $x = y^2 + z^2$ and $x = 1 - y^2$.

B U Department of Mathematics

Math 102 Calculus II

Spring 2006 Final

1. What horizontal plane is tangent to the surface $z = x^2 - 4xy - 2y^2 + 12x - 12y - 1$ and what is the point of the tangency?

Solution:

Begin by rewriting the surface equation as

$$F(x, y, z) = x^2 - 4xy - 2y^2 + 12x - 12y - 1 - z.$$

Obviously, the gradient of this function at the point (x_0, y_0, z_0) is a normal vector to the surface at (x_0, y_0, z_0), where

$$\vec{\nabla} F(x_0, y_0, z_0) = (2x_0 - 4y_0 + 12)\hat{\imath} + (-4x_0 - 4y_0 - 12)\hat{\jmath} + (-1)\hat{k}$$

For $\vec{\nabla} F(x_0, y_0, z_0)$ to be the normal vector to the horizontal plane, the first two components of $\vec{\nabla} F(x_0, y_0, z_0)$ should be zero. i.e.

$$\begin{aligned} 2x_0 - 4y_0 + 12 &= 0 \\ -4x_0 - 4y_0 - 12 &= 0 \end{aligned}$$

From these equations and the equation of surface, the point of tangency is $(-4, 1, -31)$. Hence the equation of the tangent plane is

$$\begin{aligned} \vec{\nabla} F(x_0, y_0, z_0) \quad \cdot \quad < x - x_0, y - y_0, z - z_0 > &= 0 \\ < 0, 0, -1 > \quad \cdot \quad < x + 4, y - 1, z + 31 > &= 0 \end{aligned}$$

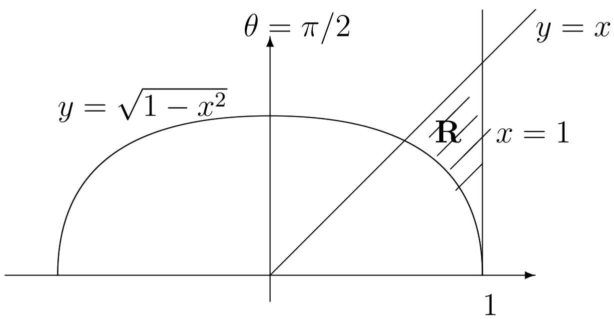
$$\Rightarrow z = -31$$

2. Evaluate $\displaystyle\int_{\sqrt{2}/2}^{1}\int_{\sqrt{1-x^2}}^{x}\frac{1}{\sqrt{x^2+y^2}}\,dydx.$

Solution:

We will use polar coordinates to evaluate this integral. So, first we will identify the region of integration.

Consider that for fixed x, y runs from $\sqrt{1-x^2}$ to x, and for fixed y, x runs from $\dfrac{\sqrt{2}}{2}$ to 1. So, the region is bounded below by a semicircle of radius 1 centered at the origin, and is bounded above by the line $x = y$. The region of the integration is the shaded area in the figure:



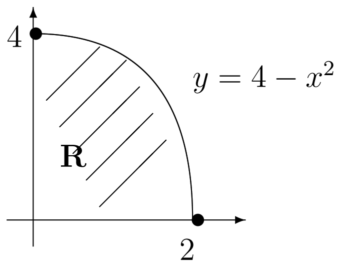
Now let us describe this area in polar coordinates. Consider that r runs from the circle with radius 1 to the line $x = 1$. The equation of the circle is $r = 1$, and the equation for $x = 1$ is $r\cos\theta = 1 \Rightarrow r = \dfrac{1}{\cos\theta}$. So, r varies from 1 to $\dfrac{1}{cos\theta}$ and θ varies from 0 to $\dfrac{\pi}{4}$. Thus,

$$
\begin{aligned}
\int_{\sqrt{2}/2}^{1}\int_{\sqrt{1-x^2}}^{x}\frac{1}{\sqrt{x^2+y^2}}\,dydx &= \int_0^{\pi/4}\int_1^{1/\cos\theta}\frac{1}{r}r\,drd\theta \\
&= \int_0^{\pi/4}(\frac{1}{\cos\theta}-1)d\theta \\
&= \int_0^{\pi/4}(\frac{d\theta}{\cos\theta}-\int_0^{\pi/4}d\theta \\
&= \ln|\sec\theta+\tan\theta|-\theta\Big|_0^{\pi/4} \\
&= \ln|\sqrt{2}+2|-\frac{\pi}{4}
\end{aligned}
$$

3. Find the volume of the solid in the first octant bounded by the coordinate planes and the surface $z = 4 - x^2 - y$.

Solution:

The projection of the solid G on the xy-plane is shown in the figure:



The volume of the solid can be found as follows:

$$
\begin{aligned}
V = \textstyle\iiint dV &= \iint_R \left[\int_0^{4-x^2-y} dz \right] dA \\
&= \int_0^2 \int_0^{4-x^2} \int_0^{4-x^2-y} dz\ dy\ dx \\
&= \int_0^2 \int_0^{4-x^2} (4 - x^2 - y)\ dy\ dx \\
&= \int_0^2 \left[4y - x^2 y - \frac{y^2}{2} \right]_{y=0}^{4-x^2} dx \\
&= \int_0^2 [8 - 4x^2 + \frac{x^4}{2}] dx \\
&= 8x - \frac{4x^3}{3} + \frac{x^5}{10} \Big|_0^2 \\
&= \frac{128}{15}
\end{aligned}
$$

4. Let C be the triangle whose edges are x=0, x+y = 1, y=0. Apply Green's theorem to evaluate the line integral

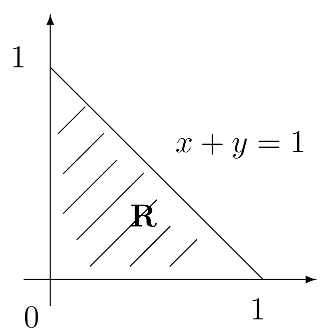
$$\int_C y^2 dx + x^2 dy$$

Solution:

Since $f(x, y) = y^2$ and $g(x, y) = x^2$, it follows from Green's theorem that

$$\begin{aligned}\int_C y^2 dx + x^2 dy &= \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(y^2)\right] dA \\ &= \iint_R (2x - 2y) dA\end{aligned}$$

As given in the question, the region of integration is



Thus,

$$\begin{aligned}\int_C y^2 dx + x^2 dy &= \int_0^1 \int_0^{1-x} (2x - 2y)\ dy\ dx \\ &= \int_0^1 \left[2xy - y^2\right]_{y=0}^{1-x} dx \\ &= \int_0^1 [2x - 2x^2 - 1 + 2x - x^2]\ dx \\ &= x^2 - \frac{2x^3}{3} - x + x^2 - \frac{x^3}{3}\Big|_0^1 \\ &= 0\end{aligned}$$

5. Let C be the curve $x = \frac{\pi}{2}\ t^5 + \frac{\pi}{2}\ t^4,\ y = t^8 + t^9 + 1,\ 0 \le t \le 1.$

Find the line integral

$$\int_C (y\cos xy + \frac{1}{y}\ e^{x/y})\ dx + (x\cos xy - \frac{x}{y^2}\ e^{x/4})\ dy$$

Solution:

When $t = 0$, we have $x = 0, y = 1$ and when $t = 1$, we have $x = \pi, y = 3$. The curve C is from $(x_0, y_0) = (0, 1)$ to $(x_1, y_1) = (\pi, 3)$.

Put $P(x, y) = y\cos xy + \frac{1}{y}\ e^{x/y}, Q(x, y) = x\cos xy - \frac{x}{y^2}\ e^{x/y}$. Then

$$\begin{aligned}
\frac{\partial P}{\partial y} &= y(-\sin xy)x + \cos xy - \frac{1}{y^2}\ e^{x/y} + \frac{1}{y}\left(-\frac{x}{y^2}\right)\ e^{x/y} \\
&= -xy\sin xy + \cos xy - \frac{1}{y^2}\ e^{x/y} - \frac{x}{y^3}\ e^{x/y}. \\
\frac{\partial Q}{\partial x} &= x(-\sin xy)y + \cos xy - \frac{1}{y^2}\ e^{x/y} - \frac{x}{y^2}\left(\frac{1}{y}\right)\ e^{x/y} \\
&= -xy\sin xy + \cos xy - \frac{1}{y^2}\ e^{x/y} - \frac{x}{y^3}\ e^{x/y}.
\end{aligned}$$

So, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ and $F = P\ \hat{\imath} + Q\ \hat{\jmath}$ is conservative.

Let's find a potential function ϕ for F. We have

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= y\cos xy + \frac{1}{y}\ e^{x/y} \\
\phi &= \int (y\cos xy + \frac{1}{y}\ e^{x/y})\ dx \\
\phi(x, y) &= \sin xy + e^{x/y} + f(y)
\end{aligned}$$

We also have

$$\begin{aligned}
Q(x, y) &= x\cos xy - \frac{x}{y^2}e^{x/y} = \frac{\partial \phi}{\partial y} \\
\frac{\partial \phi}{\partial y} &= (\cos xy)x + \frac{x}{y^2}e^{x/y} + f'(y) \\
&\Rightarrow\ f'(y) = 0, f(y) = c,
\end{aligned}$$

So,

$$\begin{aligned}
\phi(x,y) &= \sin xy + e^{x/y} + c \\
\int_C P dx + Q dy &= \phi(x_1, y_1) - \phi(x_0, y_0) \\
&= \phi(\pi, 3) - \phi(0, 1) \\
&= (\sin 3\pi + e^{\pi/3} + c) - (\sin \pi + e^{0/1} + c) \\
&= e^{\pi/3} - 1
\end{aligned}$$

Boğaziçi University
Department of Mathematics
Math 102 Calculus II

1	2	3	4	5	6	$\sum$
20 points each						**120**

Date: June 2, 2008	Full Name : ÖZDEMİR ASAF
Time: 10:00-12:00	Math 102 Number :
	Student ID :

Spring 2008 – Final Examination

IMPORTANT

1. Write your name, surname on top of each page. **2.** The exam consists of 4 questions some of which have more than one part. **3.** Read the questions carefully and write your answers neatly under the corresponding questions. **4.** Show all your work. Correct answers without sufficient explanation might not get full credit. **5.** Calculators are not allowed.

---

1. If $f = f(x-y, y-z, z-x)$, then show that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0.$$

Solution: We let $u = x-y$, $v = y-z$, $w = z-x$, then $f = f(u,v,w)$. Since

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = -1, \quad \frac{\partial u}{\partial z} = 0,$$

$$\frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 1, \quad \frac{\partial v}{\partial z} = -1,$$

and

$$\frac{\partial w}{\partial x} = -1, \quad \frac{\partial w}{\partial y} = 0, \quad \frac{\partial w}{\partial z} = 1.$$

We get

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}.\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}.\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w},$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}.\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}.\frac{\partial w}{\partial y} = -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v},$$

and

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u}.\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}.\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}.\frac{\partial w}{\partial z} = -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}.$$

Then the result follows.

2. Determine whether the following limits exist. If so, find them. If not, give an argument.

(10)**a.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$

Solution: We have

$$
\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} &= \lim_{(x,y)\to(0,0)} \frac{(x^2 - xy)(\sqrt{x} + \sqrt{y})}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})} \\
&= \lim_{(x,y)\to(0,0)} \frac{x(x-y)(\sqrt{x} + \sqrt{y})}{x - y} \\
&= \lim_{(x,y)\to(0,0)} x(\sqrt{x} + \sqrt{y}) \\
&= 0(\sqrt{0} + \sqrt{0}) \\
&= 0
\end{aligned}
$$

by the continuity of the functions x, $\sqrt{x}$, and $\sqrt{y}$.

(10)**b.** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + y^2}$

Solution: Suppose that the limit exists. Then if we approach to the point $(0,0)$ along the line $y = mx$, then we get

$$
\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + y^2} &= \lim_{x\to 0} \frac{2mx^2}{x^2 + m^2x^2} \\
&= \lim_{x\to 0} \frac{2mx^2}{x^2(m^2 + 1)} \\
&= \frac{2m}{m^2 + 1}.
\end{aligned}
$$

So for $m = 1$ and $m = 2$, we get different limits and this gives a contradiction. Therefore we conclude that the limit

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + y^2}$$

does NOT exist.

3. The surfaces

$$f(x, y, z) = x^2 + y^2 - 2 = 0$$

and

$$g(x, y, z) = x + z - 4 = 0$$

meet in an ellipse E. Find the parametric equations for the line tangent to E at the point $P(1, 1, 3)$.

Solution: The tangent line is orthogonal to both ∇f and ∇g at P, and therefore parallel to $v = \nabla f \times \nabla g$. The components of v and the coordinates of P give an equation for the line. We have

$$\nabla f(1, 1, 3) = (2x\mathrm{i} + 2y\mathrm{j})\big|_{(1,1,3)} = 2\mathrm{i} + 2\mathrm{j}$$

$$\nabla g(1, 1, 3) = (\mathrm{i} + 2\mathrm{k})\big|_{(1,1,3)} = \mathrm{i} + \mathrm{k}$$

and

$$v = (2\mathrm{i} + 2\mathrm{j}) \times (\mathrm{i} + \mathrm{k}) = 2\mathrm{i} - 2\mathrm{j} - 2\mathrm{k}.$$

So, the line is given by

$$x = 1 + 2t, \quad y = 1 - 2t, \quad z = 3 - 2t.$$

4. Evaluate the following integral

$$\int\int_R xydA$$

where R is the region bounded by the lines $y = x$, $y = 2x$, and $x + y = 2$.

Solution: After sketching the graph of the region R, one should have

$$\int\int_R xydA = \int_0^{2/3}\int_x^{2x} xydydx + \int_{2/3}^{1}\int_x^{2-x} xydydx = 13/81.$$

5.

(5)**a.** Find the line integral of $f(x, y, z) = x + y + z$ over the straight line segment from $(1, 2, 3)$ to $(0, -1, 1)$.

Solution: The straight line segment from $(1, 2, 3)$ to $(0, -1, 1)$ can be given as

$$r(t) = (\mathrm{i} + 2\mathrm{j} + 3\mathrm{k}) + t(-\mathrm{i} - 3\mathrm{i} - 2\mathrm{k}) = (1 - t)\mathrm{i} + (2 - 3t)\mathrm{j} + (3 - 2t)\mathrm{k},$$

where $t \in [0, 1]$. This gives

$$r'(t) = -\mathrm{i} - 3\mathrm{j} - 2\mathrm{k} \text{ and } ||r'(t)|| = \sqrt{14}.$$

Then we have

$$\int_C f(x, y, z)ds = \int_0^1 f(r(t))||r'(t)||dt = \int_0^1 (6 - 6t)\sqrt{14}dt = 3\sqrt{14}.$$

(15)**b.** Find the line integral of $f(x, y, z) = x + \sqrt{y} - z^2$ over **the path** from $(0, 0, 0)$ to $(1, 1, 1)$ given by

$$C_1 : \quad \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \leq t \leq 1$$

and

$$C_2 : \quad \mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}, \quad 0 \leq t \leq 1.$$

Solution: We observe that C_1 runs from the point $(0, 0, 0)$ to the point $(1, 1, 0)$ and C_2 runs from the point $(1, 1, 0)$ to the point $(1, 1, 1)$ and **the path** from $(0, 0, 0)$ to $(1, 1, 1)$ is a piecewise regular path with components C_1 and C_2. Therefore

$$\int_C f ds = \int_{C_1} f ds + \int_{C_2} f ds$$

with suitable orientations of C_1 and C_2. We compute

$$C_1 : \quad r(t) = t\mathrm{i} + t^2\mathrm{j}, \quad t \in [0, 1] \Rightarrow r'(t) = \mathrm{i} + 2t\mathrm{j} \Rightarrow ||r'(t)|| = \sqrt{1 + 4t^2}.$$

Then

$$\int_{C_1} f(x, y, z)ds = \int_0^1 f(r(t))||r'(t)||dt = \int_0^1 2t\sqrt{1 + 4t^2}dt = 1/6(5\sqrt{5} - 1).$$

Similarly,

$$C_2 : \quad r(t) = \mathrm{i} + \mathrm{j} + t\mathrm{k}, \quad t \in [0, 1] \Rightarrow r'(t) = \mathrm{k} \Rightarrow ||r'(t)|| = 1,$$

and

$$\int_{C_2} f(x, y, z)ds = \int_0^1 f(r(t))||r'(t)||dt = \int_0^1 (2 - t^2)(1)dt = 5/3.$$

Therefore we get

$$\int_C f(x, y, z)ds = \int_{C_1} f(x, y, z)ds + \int_{C_2} f(x, y, z)ds = 5/6\sqrt{5} + 3/2.$$

6. Use the transformation $u = x$, $v = z - y$, $w = xy$ to find

$$\int\int\int_G (z-y)^2 xy dV$$

where G is the solid enclosed by the surfaces $x = 1$, $x = 3$, $z = y$, $z = y + 1$, $xy = 2$, $xy = 4$.

Solution: We observe that by the transformation given, the surfaces that bounds the solid G in xyz-space corresponds to planes in uvw-space.

$$x = 1 \Rightarrow u = 1,$$

$$x = 3 \Rightarrow u = 3,$$

$$z - y = 0 \Rightarrow v = 0,$$

$$z - y = 1 \Rightarrow v = 1,$$

$$xy = 2 \Rightarrow w = 2,$$

$$xy = 4 \Rightarrow w = 4.$$

In order to write the integral in uvw-space, we also need the Jacobian. We compute

$$x = u, \quad y = w/u, \quad z = v + w/u$$

and

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = -\frac{1}{u}.$$

Now, we have

$$\int\int\int_G (z - y^2) xy dV = \int\int\int_H \left(v + \frac{w}{u} - \frac{w}{u}\right)^2 u\frac{w}{u}\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dV$$

where H is the image of G under the transformation. Finally, we get

$$\begin{aligned}\int\int\int_H \left(v + \frac{w}{u} - \frac{w}{u}\right)^2 u\frac{w}{u}\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dV &= \int_2^4\int_0^1\int_1^3 v^2 w\left| -\frac{1}{u}\right| dudvdw \\ &= 2\ln 3.\end{aligned}$$

B U Department of Mathematics

Math 102 Calculus II

Summer 2001 Final

1. The point $P = (1, -1, 2)$ lies on both the paraboloid $f(x, y, z) = x^2+y^2-z = 0$ and the ellipsoid $g(x, y, z) = 2x^2+3y^2+z^2-9 = 0$. Write an equation of the plane containing the point P that is normal to the curve of intersection between the paraboloid and the ellipsoid.

Solution:

We first determine the line formed by the intersection of the given surfaces. We solve for the curve that satisfies both the equations:

$$\begin{cases} x^2 + y^2 - z &= 0 \\ 2x^2 + 3y^2 + z^2 - 9 &= 0 \end{cases}$$

Subtracting twice the first equation from the second one gives $y^2 + z^2 + 2z - 9 = 0$ or $y^2 = -z^2 - 2z + 9$. Plugging this y^2 into the first equation gives $x^2 - z^2 - 3z + 9 = 0$ or $x^2 = z^2 + 3z - 9$. We have obtained expressions for x^2 and y^2 in terms of z. Now we set $z = t$ to get the set of equations that determine the curve of intersection:

$$\begin{cases} x^2 &= t^2 + 3t - 9 \\ y^2 &= -t^2 - 2t + 9 \\ z &= t \end{cases}$$

$\mathbf{r}'(t) = \dfrac{dx}{dt}\,\mathbf{i} + \dfrac{dy}{dt}\,\mathbf{j} + \dfrac{dz}{dt}\,\mathbf{k}$ at the point corresponding to $(x, y, z) = P = (1, -1, 2)$ will give a vector that is tangent to the curve of intersection. We can evaluate $\mathbf{r}'(t)$ at the given point by the help of implicit differentiation, with $x = 1$, $y = -1$, $z = 2$, $t = 2$. We have:

$$2x\frac{dx}{dt} = 2t + 3 \Rightarrow 2\left.\frac{dx}{dt}\right|_{\substack{x=1\\t=2}} = 7 \Rightarrow \left.\frac{dx}{dt}\right|_{\substack{x=1\\t=2}} = \frac{7}{2}.$$

Similarly,

$$2x\frac{dy}{dt} = -2t - 2 \Rightarrow -2\left.\frac{dx}{dt}\right|_{\substack{x=1\\t=2}} = -6 \Rightarrow \left.\frac{dx}{dt}\right|_{\substack{x=1\\t=2}} = 3;$$

and $\dfrac{dz}{dt} = 1$.

Thus, the vector $\frac{7}{2}\,\mathbf{i} + 3\,\mathbf{j} + \,\mathbf{k}$ is normal to the desired plane passing through $P = (1, -1, 2)$. So the equation of the plane is:

$$\frac{7}{2}(x-1) + 3(y+1) + (z-2) = 0$$

or,

$$7x + 6y + 2y = 5.$$

2. Consider the region Q bounded by the curves $xy = 1$, $xy = 3$, $x^2 - y^2 = 1$, $x^2 - y^2 = 4$. Let T be the transformation from x-y plane to u-v plane given by $u = xy$, $y = x^2 - y^2$. ***(i)*** Find the corresponding region Q' in the u-v plane. ***(ii)*** Calculate the integral $I = \iint_Q (x^2 + y^2)dxdy$.

[*Hint*: $\dfrac{\partial(x, y)}{\partial(u, v)} = \left(\dfrac{\partial(u, v)}{\partial(x, y)}\right)^{-1}$, or, you may want to use the inverse transformation $x = \dfrac{1}{\sqrt{2}}\sqrt{\sqrt{4u^2 + v^2} + v}$, $y = \dfrac{1}{\sqrt{2}}\sqrt{\sqrt{4u^2 + v^2} - v}$, at your own risk!]

Solution:

(i) The region corresponds to the rectangle bounded by the lines $u = 1$, $u = 3$, $v = 1$, $v = 4$.

(ii) First we compute the Jacobian:

$$\frac{\partial(x, y)}{\partial(u, v)} = \left(\frac{\partial(u, v)}{\partial(x, y)}\right)^{-1} = \left(\left|\begin{array}{cc} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{array}\right|\right)^{-1} = \left(\left|\begin{array}{cc} y & 2x \\ x & -2y \end{array}\right|\right)^{-1} = \frac{-1}{2(x^2 + y^2)}$$

Now, we can evaluate the integral in the u-v plane:

$$I = \iint_Q (x^2 + y^2)dxdy = \iint_{Q'} (x^2 + y^2)\left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv = \int_1^4 \int_1^3 \frac{1}{2}\, dudv = 3$$

3. Find the volume V of the solid bounded below by the sphere $\rho = 2\cos\phi$ and above by the cone $z = \sqrt{x^2 + y^2}$ using a triple integral in ***spherical coordinates.*** ***Show clearly*** how you find the limits of the integrals.

Solution:

For the points on the cone, $\phi = \tan^{-1}\left(\dfrac{\sqrt{x^2 + y^2}}{z}\right) = \dfrac{\pi}{4}$. Since all the points in the solid are below the cone, ϕ must be greater than $\pi/4$. Also, since they have positive z coordinates, ϕ must be less than $\pi/2$. So

$$\frac{\pi}{4} \le \phi \le \frac{\pi}{2}.$$

The radius, ρ, is bounded above by the sphere, so

$$0 \le \rho \le 2\cos\phi.$$

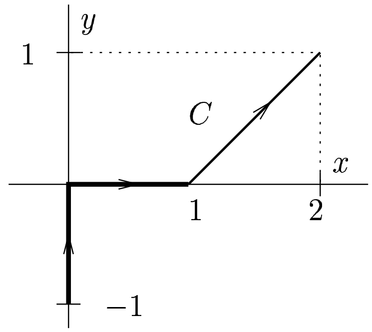
Since there no restriction for θ:

$$0 \le \theta \le 2\pi.$$

Therefore, the volume of the solid can be found by the integral:

$$\begin{aligned}
\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} \rho^2 \sin\theta \, d\rho\, d\phi\, d\theta &= \int_0^{2\pi}\int_{\pi/4}^{\pi/2} \frac{8}{3}\cos^3\phi \sin\phi \, d\phi\, d\theta \\
&= \int_0^{2\pi} \left[-\frac{2}{3}\cos^4\phi\right]_{\pi/4}^{\pi/2} d\theta \\
&= \int_0^{2\pi} \frac{1}{6}\, d\theta = \frac{\pi}{3}.
\end{aligned}$$

4. Given the vector fields: $\mathbf{F}_1(x,y,z) = (2xyz - y)\,\mathbf{i} = (x^2z + 3y^2 - x)\,\mathbf{j} + x^2y\,\mathbf{k}$, $\mathbf{F}_2(x,y,z) = y^2\,\mathbf{i} = x^2\,\mathbf{j} + 5\,\mathbf{k}$. ***(i)*** One of $\mathbf{F}_1$ and $\mathbf{F}_1$ is conservative, the other is not. Find which is which. ***(ii)*** For the conservative $\mathbf{F}$ calculate the potential. ***(iii)*** For the conservative $\mathbf{F}$ calculate the line integral over the path C in the x-y plane, as shown in the figure. ***(iv)*** For the nonconservative $\mathbf{F}$ calculate the line integral over the same path C.



Solution:

(i) The field $\mathbf{F}_1(x,y,z) = (2xyz - y)\,\mathbf{i} = (x^2z + 3y^2 - x)\,\mathbf{j} + x^2y\,\mathbf{k}$ is conservative since

$$\frac{\partial(2xyz - y)}{\partial y} = 2xz - 1 = \frac{\partial(x^2z + 3y^2 - x)}{\partial x}$$

$$\frac{\partial(x^2z + 3y^2 - x)}{\partial z} = x^2 = \frac{\partial x^2y}{\partial y}$$

$$\frac{\partial(2xyz - y)}{\partial z} = 2xy = \frac{\partial x^2y}{\partial y}$$

The field $\mathbf{F}_2(x,y,z) = y^2\,\mathbf{i} = x^2\,\mathbf{j} + 5\,\mathbf{k}$ is not conservative, because:

$$\frac{\partial y^2}{\partial y} = 2y \neq \frac{\partial x^2}{\partial x} = 2x.$$

(ii) Let $f(x,y,z)$ be a potential for the field $\mathbf{F}_1(x,y,z) = (2xyz - y)\,\mathbf{i} = (x^2z + 3y^2 - x)\,\mathbf{j} + x^2y\,\mathbf{k}$. Then $\dfrac{\partial f}{\partial x} = 2xyz - y$, so $f(x,y,z) = x^2yz - xy + h(y,z)$ for some function h. Taking the partial derivatives of f with respect to y and z, we find $\dfrac{\partial h}{\partial y} = 3y^2$ and $\dfrac{\partial h}{\partial z} = 0$. So $h = y^3 + c$ for some positive c, and $f(x,y,z) = x^2yz - xy + y^3 + c$ is a potential function.

(iii) Since the field $\mathbf{F}_1$ is path conservative, the line integral is equal to the difference of the values of the potential f, which we have found above, at the end points. $\displaystyle\int_C \mathbf{F}_1 \cdot d\mathbf{r} = f(1,1,0) - f(0,-1,0) = 1.$

(iv) The path C is the union of the smooth curves C_1, C_2 and C_3, where

$$\begin{array}{lll} C_1: & t\,\mathbf{j}, & -1 \le t \le 0 \\ C_2: & t\,\mathbf{i}, & 0 \le t \le 1 \\ C_3: & \mathbf{i} + t\,\mathbf{j}, & 0 \le t \le 1 \end{array}$$

So,

$$\begin{aligned} \int_C \mathbf{F}_2 \cdot d\mathbf{r} &= \int_{C_1} \mathbf{F}_2 \cdot d\mathbf{r} + \int_{C_2} \mathbf{F}_2 \cdot d\mathbf{r} + \int_{C_3} \mathbf{F}_2 \cdot d\mathbf{r} \\ &= \int_{-1}^{0} (t^2\,\mathbf{i}) \cdot \mathbf{j}\,dt + \int_0^1 (t^2\,\mathbf{j}) \cdot \mathbf{i}\,dt + \int_0^1 (t^2\,\mathbf{i} + \mathbf{j}) \cdot \mathbf{j}\,dt \\ &= 0 + 0 + 1 = 1 \end{aligned}$$

5. A cardioid in polar coordinates is given by $r = 1 + \cos\theta$, $0 \le \theta \le 2\pi$. Find the area of the cardioid ***by using the Green area formula***, $A = \frac{1}{2}\int_C x dy - y dx$, using θ as the parameter. (Do not use any other method to calculate the area!)

Solution:

We parametrize the boundary curve by $-\pi \le \theta \le \pi$ as follows:

$$\begin{aligned} x(\theta) &= r(\theta)\cos(\theta) = (1 + \cos(\theta)\cos(\theta) \\ y(\theta) &= r(\theta)\sin(\theta) = (1 + \cos(\theta)\sin(\theta). \end{aligned}$$

Rearranging these equations, we get :

$$x(\theta) = \frac{1}{2}\left(1 + \cos 2\theta + 2\cos\theta\right), \quad y(\theta) = \frac{1}{2}\left(\sin 2\theta + 2\sin\theta\right)$$

It follows that:

$$\frac{d}{d\theta}x(\theta) = -\left(\sin 2\theta + \sin\theta\right), \quad \frac{d}{d\theta}y(\theta) = \left(\cos 2\theta + \cos\theta\right)$$

Thus,

$$\begin{aligned} A &= \frac{1}{2}\int_C x dy - y dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi} \frac{1}{2}\left(1 + \cos 2\theta + 2\cos\theta\right)\left(\cos 2\theta + \cos\theta\right)\,d\theta \\ &\quad + \frac{1}{2}\int_{-\pi}^{\pi} \frac{1}{2}\left(\sin 2\theta + 2\sin\theta\right)\left(\sin 2\theta + \sin\theta\right)\,d\theta \\ &= \frac{1}{2}\left(\frac{3\pi}{2} + \frac{3\pi}{2}\right) = \frac{3\pi}{2}. \end{aligned}$$

6. Consider the line integral $I = \int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \left(\frac{1}{4}x^2 y + \frac{1}{3}y^3\right)\,\mathbf{i} + x\,\mathbf{j}$. Find the smooth, simple, closed curve C with counterclockwise orientation, that maximizes I. [*Hint:* Convert I to a surface integral. In which region of the x-y plane is the integrand of the surface integral positive?]

Solution:

Using Green's theorem, we get

$$I = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_Q \frac{\partial x}{\partial x} - \frac{\partial}{\partial dy}\left(\frac{1}{4}x^2 y + \frac{1}{3}y^3\right)\, dA = \iint_Q 1 - \left(\frac{1}{4}x^2 + y^2\right) dA.$$

The integrand is positive when $\dfrac{x^2}{4} + y^2 \leq 1$. The boundary curve of this region, $\dfrac{x^2}{4} + y^2 = 1$, is the curve that maximizes the line integral. The curve can also be parametrized as:

$$x = 2\cos t,\ \ y = \sin t,\ \ 0 \leq t \leq 2\pi.$$

B U Department of Mathematics

Math 102 Calculus II

Summer 2002 Final

1. A point moves along the parabola $y = x^2$ in such a way that the horizontal component of its velocity is always 3. Find the tangential and normal components of its acceleration at the point $P(1, 1)$.

Solution:

Let $x = 3t$, $y = 9t^2$ be the parametric equations of the parabola. So the point $P(1, 1)$ corresponds to $t = \dfrac{1}{3}$.

Now $\boldsymbol{r}(t) = 3t\boldsymbol{i} + 9t^2\boldsymbol{j}$, then $\boldsymbol{v}(t) = 3\boldsymbol{i} + 18t\boldsymbol{j}$ and $\|\boldsymbol{v}(t)\| = \sqrt{9 + (18t)^2}$ so that $\|\boldsymbol{v}(t)\||_{t=1/3} = \sqrt{75} = 3\sqrt{5}$. Hence, $\boldsymbol{T} = \dfrac{\boldsymbol{v}(t)}{\|\boldsymbol{v}(t)\|} = \dfrac{3}{3\sqrt{5}}\boldsymbol{i} + \dfrac{6}{3\sqrt{5}}\boldsymbol{j} = \left\langle \dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}} \right\rangle$.

Also, $\boldsymbol{a}(t) = 0\boldsymbol{i} + 18\boldsymbol{j}$ and $\boldsymbol{a}(t) = a_T\boldsymbol{T} + a_N\boldsymbol{N}$. This gives $a_T = \boldsymbol{a} \cdot \boldsymbol{T} = \dfrac{36}{\sqrt{5}}$,

so $a_N\boldsymbol{N} = \boldsymbol{a}(t) - a_T\boldsymbol{T} = \langle 0, 18 \rangle - \dfrac{36}{\sqrt{5}} \left\langle \dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}} \right\rangle = \left\langle \dfrac{-36}{5}, \dfrac{18}{5} \right\rangle$. Therefore,

$$\boldsymbol{a}(t) = \left\langle \frac{36}{5}, \frac{72}{5} \right\rangle + \left\langle \frac{-36}{5}, \frac{18}{5} \right\rangle.$$

2. The surface of a lake is represented by a region D in the xy-plane such that the depth (in feet) under the point (x, y) is given by

$$f(x, y) = 300 - 2x^2 - 3y^2$$

a) In what direction should a boat at the point $P(4, 9)$ sail, in order for the depth of the water to decrease most rapidly?
b) In what direction at the point $P(4, 9)$ does the depth remain the same?

Solution:

a) The gradient of f is : $\boldsymbol{\nabla} f = \langle -4x, -6y \rangle$. Evaluated at $P(4, 9)$ it will be $\boldsymbol{\nabla} f|_{P(4,9)} = \langle -16, -54 \rangle$. The water depth will decrease most rapidly in this direction.

b) The depth will remain the same in the directions that are perpendicular to $\langle -16, -54 \rangle$, i.e. $\pm \langle 54, -16 \rangle$.

3. Given the force field $\boldsymbol{F}(x, y) = x(x^2 + y^2)^{\frac{-3}{2}}\,\boldsymbol{i} + y(x^2 + y^2)^{\frac{-3}{2}}\,\boldsymbol{j}$,

a) Determine whether $\boldsymbol{F}$ is a conservative force field or not; if it is, find a potential energy function for it.

b) Find the work done by this force from $A(1, 0)$ to $B(e^{2\pi}, 0)$ along the curve C described by the parametric equations

$$\begin{aligned} x &= e^t \cos t \\ y &= e^t \sin t. \end{aligned}$$

Solution:

a) Set $M = \dfrac{x}{(x^2+y^2)^{\frac{3}{2}}}$, and $N = \dfrac{y}{(x^2+y^2)^{\frac{3}{2}}}$. Then

$$N_x = y\left(-\frac{3}{2}\right)\left(x^2+y^2\right)^{-5/2}2x, \qquad M_y = x\left(-\frac{3}{2}\right)\left(x^2+y^2\right)^{-5/2}2y$$

So $N_x = M_y$. $\boldsymbol{F}$ is conservative, on any region not including the origin. Therefore, $\boldsymbol{F} = \boldsymbol{\nabla} w$.

$w_x = M \Longrightarrow w(x,y) = \displaystyle\int M dx = \frac{1}{2}\frac{(x^2+y^2)^{-1/2}}{-1/2} + \varphi(y)$

$w = -\left(x^2+y^2\right)^{-1/2} + \varphi(y)$

$w_y = N \Longrightarrow \dfrac{1}{2}\left(x^2+y^2\right)^{-3/2}2y + \varphi' = \dfrac{y}{(x^2+y^2)^{3/2}}$

Therefore, $\varphi' = 0 \Longrightarrow \varphi = C$.

$$w(x,y) = -\frac{1}{\sqrt{x^2+y^2}} + C.$$

b)

$$\begin{aligned}\text{work} &= w(B) - w(A)\\ &= -\frac{1}{\sqrt{(e^{2\pi})^2}} + \frac{1}{\sqrt{1}}\\ &= 1 - \frac{1}{e^{2\pi}}\end{aligned}$$

4. Find the volume of the solid which is bounded above and below by the cone $z^2 = 2x^2 + 2y^2$ and on the sides by the cylinder $x^2 + y^2 - 4y = 0$.

Solution:

When the cone and the cylinder intersect, $x^2 + y^2 = \dfrac{z^2}{2} = 4y$.

$x^2 + y^2 = 4y \Longrightarrow r = \sin\theta$ (circle) and $z^2 = 2r^2 \Longrightarrow z = \pm r\sqrt{2}$

$$\begin{aligned}V &= 2\cdot\ \text{(volume of upper half)}\\ &= 2\underbrace{\iint}_{\substack{\text{projection}\\ \text{onto } xy\text{-plane}}}\int_0^{r\sqrt{2}} r\,dz\,dr\,d\theta\\ &= 2\int_0^{\pi}\int_0^{4\sin\theta} r^2\sqrt{2}dr\,d\theta = 2\int_0^{\pi}\frac{\sqrt{2}}{3}4^3\sin^3\theta d\theta\\ &= \frac{2\sqrt{2}\cdot 4^3}{3}\int_0^{\pi}(1-\cos^2\theta)\sin\theta d\theta\\ &= \frac{128\sqrt{2}}{3}\left(-\cos\theta + \frac{\cos^3\theta}{3}\right)\Big|_0^{\pi} = \frac{512\sqrt{2}}{9}.\end{aligned}$$

5. Evaluate the surface integral

$$\iint\limits_S y\,z\,dS$$

where S is the portion of the paraboloid $x = 1 - z^2$ bounded by the yz-plane and the planes $y = -2$ and $y = 2$.

Solution:

$$x = 1 - z^2 \Longrightarrow \frac{\partial x}{\partial y} = 0, \text{ and } \frac{\partial x}{\partial z} = -2z.$$

$$\begin{aligned} dS &= \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2} \\ &= \sqrt{1+4z^2}dy\,dz \end{aligned}$$

On the yz-plane, $x = 0$, so $z = \pm 1$.

$$I = \iint\limits_S y\,z\,dS = \int_{-1}^{1}\int_{-2}^{2} yz\sqrt{1+4z^2}\,dy\,dz$$

$$I = \int_{-1}^{1} z\sqrt{1+4z^2}\,\left(\frac{y^2}{2}\right)\Bigg|_{-2}^{2} dz = \int_{-1}^{1} z\sqrt{1+4z^2}\,(0)\,dz = 0$$

6. Find the flux of the vector field

$$\boldsymbol{F}(x, y, z) = 3x\boldsymbol{i} - z^2y\boldsymbol{j} + 2z\boldsymbol{k}$$

outward across the rectangular box with vertices

$$(1,0,0), (1,3,0), (-2,0,0), (-2,3,0), (1,0,5), (1,3,5), (-2,0,5) \text{ and } (-2,3,5).$$

Solution:

The flux of $\boldsymbol{F}$ across S is $\iint\limits_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS$.

By the divergence theorem, this integral equals the triple integral on D where D is the closed surface determined by the rectangular box.

$S = \iiint\limits_D \boldsymbol{\nabla}\cdot\boldsymbol{F}dV$. Also, $\boldsymbol{\nabla}\cdot\boldsymbol{F} = 3 - z^2 + 2$.

$$\begin{aligned} \text{Flux} &= \int_{-2}^{1}\int_{0}^{3}\int_{0}^{5} (5 - z^2)\,dz\,dy\,dx \\ &= \int_{-2}^{1}\int_{0}^{3} \left(5z - \frac{z^3}{3}\right)\Bigg|_0^5 dy\,dx \\ &= 3\cdot 3\left(25 - \frac{125}{3}\right) = -150. \end{aligned}$$

7. Evaluate the surface integral

$$\iint_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, dS$$

where $\boldsymbol{F}$ is the vector field $\boldsymbol{F}(x, y, z) = -yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k}$, S is that part of the sphere $x^2 + y^2 + z^2 = 5$ below the plane $z = 2$, and $\boldsymbol{n}$ is a unit outward normal vector to the surface of the sphere.

Solution:

By Stokes' Theorem, the surface integral equals

$$I = \oint_C \boldsymbol{F} \cdot \boldsymbol{dR} = \oint_C -yz\, dx + xz\, dy + xy\, dz$$

where C is the circle formed by the intersection of the sphere and the plane. The circle C is given by $z = 2$, $x^2 + y^2 = 1$.

Therefore, $C : \begin{cases} x &= \cos t \\ y &= \sin t \\ z &= 2 \end{cases}$ and $0 \le t \le 2\pi \Longrightarrow \begin{cases} dx &= -\sin t\, dt \\ dy &= \cos t\, dt \\ dz &= 0. \end{cases}$

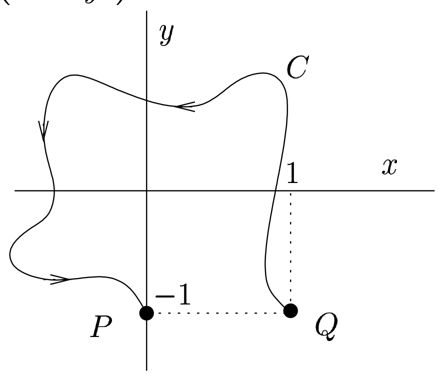
$$I = \int_0^{2\pi} 2\sin^2 t\, dt + 2\cos^2 t\, dt + 0 = 2\int_0^{2\pi} dt = 4\pi.$$

B U Department of Mathematics

Math 102 Calculus II

Summer 2003 Final

1. Let $\boldsymbol{F}(x,y)=\left(\dfrac{-y}{x^2+y^2}\right)\boldsymbol{i}+\left(\dfrac{-x}{x^2+y^2}\right)\boldsymbol{j}$ on $\mathbb{R}^2\backslash\{0,0\}$. Compute $\displaystyle\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ if C is the curve from Q to P as in the figure.



Solution:

Let $\boldsymbol{F}=f(x,y)\boldsymbol{i}+g(x,y)\boldsymbol{j}$ that is $f(x,y)=\left(\dfrac{-y}{x^2+y^2}\right)$ and $g(x,y)=\left(\dfrac{-x}{x^2+y^2}\right)$. Then

$$f_y=\frac{y^2-x^2}{(x^2+y^2)^2}\quad\text{and}\quad g_x=\frac{y^2-x^2}{(x^2+y^2)^2}.$$

They are equal, hence there exists a function $h(x,y)$ such that $\boldsymbol{\nabla}h=\boldsymbol{F}$. This function satisfies $h_x=-\dfrac{y}{x^2+y^2}$ and $h_y=\dfrac{x}{x^2+y^2}$. Integrating the first one:

$$h_x=-\frac{y}{x^2+y^2}\Rightarrow h=-\arctan\left(\frac{x}{y}\right)+k(y),$$

where k is a function of y. Then,

$$h_y=\frac{x}{x^2+y^2}+k'(y)=\frac{x}{x^2+y^2}\Rightarrow k(y)=\text{constant}.$$

Hence, $h(x,y)=-\arctan\left(\dfrac{x}{y}\right)+k$, where k is a constant.

h is discontinuous at the points where $y=0$. But since $\boldsymbol{F}$ is conservative, $\displaystyle\int_C \boldsymbol{F}\cdot d\boldsymbol{s}$ is independent of the path. By the fundamental theorem for line integrals only end points of C are important. Now,

$$\int_C \boldsymbol{F}\cdot d\boldsymbol{s}=\int_C \boldsymbol{\nabla}h\cdot d\boldsymbol{s}=h(0,-1)-h(1,-1)=-\frac{\pi}{4}.$$

2. Let a space curve parametrized by t be given by:

$$\begin{aligned} x &= t-1 \\ y &= 2t+1 \\ z &= t^2+2 \end{aligned}$$

Compute the curvature of this curve at the point $(-1,1,2)$ using the definition of curvature.

Solution:

Let $\boldsymbol{\sigma}(t)$ denote the curve in the question. Then $\boldsymbol{\sigma}(t) = \langle t-1, 2t+1, t^2+2\rangle$. Now the unit tangent vector at an arbitrary point is $\boldsymbol{T}(t) = \dfrac{\boldsymbol{\sigma}'(t)}{\|\boldsymbol{\sigma}'(t)\|}$. By definition, the curvature of a curve $\boldsymbol{\sigma}(t)$ at a point s is $\|\boldsymbol{T}'(s)\|$.

$$\boldsymbol{\sigma}'(t) = \langle 1, 2, 2t\rangle \Rightarrow \|\boldsymbol{\sigma}'(t)\| = \left(5+4t^2\right)^{1/2}.$$

Then computing the derivative of the unit tangent:

$$\boldsymbol{T}'(t) = \left\langle -4t(5+4t^2)^{-3/2}, -8t(5+4t^2)^{-3/2}, 2(5+4t^2)^{-1/2} - 8t^2(5+4t^2)^{-3/2}\right\rangle.$$

The point $(-1,1,2)$ is achieved when $t=0$, i.e. $\boldsymbol{\sigma}(0) = (-1,1,2)$. So we need to find $\|\boldsymbol{T}'(0)\|$ to find the curvature at this point:

$$\boldsymbol{T}'(0) = \langle 0, 0, 2\sqrt{5}\rangle \Rightarrow \|\boldsymbol{T}'(0)\| = 2\sqrt{5},$$

which is the curvature at $(-1,1,2)$.

3. Let $f(x,y) = \sin x + \cos y$, $0 < x, y < \pi$. Find the local maximum, minimum and saddle points of f in the given domain.

Solution:

The given domain excludes its boundary. Given f is differentiable everywhere, so for the given domain if it has a local maximum, minimum and saddle point, they must be attained at the points $f_x = f_y = 0$. But $f_y = -\sin y \neq 0$ in the given domain. Hence there is no local maximum, minimum or saddle point of f if $0 < x, y < \pi$.

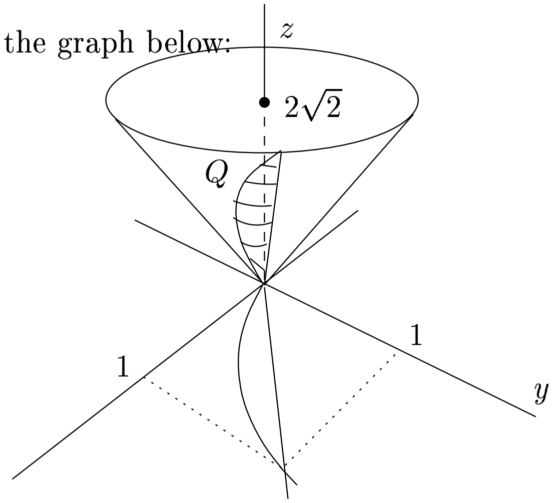
4. Find the surface area of the portion of the cone $z^2 = 4x^2 + 4y^2$ that is above the region in the first quadrant bounded by the line $y = x$ and the parabola $y = x^2$.

Solution:

On the cone we have $z^2 = 4x^2+4y^2 \Rightarrow z = \mp\sqrt{4x^2+4y^2}$. Since the surface area of the portion that is above the given region is asked, we have to consider $z \geq 0$. So, our function is $z = f(x,y) = \sqrt{4x^2+4y^2}$. Let Q denote the region given and A_s denote the surface area. Then:

$$A_s = \iint_Q \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1\right]^{1/2} dA$$

The region is shown in the graph below:



So, the region is formed by the points (x, y) such that $0 \leq x \leq 1$ and $x^2 \leq y \leq x$.
Now, $\dfrac{\partial f}{\partial x} = \left(4x^2+4y^2\right)^{-1/2} 4x$ and $\dfrac{\partial f}{\partial y} = \left(4x^2+4y^2\right)^{-1/2} 4y$. So,

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1 = \frac{16x^2+16y^2}{4x^2+4y^2} + 1 = 5.$$

Thus the surface area is:

$$\begin{aligned} A_s &= \iint_Q \sqrt{5}\, dA = \sqrt{5}\int_0^1\int_{x^2}^{x} dydx \\ &= \sqrt{5}\int_0^1 (x-x^2)dx = \sqrt{5}\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1 = \frac{\sqrt{5}}{6}. \end{aligned}$$

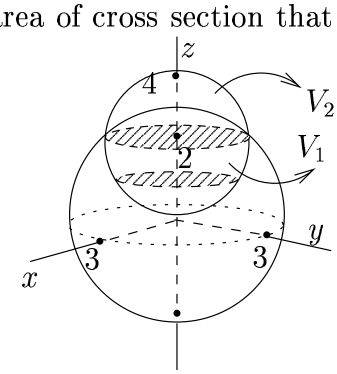
5. Find the volume that is enclosed by the spheres $x^2+y^2+z^2 = 9$ and $x^2+y^2+(z-2)^2 = 4$.

Solution:

First, we have to find out where these spheres intersect. Clearly at the intersection z must satisfy $z^2-(z-2)^2 = 5 \Rightarrow 4z-4 = 5 \Rightarrow z = \dfrac{9}{4}$.
We then substitute this in one of the sphere equations to find the equation that x and y satisfy:

$$x^2+y^2+\left(\frac{9}{4}-2\right)^2 = 4 \Rightarrow x^2+y^2 = 4 - \frac{1}{16} = \frac{63}{16}.$$

So, we have to divide the enclosed volume into two parts. The below part is a portion of the first sphere and the above part is a portion of the second. Let's call the first part V_1 and the second part V_2.
For finding V_1 consider the area of cross section that varies with z as shown in the figure:



If we call it $A(z)$ then $A(z) = (9 - z^2)\pi$
First volume is consequently

$$V_1 = \int_{9/4}^{3} (9 - z^2)\pi dz = \frac{99}{64}\pi.$$

For V_2, similarly, $B(z) = (4 - (z-2)^2)\pi$. Then:

$$V_2 = \int_0^{9/4} (4 - (z-2)^2)\pi = \frac{405}{64}\pi.$$

Then the total volume is just the sum of these two:

$$\text{Volume} = V_1 + V_2 = \left(\frac{405}{64} + \frac{99}{64}\right)\pi = \frac{504}{64}\pi = 8\pi.$$

6. **(a)** Suppose $f_x = 0$ for all (x, y) in some circular domain Ω in the plane. What can you conclude about f?

   (b) Suppose $f_x = f_y = 0$ in Ω. Show that f is constant in Ω.

Solution:

(a) We claim that for a fixed $k \in \mathbb{R}$, $f(x, k)$ is constant for all $x \in \mathbb{R}$ such that $(x, k) \in \Omega$.
Consider a line $l = \{(x, y) | \ y = k\}$ that is parallel to $x-$axis and intersects Ω. On $\Omega \cap l$ we can consider f as a function of x only, since y is fixed. Since Ω is a circular domain $\Omega \cap l$ is connected we can apply the Mean Value Theorem (MVT). Let x_1, x_2 be two distinct points such that $(x_1, k), (x_2, k) \in \Omega \cap l$. Then, by MVT there exists a c such that $x_1 < c < x_2$ that satisfies:

$$f'(c) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = 0 \Rightarrow f(x_1) = f(x_2).$$

Here $f'(c) = 0$ as l is a line parallel to the $x-$axis. Hence, for fixed k we have proved that $f(x, k)$ is constant.
(b) Since $f_y = 0$ too, by part **(a)** for a fixed s we know that $f(s, y)$ is constant. Let $(x_0, y_0) \in \Omega$ and $f(x_0, y_0) = m$. Let $(x_1, y_1) \in \Omega$ be arbitrary then:

$$\begin{aligned} f_x = 0 \Rightarrow f(x_1, y_1) = f(x_0, y_1) \quad &\text{by part } \textbf{(a)}, \\ f_y = 0 \Rightarrow f(x_0, y_1) = f(x_0, y_0) \quad &\text{by part } \textbf{(b)}. \end{aligned}$$

Thus $f(x_1, y_1) = f(x_0, y_0) = m$ which immediately implies f is constant.

B U Department of Mathematics

Math 102 Calculus II

Date: August 4, 2004	Full Name :
Time: 11:30-13:30	Math 102 Number :
	Student ID :

Summer 2004 Final Exam

IMPORTANT

1. Write your name, surname on top of each page. **2.** The exam consists of 6 questions some of which have more than one part. **3.** Read the questions carefully and write your answers neatly under the corresponding questions. **4.** Show all your work. Correct answers without sufficient explanation might not get full credit. **5.** Calculators are not allowed.

Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
20 pts	40 pts	40 pts	35 pts	35 pts	30 pts	100 pts

1.) [20] Find an equation of the plane perpendicular to the line $\left\{\begin{array}{lcl} x & = & 2+3t \\ y & = & -1+2t \\ z & = & 3-2t \end{array}\right.$ and containing the point $(3,-2,3)$.

Solution:

The direction vector of the line $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ is the normal vector of the plane asked. $(3,-2,3)$ is also on the plane, so the equation of the plane is

$$3(x-3) + 2(y+2) + (-2)(z-3) = 0$$

or

$$3x + 2y - 2z = -1.$$

2.) Given the force field $\vec{F}(x,y,z) = \ln(yz)\,\vec{i} + \frac{x}{y}\,\vec{j} + \frac{x}{z}\,\vec{k}$,

a. [25] Determine whether $\vec{F}$ is a conservative field or not; if it is, find a potential energy function for it.

b. [15] Find the work done by this force from $(0,1,1)$ to $(\pi, \sqrt{e}, \sqrt{e})$ along the curve $\mathbf{C}$ described by the parametric equations

$$\mathbf{C}: \left\{\begin{array}{lcl} x & = & \pi t^2 \\ y & = & 1 + (\sqrt{e}-1)t^3 \\ z & = & 1 + (\sqrt{e}-1)\sin\left(\frac{\pi t}{2}\right) \end{array}\right. .$$

Solution:

a) Let $M = \ln(yz)$, $N = \frac{x}{y}$, $P = \frac{x}{z}$.

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{1}{y} = \frac{\partial N}{\partial x} \\
\frac{\partial M}{\partial z} &= \frac{1}{z} = \frac{\partial P}{\partial x} \\
\frac{\partial N}{\partial z} &= 0 = \frac{\partial P}{\partial y}
\end{aligned}
$$

Hence $\mathbf{F}$ is conservative.

Then $\exists f(x,y,z)$ a potential function such that $\nabla f = \mathbf{F}$

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \ln(yz) \\
&\Rightarrow f = \int \ln(yz)dx = x\ln(yz) + \Psi_1(y,z) = x\ln y + x\ln z + \Psi_1 \\
\frac{\partial f}{\partial y} &= \frac{x}{y} \\
&\Rightarrow f = \int \frac{x}{y}dy = x\ln y + \Psi_2(x,z) \\
\frac{\partial f}{\partial z} &= \frac{x}{z} \\
&\Rightarrow f = \int \frac{x}{z}dz = x\ln z + \Psi_3(x,y)
\end{aligned}
$$

So $f(x,y,z) = xln(yz) + C$

b) Since $\mathbf{F}$ is conservative, work $= \int_C \mathbf{F}d\mathbf{r} = f(\pi, \sqrt{e}, \sqrt{e}) - f(0,0,1) = \pi\ln(\sqrt{e}\sqrt{e}) - 0\ln(1\cdot 1) = \pi$

3.)[40] Find the points on the hyperboloid $x^2 - y^2 + 2z^2 = 1$ where the normal line is parallel to the line that joins the points $(3,-1,0)$ and $(5,3,6)$.

Solution:

Let $g(x,y,z) = x^2 - y^2 + 2z^2 - 1$, then the hyperboloid is the level surface of $g(x,y,z) = 0$.

The gradient of g at $P(x_0, y_0, z_0)$ is :

$$\nabla g|_P = 2x_0\mathbf{i} - 2y_0\mathbf{j} + 4z_0\mathbf{k}$$

The direction of the line through $(3,-1,0)$ and $(5,3,6)$ is:

$$\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$$

Then

$$
\begin{aligned}
2x_0 &= 2\lambda \\
-2y_0 &= 4\lambda \\
4z_0 &= 6\lambda
\end{aligned}
$$

for some $\lambda \in \mathbb{R}$, i.e, $\nabla g\big|_P = \lambda \mathbf{v}$ for some $\lambda \in \mathbb{R}$.

Then we find

$$\begin{aligned} x_0 &= \lambda \\ y_0 &= -2\lambda \\ z_0 &= 3/2\lambda \end{aligned}$$

and we plug these into the equation of the hyperboloid to find λ,

$$\begin{aligned} \lambda^2 - 4\lambda^2 + 2\left(\frac{9\lambda^2}{4}\right) - 1 &= 0 \\ \left(-3 + \frac{9}{2}\right)\lambda^2 &= 1 \end{aligned}$$

$$\lambda^2 = \frac{2}{3}, \quad \lambda = \pm\sqrt{\frac{2}{3}}.$$

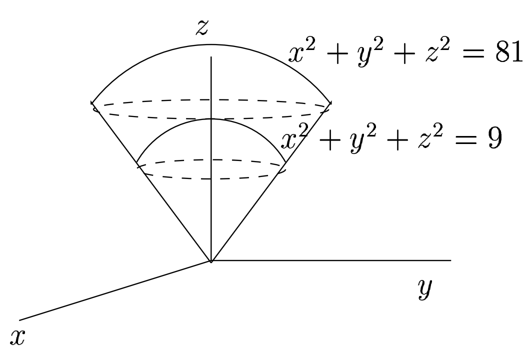
So the points are $P_1 : \left(\frac{\sqrt{2}}{\sqrt{3}}, \frac{-2\sqrt{2}}{\sqrt{3}}, \frac{3\sqrt{2}}{2\sqrt{3}}\right)$ and $P_2 : \left(\frac{-\sqrt{2}}{\sqrt{3}}, \frac{2\sqrt{2}}{\sqrt{3}}, \frac{-\sqrt{3}}{\sqrt{2}}\right)$

4.) [35] Evaluate the triple integral

$$\iiint_D \frac{1}{x^2+y^2+z^2}\, dV$$

where **D** is the solid region above the xy-plane bounded by the cone $z = \sqrt{3x^2+3y^2}$ and the spheres $x^2+y^2+z^2 = 9$ and $x^2+y^2+z^2 = 81$.

Solution:



$$\begin{aligned} x^2+y^2+z^2 &= \rho^2 \\ z &= \sqrt{3(x^2+y^2} \\ \sqrt{3\rho^2\sin^2\phi} &= \rho\cos\phi \\ 3\rho^2\sin^2\phi &= \rho^2\cos^2\phi \\ 3\sin^2\phi &= 1-\sin^2\phi \\ 0 &= 1-4\sin^2\phi \\ \frac{1}{2} &= \sin\phi \\ \phi &= \frac{1}{6} \end{aligned}$$

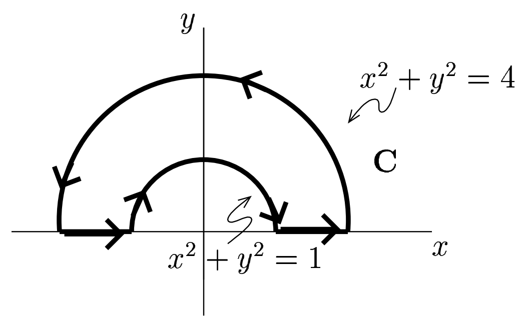
Using spherical coordinates:

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\pi/6}\int_3^9 \frac{1}{\rho^2}\rho^2 \sin\phi d\rho d\phi d\theta &= 6\int_0^{2\pi}\int_0^{\pi/6} \sin\phi d\rho d\phi d\theta \\
&= 6\int_0^{2\pi} -\cos\phi|_0^{\pi/6}\, d\theta \\
&= (-6\sqrt{3}+12)\pi
\end{aligned}$$

5.) [35] Evaluate the line integral

$$\oint_C \sqrt{x}\, dx + \ln(x^2+y^2)\, dy$$

where **C** is the curve given in the figure below:



Solution:

In this problem, let $M = \sqrt{x}$, $N = \ln(x^2+y^2)$. Then, $M(x,y)$, $N(x,y)$ are continuous functions, $\frac{\partial M}{\partial y}$, $\frac{\partial N}{\partial x}$ are also continuous, C is a simple, closed curve. So we can use Green's theorem:

$$\begin{aligned}
\oint_C \sqrt{x}dx + \ln(x^2+y^2)dy &= \iint_R (\frac{N}{x} - \frac{M}{y})dA \\
&= \iint_R \frac{2x}{x^2+y^2}dA \\
&= \int_0^{\pi}\int_1^2 \frac{2x\cos\theta}{r^2} r dr d\theta \\
&= \int_0^{\pi} 2\cos\theta r|_0^2 d\theta \\
&= 2\int_0^{\pi} \cos\theta d\theta \\
&= 2\sin\theta|_0^{\pi} \\
&= 0
\end{aligned}$$

6.) [30] Given that $\vec{\nabla} f(x_0,y_0) = \vec{i} - 2\vec{j}$ and $D_{\vec{u}}f(x_0,y_0) = -2$ where $D_{\vec{u}}f$ is the directional derivative of f in the direction of the vector $\vec{u}$. Find the unit vector(s) $\vec{u}$.

Solution:

let $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$, $a, b \in \mathbb{R}$.

$$\begin{aligned} D_{\mathbf{u}} f(x_0, y_0) = -2 &= \nabla f(x_0, y_0) \cdot u \\ &= (\mathbf{i} - 2\mathbf{j}) \cdot (a\mathbf{i} + b\mathbf{j}) \\ -2 &= a - 2b \end{aligned}$$

Also since $\mathbf{u}$ is a unit vector, $a^2 + b^2 = 1$. Using $a = -2 + 2b$ we get

$$\begin{aligned} (-2 + 2b)^2 + b^2 &= 1 \\ 4 - 8b + 4b^2 + b^2 &= 1 \\ 5b^2 - 8b + 4b^2 + b^2 &= 0 \end{aligned}$$

$$b_1 = 1, b_2 = \frac{3}{5}$$

$b = 1 \Rightarrow a = 0$ and $b = \frac{3}{5} \Rightarrow a = -\frac{4}{5}$. So either $\mathbf{u} = 0\mathbf{i} + 1\mathbf{j} = \mathbf{j}$ or $\mathbf{u} = -\frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$.

Boğaziçi University
Department of Mathematics
Math 102 Calculus II

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6	6	8	8	8	12	12	60

Date: August 2, 2008	Full Name :
Time: 09:10-11:30	Math 102 Number :
	Student ID :

Summer 2008 – Final Examination

IMPORTANT

1. Write your name, surname on top of each page. **2.** The exam consists of 7 questions some of which have more than one part. **3.** Read the questions carefully and write your answers neatly under the corresponding questions. **4.** Show all your work. Correct answers without sufficient explanation might not get full credit. **5.** Calculators are not allowed.

1. Suppose the vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ satisfy $\mathbf{u}+\mathbf{v}+\mathbf{w}=\mathbf{0}$. Show that $\mathbf{u}\times\mathbf{v}=\mathbf{v}\times\mathbf{w}=\mathbf{w}\times\mathbf{u}$.

Solution: We have

$$\mathbf{u}\times\mathbf{v}=\mathbf{u}\times(-\mathbf{u}-\mathbf{w})=-\mathbf{u}\times\mathbf{u}-\mathbf{u}\times\mathbf{w}=-\mathbf{u}\times\mathbf{w}=\mathbf{w}\times\mathbf{u}\,.$$

The identity $\mathbf{u}\times\mathbf{v}=\mathbf{v}\times\mathbf{w}$ is proved similarly.

2. Determine whether the limit $\lim_{(x,y)\to(0,0)} \dfrac{2x^2y}{x^4+y^2}$ exists. If so, find its value.

Solution: Considering, for instance, the curves

$$C_1 \;:\; x=t,\; y=0 \qquad \text{and} \qquad C_2 \;:\; x=t,\; y=t^2$$

we have

$$\lim_{(x,y)\xrightarrow{C_1}(0,0)} \frac{2x^2y}{x^4+y^2}=\lim_{t\to 0}\frac{0}{t^4}=0$$

and, on the other hand,

$$\lim_{(x,y)\xrightarrow{C_2}(0,0)} \frac{2x^2y}{x^4+y^2}=\lim_{t\to 0}\frac{2t^4}{2t^4}=1\,.$$

Therefore the given limit does not exist.

3. Locate and classify the extrema for the function $\quad F(x,y) = x^2 - xy + y^4$.

Solution: First

$$\nabla F(x,y) = \langle 2x - y, -x + 4y^3 \rangle = \mathbf{0}$$

if and only if

$$(x,y) \in \left\{ (0,0)\,, \left(-\frac{\sqrt{2}}{8}, -\frac{\sqrt{2}}{4}\right), \left(\frac{\sqrt{2}}{8}, \frac{\sqrt{2}}{4}\right) \right\}.$$

Now,

$$D(x,y) = \begin{vmatrix} F_{xx}(x,y) & F_{xy}(x,y) \\ F_{yx}(x,y) & F_{yy}(x,y) \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 12y^2 \end{vmatrix} = 24y^2 - 1$$

so that

$$D(0,0) = -1 < 0 \qquad \Rightarrow \qquad \text{saddle point at } (0{,}0)\,.$$

On the other hand,

$$D\left(-\frac{\sqrt{2}}{8}, -\frac{\sqrt{2}}{4}\right) = D\left(\frac{\sqrt{2}}{8}, \frac{\sqrt{2}}{4}\right) = 24\left(\frac{\sqrt{2}}{4}\right)^2 - 1 = 2 > 0$$

and

$$F_{xx} \equiv 0$$

implies that F has relative minimum at the points $\left(-\frac{\sqrt{2}}{8}, -\frac{\sqrt{2}}{4}\right)$ and $\left(\frac{\sqrt{2}}{8}, \frac{\sqrt{2}}{4}\right)$.

4. Use Lagrange multipliers to investigate the function $\quad f(x,y) = \dfrac{1}{x} + \dfrac{1}{y} \quad$ for extrema over the curve $\quad \dfrac{1}{x^2} + \dfrac{1}{y^2} = \dfrac{1}{a^2} \quad$ where a is a positive constant.

Solution: For $g(x,y) = \dfrac{1}{x^2} + \dfrac{1}{y^2} - \dfrac{1}{a^2}$, $\nabla f(x,y) = \lambda \nabla g(x,y)$ holds if and only if

$$-\left\langle \frac{1}{x^2}, \frac{1}{y^2} \right\rangle = -2\lambda \left\langle \frac{1}{x^3}, \frac{1}{y^3} \right\rangle$$

yielding

$$x = y = 2\lambda\,.$$

The constraint equation $g(x,y) = 0$, in turn, gives $\lambda = \mp a/\sqrt{2}$. Finally, the computation

$$f(\sqrt{2}a, \sqrt{2}a) = \frac{\sqrt{2}}{a} \qquad \text{and} \qquad f(-\sqrt{2}a, -\sqrt{2}a) = -\frac{\sqrt{2}}{a}$$

shows that, subject to the constraint $g(x,y) = 0$, f attains its maximum at $(\sqrt{2}a, \sqrt{2}a)$, and minimum at $(-\sqrt{2}a, -\sqrt{2}a)$.

5. Find the volume V of the solid bounded by the plane $z = 0$, and the surfaces $x^2 + y^2 = a^2$ and $az = a^2 - x^2$.

Solution:

$$V = 4\int_0^a\int_0^{\frac{a^2-x^2}{a}}\int_0^{\sqrt{a^2-x^2}} \mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x = 4\int_0^a\int_0^{\frac{a^2-x^2}{a}} \sqrt{a^2-x^2}\,\mathrm{d}z\,\mathrm{d}x = \frac{4}{a}\int_0^a \left(a^2-x^2\right)^{3/2}\,\mathrm{d}x$$
$$\stackrel{(*)}{=} \frac{4a^4}{a}\int_0^{\pi/2}\cos^4\theta\,\mathrm{d}\theta = 4a^3\int_0^{\pi/2}\left(\frac{\cos 2\theta+1}{2}\right)^2\mathrm{d}\theta = a^3\int_0^{\pi/2}\left(\cos^2 2\theta + 2\cos 2\theta + 1\right)\mathrm{d}\theta$$
$$= a^3\int_0^{\pi/2}\left(\frac{\cos 4\theta+1}{2} + 2\cos 2\theta + 1\right)\mathrm{d}\theta = a^3\left(\frac{\sin 4\theta}{8} + \sin 2\theta + \frac{3}{2}\theta\right)\Big|_{\theta=0}^{\theta=\pi/2} = \frac{3\pi}{4}a^3$$

where in $(*)$ we used the substitution $x = a\sin\theta$.

6. The plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ (a, b, c are positive constants) divides the solid ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \le 1$ into two unequal pieces. In this problem, you will find the volume V of the smaller piece G as follows.

(a) (*4 pts.*) Let G_1 be the image of the solid G under the change of variables $u = x/a$, $v = y/b$, and $w = z/c$; and let V_1 be its volume. Find a relation between V and V_1.

Solution: The Jacobian of the change of variables $u = x/a$, $v = y/b$, and $w = z/c$ is given by

$$\frac{\partial(x,y,x)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

so that

$$V = \iiint_G \mathrm{d}V_{x,y,z} = \iiint_{G_1} \left|\frac{\partial(x,y,x)}{\partial(u,v,w)}\right| \mathrm{d}V_{u,v,w} = abc\iiint_{G_1} \mathrm{d}V_{u,v,w} = abc\,V_1\,.$$

(b) (*4 pts.*) Let G_2 be the the spherical cap bounded above by the unit sphere, and below by the plane $z = 1/\sqrt{3}$; and let V_2 be its volume. Find a relation between V_1 and V_2.

Solution: Since $n = \langle a,b,c\rangle := \langle 1,1,1\rangle$ is a normal to the plane $M : u + v + w = 1$, the distance of M to the origin $(u_0, v_0, w_0) := (0,0,0)$ is

$$D = \frac{|au_0 + bv_0 + cw_0 - 1|}{\sqrt{a^2+b^2+c^2}} = \frac{1}{\sqrt{3}}\,.$$

Accordingly, the solid $G_1 = \{(u,v,w) \in \mathbb{R}^3 : u^2+v^2+w^2 \le 1 \ \ \&\ \ u+v+w \ge 1\}$ is the (*smaller*) spherical cap obtained by cutting the unit sphere $u^2+v^2+w^2 = 1$ via a plane $1/\sqrt{3}$ units away from the origin. Therefore $V_1 = V_2$.

(c) (*4 pts.*) Find the volume V_2 of the spherical cap G_2.

Solution:

$$
\begin{aligned}
V_2 &= \int_0^{2\pi}\int_0^{\sqrt{2/3}}\int_{1/\sqrt{3}}^{\sqrt{1-r^2}} \mathrm{d}w\, r\, \mathrm{d}r\, \mathrm{d}\theta = \int_0^{2\pi}\int_0^{\sqrt{2/3}} \left(\sqrt{1-r^2} - \frac{1}{\sqrt{3}}\right) r\, \mathrm{d}r\, \mathrm{d}\theta \\
&= \int_0^{2\pi} \left[-\frac{1}{3}(1-r^2)^{3/2} - \frac{r^2}{2\sqrt{3}}\right]_{r=0}^{r=\sqrt{2/3}} \mathrm{d}\theta = 2\pi\left(-\frac{1}{9\sqrt{3}} - \frac{1}{3\sqrt{3}} + \frac{1}{3}\right) \\
&= \frac{2\pi}{9\sqrt{3}}\left(3\sqrt{3} - 4\right)
\end{aligned}
$$

7. (a) (*6 pts.*) Using Green's theorem, find a simple closed curve C with counterclockwise orientation that maximizes the value of

$$I = \int_C \frac{y^3}{3}\, dx + \left(x - \frac{x^3}{3}\right) dy$$

and explain your reasoning.

Solution: Using Green's theorem, we have

$$I = \iint_R \left[\frac{\partial}{\partial x}\left(x - \frac{x^3}{3}\right) - \frac{\partial}{\partial y}\left(\frac{y^3}{3}\right)\right] \mathrm{d}A = \iint_R \left(1 - x^2 - y^2\right) \mathrm{d}A$$

where R is the region enclosed by the simple closed curve C. The region that maximizes this double integral is therefore

$$R = \left\{(x, y) \in \mathbb{R}^2 : 1 - x^2 - y^2 \geq 1\right\} = \text{the closed unit disk}\,.$$

Thus the curve C that maximizes I is the unit circle oriented in the counterclockwise direction, that is

$$C \ :\ x = \cos t, \quad y = \sin t, \quad 0 \leq t \leq 2\pi\,.$$

(b) (*4 pts.*) Find the maximum value of I by evaluating the double integral you found in part (a).

Solution: Passing to polar coordinates we get

$$I_{\max} = \iint_{x^2+y^2\leq 1} \left(1 - x^2 - y^2\right) \mathrm{d}A = \int_0^{2\pi}\int_0^1 \left(1 - r^2\right) r\, \mathrm{d}r\, \mathrm{d}\theta = 2\pi\left[\frac{r^2}{2} - \frac{r^4}{4}\right]_{r=0}^{r=1} = \frac{\pi}{2}\,.$$

(c) (*2 pts.*) The double integral you evaluated in part (b) corresponds to the volume of a solid above the xy-plane. Describe this solid.

Solution: This is the solid bounded above by the elliptic paraboloid $z = 1 - x^2 - y^2$ and below by the plane $z = 0$.