10-1-2005

The Mean Value Theorem

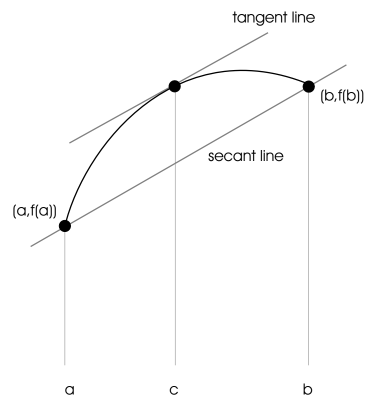
A **secant line** is a line drawn through two points on a curve.

The Mean Value Theorem relates the slope of a secant line to the slope of a tangent line.

The Mean Value Theorem. If f is continuous on $a \le x \le b$ and differentiable on $a < x < b$, there is a number c in $a < x < b$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

The picture below shows why this makes sense. I've drawn a secant line through the points $(a, f(a))$ and $(b, f(b))$. The Mean Value Theorem says that somewhere in between a and b, there is a point c on the curve where the tangent line has the same slope as the secant line.



Lines with the same slope are parallel. To find a point where the tangent line is parallel to the secant line, take the secant line and "slide" it (without changing its slope) until it's tangent to the curve.

If you experiment with some curves, you'll find that it's always possible to do this (provided that the curve is continuous and differentiable as stipulated in the theorem).

Example. Consider $f(x) = x^3 + 3x^2$ on the interval $-5 \le x \le 1$. Since f is a polynomial, f is continuous on $-5 \le x \le 1$ and differentiable on $-5 < x < 1$.

$$\frac{f(1) - f(-5)}{1 - (-5)} = \frac{4 - (-50)}{1 - (-5)} = 9,$$

so I should be able to find a number c between -5 and 1 such that $f'(c) = 9$.

$f'(x) = 3x^2 + 6x$, so $f'(c) = 3c^2 + 6c$. Set $f'(c)$ equal to 9 and solve for c:

$$3c^2 + 6c = 9, \quad c^2 + 2c = 3, \quad c^2 + 2c - 3 = 0, \quad (c+3)(c-1) = 0, \quad c = -3 \quad \text{or} \quad c = 1.$$

$c = 11$ is *not* in the interval $-5 < x < 1$ — it's an endpoint — but $c = -3$ is. $c = -3$ is a number satisfying the conclusion of the Mean Value Theorem. □

Notes:

1. There may be more than one value for c which works.

2. In general, *finding* a value of c that works may be difficult. The theorem only guarantees that such a c exists.

You have to ensure that the hypotheses of the theorem are satisfied before you apply it.

Example. Consider $f(x) = \dfrac{1}{x^2}$ on the interval $-1 \le x \le 1$. Then

$$\frac{f(1) - f(-1)}{1 - (-1)} = 0.$$

However, $f'(x) = -\dfrac{2}{x^3}$, and $f'(c) = -\dfrac{2}{c^3} = 0$ has no solution.

This does not contradict the Mean Value Theorem, because f blows up at $x = 0$, which is in the middle of the interval $-1 \le x \le 1$. □

Example. Calvin Butterball runs a 100 yard dash in 20 seconds. Assume that the function $s(t)$ which gives his position relative to the starting line is continuous and differentiable. Show that Calvin must have been running at 5 yards per second at some point during his run.

When $t = 0$, he's at the starting line, so $s = 0$. When $t = 20$, he's at the finish line, so $s = 100$. Applying the Mean Value Theorem to s for $0 \le t \le 20$, I find that there is a point c between 0 and 20 such that

$$s'(c) = \frac{100 - 0}{20 - 0} = 5.$$

That is, Calvin's velocity at $t = c$ was 5 yards per second, which is what I wanted to show. □

The Mean Value Theorem is often used to prove mathematical results. Here's an example. You know that the derivative of a constant is zero. The converse is also true.

Theorem. If f is continuous on the closed interval $[a, b]$ and $f'(x) = 0$ for all x in the open interval (a, b), then f is constant on the closed interval $[a, b]$.

Proof. To prove this, let d be any number such that $a < d \le b$. The Mean Value Theorem applies to f on the interval $[a, d]$, so there is a number c such that $a < c < d$ and

$$\frac{f(d) - f(a)}{d - a} = f'(c).$$

By assumption, $f'(c) = 0$. Therefore,

$$\frac{f(d) - f(a)}{d - a} = 0, \quad \text{so} \quad f(d) - f(a) = 0, \quad \text{and} \quad f(d) = f(a).$$

Since d was an aribtrary number such that $a < d \le b$, it follows that $f(a) = f(x)$ for all x in $[a, b]$. This means that f is constant on the interval. □

Example. I know that $\dfrac{d}{dx} x^3 = 3x^2$. If $f(x)$ is any other function such that $\dfrac{d}{dx} f(x) = 3x^2$, then

$$\frac{d}{dx}\left(f(x) - x^3\right) = 3x^2 - 3x^2 = 0.$$

By the theorem, $f(x) - x^3 = c$, where c is a constant. Therefore, $f(x) = x^3 + c$. In other words, the only functions whose derivatives are $3x^2$ are functions like

$$x^3, \quad x^3 + 13, \quad x^3 - \sqrt{7}, \quad \text{and so on.}$$

When I discuss **antiderivatives** later on, I'll express this fact by writing

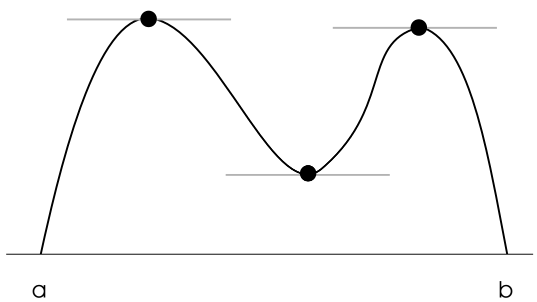
$$\int 3x^2\,dx = x^3 + c. \quad \square$$

In the case where the Mean Value Theorem applies and $f(a) = f(b)$, I get

$$\frac{f(b) - f(a)}{b - a} = 0.$$

The MVT says there is a point c in $a < x < b$ such that $f'(c) = 0$. This is called **Rolle's Theorem**, and a special case may be stated more informally as follows:

- For a "nice" function, there is at least one horizontal tangent between every pair of roots.



In the picture above, there are three critical points between the roots at a and b.

Example. By the Mean Value Theorem, the function $f(x) = x(x-20)(x-200)(x-2000)$ has critical points — places where $f' = 0$ — between 0 and 20, between 20 and 200, and between 200 and 2000. $\square$

Example. Prove that the function $f(x) = x^5 + 7x^3 + 13x - 18$ has exactly one root.

Note that a graph is *not* a proof!

Since $f(10) = 107112$ and $f(-10) = -107148$, and since f is continuous, the Intermediate Value Theorem implies that there is a root between -10 and 10. Thus, f has at least one root.

Suppose that f has more than one root. Suppose, in particular, that a and b are distinct roots of f.

By Rolle's theorem, f must have a horizontal tangent between a and b.

However, the derivative is $f'(x) = 5x^4 + 21x^2 + 13$. Since the powers of x are even, $f'(x) > 0$ for all x: There are no horizontal tangents.

This contradiction shows that there can't be more than one root. Since I already know that there's at least one root, there must be exactly one root. $\square$

Example. Here is another mathematical result which follows from the Mean Value Theorem; it will be useful in **graphing curves**.

- If f is differentiable on $a < x < b$ and $f'(x) > 0$ on $a < x < b$, then f **increases** on $a < x < b$.

To say that f **increases** means that f *goes up from left to right*.

To see this, take p and q between a and b; say $a < p < q < b$. I want to show $f(p) < f(q)$. By the MVT,

$$\frac{f(q) - f(p)}{q - p} = f'(c)$$

for some c between p and q.

But $f'(c) > 0$, so

$$\frac{f(q) - f(p)}{q - p} > 0, \quad f(q) - f(p) > 0, \quad f(q) > f(p).$$

This proves that f increases on the interval. □

Example. Here's another example of how the Mean Value Theorem can be used to prove a mathematical result — in this case, an inequality. Apply the MVT to $f(x) = \tan x$ on the interval $0 \le x \le k$, where $k < \dfrac{\pi}{2}$ to avoid running into the vertical asymptote. I get

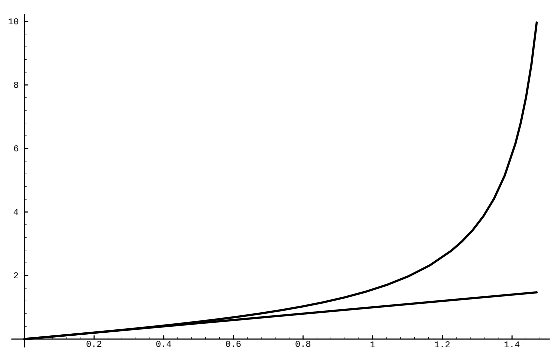
$$\frac{\tan k - \tan 0}{k - 0} = (\sec c)^2$$

for some c between 0 and k.

Now $(\sec c)^2 \ge 1$ and $\tan 0 = 0$, so

$$\frac{\tan k}{k} \ge 1 \quad \text{and hence} \quad \tan k \ge k.$$

A picture which illustrates this (not to scale) follows:



The curve is the graph of $y = \tan x$ and the line is $y = x$. You can see that the curve appears to lie above the line. □

Example. (**Using the Mean Value Theorem to estimate a function value**) Suppose that f is a differentiable function,

$$f(3) = 2, \quad \text{and} \quad 3 \le f'(x) \le 4 \quad \text{for all} \quad x.$$

Prove that $8 \le f(5) \le 10$.

Apply the Mean Value Theorem to f on the interval $3 \le x \le 5$:

$$\frac{f(5) - f(3)}{5 - 3} = f'(c) \quad \text{where} \quad 3 < c < 5.$$

Then since $3 \le f'(c) \le 4$, I have

$$\begin{aligned} 3 &\le \frac{f(5) - f(3)}{5 - 3} \le 4 \\ 3 &\le \frac{f(5) - 3}{2} \le 4 \\ 6 &\le f(5) - 3 \le 8 \\ 8 &\le f(5) \le 10 \quad \square \end{aligned}$$