MATH1010 University Mathematics
Supplementary Exercise on Integration

5   Integration

5.1   Indefinite integral

Exercise 5.1.1. *Evaluate*

1. $\displaystyle\int (3-x^2)^3\,dx$

2. $\displaystyle\int x^2(5-x)^4\,dx$

3. $\displaystyle\int \frac{x+1}{\sqrt{x}}\,dx$

4. $\displaystyle\int \left(8t - \frac{2}{t^{\frac{1}{4}}}\right)\,dt$

5. $\displaystyle\int 3\csc^2 x\,dx$

6. $\displaystyle\int 4\tan\theta\sec\theta\,d\theta$

5.3   Indefinite by substitution

Exercise 5.3.1. *Evaluate*

1. $\displaystyle\int \frac{dx}{\sqrt{2-5x}}$

2. $\displaystyle\int \frac{e^{3x}+1}{e^x+1}\,dx$

3. $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$

4. $\displaystyle\int x^2\sqrt[3]{1+x^3}\,dx$

5. $\displaystyle\int \frac{x\,dx}{(1+x^2)^2}$

6. $\displaystyle\int \frac{dx}{\sqrt{x}(1+x)}$

7. $\displaystyle\int \frac{1}{x^2}\sin\frac{1}{x}\,dx$

8. $\displaystyle\int xe^{-x^2}\,dx$

9. $\displaystyle\int \frac{e^x\,dx}{2+e^x}$

10. $\displaystyle\int \frac{dx}{e^x+e^{-x}}$

11. $\displaystyle\int \tan x\,dx$

12. $\displaystyle\int \frac{dx}{1+e^x}$

6   Further techniques of integration

6.1   Integration by trigonometric identities

Exercise 6.1.1. *Evaluate*

1. $\displaystyle\int \frac{dx}{1-\cos x}$

2. $\displaystyle\int \sin^5 x\cos x\,dx$

3. $\displaystyle\int \sin 3x\sin 5x\,dx$

4. $\displaystyle\int \cos\frac{x}{2}\cos\frac{x}{3}\,dx$

5. $\displaystyle\int \cos^3 x\,dx$

6. $\displaystyle\int \sin^4 x\,dx$

7. $\displaystyle\int \frac{dx}{\cos x\sin^2 x}$

8. $\displaystyle\int \frac{\sin x\cos^3 x}{1+\cos^2 x}\,dx$

9. $\displaystyle\int \tan^5 x\,dx$

10. $\displaystyle\int \frac{dx}{\sin^4 x\cos^4 x}\,dx$

11. $\displaystyle\int \sin 5x\cos x\,dx$

12. $\displaystyle\int \cos x\cos 2x\cos 3x\,dx$

6.2 Integration by parts

Exercise 6.2.1. *Evaluate*

1. $\int \ln x \, dx$

2. $\int x^2 \ln x \, dx$

3. $\int \left(\frac{\ln x}{x}\right)^2 dx$

4. $\int x e^{-x} dx$

5. $\int x^2 e^{-2x} dx$

6. $\int x \cos x \, dx$

7. $\int x^2 \sin 2x \, dx$

8. $\int \sin^{-1} x \, dx$

9. $\int x \tan^{-1} x \, dx$

10. $\int \ln(x + \sqrt{1 + x^2}) dx$

11. $\int x \sin^2 x \, dx$

12. $\int \sin(\ln x) dx$

6.3 Reduction formula

Exercise 6.3.1. *Prove the following reduction formulas.*

1. $I_n = \int x^n e^{ax} dx; \ I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}, \ n \geq 1$

2. $I_n = \int \sin^n x \, dx; \ I_n = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}, \ n \geq 2$

3. $I_n = \int \cos^n x \, dx; \ I_n = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}, \ n \geq 2$

4. $I_n = \int \frac{1}{\sin^n x} dx; \ I_n = -\frac{\cos x}{(n-1) \sin^{n-1} x} + \frac{n-2}{n-1} I_{n-2}, \ n \geq 2$

5. $I_n = \int x^n \cos x \, dx; \ I_n = x^n \sin x + n x^{n-1} \cos x - n(n-1) I_{n-2}, \ n \geq 2$

6. $I_n = \int \frac{dx}{(x^2 - a^2)^n}; \ I_n = -\frac{x}{2a^2(n-1)(x^2 - a^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)} I_{n-1}, \ n \geq 1$

7. $I_n = \int \frac{x^n dx}{\sqrt{x+a}}; \ I_n = \frac{2x^n \sqrt{x+a}}{2n+1} - \frac{2an}{2n+1} I_{n-1}, \ n \geq 1$

6.4 Trigonometric substitution

Exercise 6.4.1. *Evaluate*

1. $\int \frac{x^2}{1 + x^2} dx$

2. $\int \frac{dx}{(1 - x^2)^{\frac{3}{2}}}$

3. $\int \sqrt{\frac{1 + x}{1 - x}} \, dx$

4. $\int \frac{dx}{(1 + x^2)^{\frac{3}{2}}}$

5. $\int \frac{x^2 dx}{\sqrt{9 - x^2}}$

6. $\int \frac{dx}{\sqrt{4 + x^2}}$

6.5 Integration of rational functions

Exercise 6.5.1. *Evaluate*

1. $\displaystyle \int \frac{x^2 dx}{1 - x^2}$

2. $\displaystyle \int \frac{x^3}{3 + x} dx$

3. $\displaystyle \int \frac{(1+x)^2}{1 + x^2} dx$

4. $\displaystyle \int \frac{dx}{x^2 + 2x - 3}$

5. $\displaystyle \int \frac{dx}{(x^2 - 2)(x^2 + 3)}$

6. $\displaystyle \int \frac{x^2 + 1}{(x+1)^2(x-1)}, dx$

7. $\displaystyle \int \frac{x^2}{(x^2 - 3x + 2)^2}, dx$

8. $\displaystyle \int \frac{x^2 + 5x + 4}{x^4 + 5x^2 + 4}, dx$

9. $\displaystyle \int \frac{dx}{(x+1)(x^2+1)}$

10. $\displaystyle \int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx$

11. $\displaystyle \int \frac{4 - 2x}{(x^2+1)(x-1)^2} dx$

12. $\displaystyle \int \frac{dx}{x(x^2+1)^2}$

6.6 t-substitution

Exercise 6.6.1. *Use t-substitution to evaluate the following integrals.*

1. $\displaystyle \int \frac{dx}{\sin^3 x}$

2. $\displaystyle \int \frac{dx}{1 + \sin x}$

3. $\displaystyle \int \frac{dx}{\sin x \cos^4 x}$

6.7 Improper integrals

Exercise 6.7.1. *Evaluate the following improper integrals.*

1. $\displaystyle \int_4^\infty \frac{dx}{x^2}$

2. $\displaystyle \int_{-\infty}^\infty \frac{dx}{1 + x^2}$

3. $\displaystyle \int_{-1}^1 \frac{dx}{\sqrt{1 - x^2}}$

4. $\displaystyle \int_2^\infty \frac{dx}{x^2 + x - 2}$

5. $\displaystyle \int_{-\infty}^\infty \frac{dx}{(x^2 + x + 1)^2}$

6. $\displaystyle \int_0^\infty \frac{dx}{1 + x^3}$

7. $\displaystyle \int_0^\infty \frac{\tan^{-1} x}{(1 + x^2)^{\frac{3}{2}}} dx$

8. $\displaystyle \int_0^\infty e^{-x} \cos x dx$

9. $\displaystyle \int_0^{\frac{\pi}{2}} \ln(\sin x) dx$

Exercise 6.7.2. *Determine whether the following improper integrals are convergent.*

1. $\displaystyle \int_0^\infty \frac{x^2 dx}{x^4 - x^2 + 1}$

2. $\displaystyle \int_1^\infty \frac{dx}{x\sqrt[3]{x^2 + 1}}$

3. $\displaystyle \int_0^1 \frac{dx}{\ln x}$

4. $\displaystyle \int_2^\infty \frac{dx}{x \ln x}$

5. $\displaystyle \int_0^\infty \frac{\ln(1 + x)}{\sqrt{x}} dx$

6. $\displaystyle \int_0^{\frac{\pi}{2}} \tan x dx$

3

Miscellaneous

Exercise. *Evaluate*

1. $\int f(x)dx$ *where*
$$f(x) = \begin{cases} 1, & \text{if } x < 0 \\ 2x + 1, & \text{if } x \geq 0 \end{cases}$$

2. $\int f(x)dx$ *where*
$$f(x) = \begin{cases} 4x - 1, & \text{if } x < 3 \\ x^2 + 1, & \text{if } x \geq 3 \end{cases}$$

3. $\int |x|dx$

4. $\int |x^2 - 1|dx$

5. $\int |x^2 - x|dx$

6. $\int x^2 \sqrt[3]{1-x}\, dx$

7. $\int x^5(2 - 5x^3)^{\frac{2}{3}}dx$

8. $\int e^x \cos x dx$

9. $\int \dfrac{xdx}{\cos^2 x}$

10. $\int \dfrac{dx}{\sqrt{25x^2 - 4}}$

11. $\int \dfrac{dx}{(x^2 + 1)^3}$

12. $\int \dfrac{x^2}{(x^2 + 2x + 2)^2}\, dx$

13. $\int \dfrac{4dx}{(4x^2 + 1)^2}$

14. $\int \sqrt{\dfrac{4 - x}{x}}\, dx$

15. $\int \dfrac{8dx}{x^2\sqrt{4 - x^2}}\, dx$

16. $\int \dfrac{dx}{(\cos x + \sin x)^2}$

17. $\int \sqrt{1 + \sin x}\, dx$

Answers

Exercise 5.1.1

1. $27x - 9x^3 + \frac{9}{5}x^5 - \frac{1}{7}x^7 + C$

2. $\frac{625}{3}x^3 - 125x^4 + 30x^5 - \frac{10}{3}x^6 + \frac{1}{7}x^7 + C$

3. $\frac{2}{3}x^{\frac{3}{2}} + 2\sqrt{x} + C$

4. $4t^2 - \frac{8}{3}t^{\frac{3}{4}} + C$

5. $-3\cot x + C$

6. $4\sec\theta + C$

Exercise 5.3.1

1. $-\frac{2}{5}\sqrt{2-5x} + C$

2. $\frac{1}{2}e^{2x} - e^x + x + C$

3. $-\sqrt{1-x^2} + C$

4. $\frac{1}{4}(1+x^3)^{\frac{4}{3}} + C$

5. $-\frac{1}{2(1+x^2)} + C$

6. $2\tan^{-1}\sqrt{x} + C$

7. $\cos\frac{1}{x} + C$

8. $-\frac{1}{2}e^{-x^2} + C$

9. $\ln(2+e^x) + C$

10. $\tan^{-1}e^x + C$

11. $-\ln|\cos x| + C$

12. $x - \ln(1+e^x) + C$

Exercise 6.1.1

1. $-\cot\frac{x}{2} + C$

2. $\frac{1}{6}\sin^6 x + C$

3. $\frac{1}{4}\sin 2x - \frac{1}{16}\sin 8x + C$

4. $3\sin\frac{x}{6} + \frac{3}{5}\sin\frac{5x}{6} + C$

5. $\sin x - \frac{1}{3}\sin^3 x + C$

6. $\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$

7. $-\frac{1}{\sin x} + \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x} + C$

8. $-\frac{1}{2}\cos^2 x + \frac{1}{2}\ln(1+\cos^2 x) + C$

9. $\frac{\tan^4}{4} - \frac{\tan^2 x}{2} - \ln|\cos x| + C$

10. $-8\cot 2x - \frac{8}{3}\cot^3 2x + C$

11. $-\frac{1}{8}\cos 4x - \frac{1}{12}\cos 6x + C$

12. $\frac{x}{4} + \frac{\sin 2x}{8} + \frac{\sin 4x}{16} + \frac{\sin 6x}{24} + C$

Exercise 6.2.1

1. $x\ln x - x + C$

2. $\frac{x^3}{3}(\ln x - \frac{1}{3}) + C$

3. $-\frac{1}{x}((\ln x)^2 + 2\ln x + 2) + C$

4. $-(x+1)e^{-x} + C$

5. $-\frac{e^{-2x}}{4}(2x^2 + 2x + 1) + C$

6. $x\sin x + \cos x + C$

7. $-\frac{2x^2-1}{4}\cos 2x + \frac{x}{2}\sin 2x + C$

8. $x\sin^{-1}x + \sqrt{1-x^2} + C$

9. $-\frac{x}{2} + \frac{1+x^2}{2}\tan^{-1}x + C$

10. $x\ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2} + C$

11. $\frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x + C$

12. $\frac{x}{2}(\sin(\ln x) - \cos(\ln x)) + C$

Exercise 6.4.1

1. $x - \tan^{-1} x + C$

2. $\frac{x}{\sqrt{1-x^2}} + C$

3. $-\sqrt{1-x^2} + \sin^{-1} x + C$

4. $\frac{x}{\sqrt{1+x^2}} + C$

5. $\frac{9}{2} \sin^{-1} \frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C$

6. $\ln|x + \sqrt{4+x^2}| + C$

Exercise 6.5.1

1. $-x + \frac{1}{2} \ln\left|\frac{1+x}{1-x}\right| + C$

2. $9x - \frac{3}{2}x^2 + \frac{1}{3}x^3 - 27 \ln|3+x| + C$

3. $x + \ln(1+x^2) + C$

4. $\frac{1}{4} \ln\left|\frac{x-1}{x+3}\right| + C$

5. $\frac{1}{10\sqrt{2}} \ln\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right| - \frac{1}{5\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} + C$

6. $\frac{1}{x+1} + \frac{1}{2} \ln|x^2 - 1| + C$

7. $-\frac{5x-6}{x^2-3x+2} + 4 \ln\left|\frac{x-1}{x-2}\right| + C$

8. $\tan^{-1} x + \frac{5}{6} \ln \frac{x^2+1}{x^2+4} + C$

9. $\frac{1}{2} \tan^{-1} x + \frac{1}{4} \ln \frac{(x+1)^2}{x^2+1} + C$

10. $x^2 + 2 \ln|x+1| + 3 \ln|x-3| + C$

11. $\tan^{-1} x - \frac{1}{x-1} + \ln \frac{x^2+1}{(x-1)^2} + C$

12. $\frac{1}{2(x^2+1)} + \ln|x| - \frac{1}{2} \ln(x^2+1) + C$

Exercise 6.6.1

1. $-\frac{\cos x}{2 \sin^2 x} + \frac{1}{2} \ln\left|\tan \frac{x}{2}\right| + C$

2. $\tan x - \sec x + C$

3. $\frac{1}{\cos x} + \frac{1}{3 \cos^3 x} + \ln\left|\tan \frac{x}{2}\right| + C$

Exercise 6.7.1

1. $\frac{1}{4}$

2. π

3. π

4. $\frac{2}{3} \ln 2$

5. $\frac{4\pi}{3\sqrt{3}}$

6. $\frac{2\pi}{3\sqrt{3}}$

7. $\frac{\pi}{2} - 1$

8. $\frac{1}{2}$

9. $-\frac{\pi}{2} \ln 2$

Exercise 6.7.2

1. Convergent

2. Convergent

3. Divergent

4. Divergent

5. Convergent

6. Divergent

Miscellaneous Exercise

1. $F(x) + C$ where
$$F(x) = \begin{cases} x, & \text{if } x < 0 \\ x^2 + x, & \text{if } x \geq 0 \end{cases}$$

2. $F(x) + C$ where
$$F(x) = \begin{cases} 2x^2 - x, & \text{if } x < 3 \\ \frac{x^3}{3} + x + 3, & \text{if } x \geq 3 \end{cases}$$

3. $F(x) + C$ where
$$F(x) = \begin{cases} -\frac{x^2}{2}, & \text{if } x < 0 \\ \frac{x^2}{2}, & \text{if } x \geq 0 \end{cases}$$

4. $F(x) + C$ where
$$F(x) = \begin{cases} \frac{x^3}{3} - x - \frac{4}{3}, & \text{if } x < -1 \\ x - \frac{x^3}{3}, & \text{if } -1 \leq x < 1 \\ \frac{x^3}{3} - x + \frac{4}{3}, & \text{if } x \geq 1 \end{cases}$$

5. $F(x) + C$ where
$$F(x) = \begin{cases} \frac{x^3}{3} - \frac{x^2}{2}, & \text{if } x < 0 \\ -\frac{x^3}{3} + \frac{x^2}{2}, & \text{if } 0 \leq x < 1 \\ \frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{3}, & \text{if } x \geq 1 \end{cases}$$

6. $-\frac{3}{140}(9 + 12x + 14x^2)(1-x)^{\frac{4}{3}} + C$

7. $-\frac{6 + 25x^3}{1000}(2 - 5x^3)^{\frac{5}{3}} + C$

8. $\frac{e^x}{2}(\cos x + \sin x) + C$

9. $x \tan x + \ln|\cos x| + C$

10. $\frac{1}{5} \ln|5x + \sqrt{25x^2 - 4}| + C$

11. $\frac{x(3x^2 + 5)}{8(x^2 + 1)^2} + \frac{3}{8} \tan^{-1} x + C$

12. $\frac{1}{x^2 + 2x + 2} + \tan^{-1}(x + 1) + C$

13. $\tan^{-1} 2x + \frac{2x}{4x^2 + 1} + C$

14. $\sqrt{x(4 - x)} + 4 \sin^{-1} \frac{\sqrt{x}}{2} + C$

15. $-\frac{2\sqrt{4 - x^2}}{x} + C$

16. $-\frac{\cos x}{\cos x + \sin x} + C$

17. $2\sqrt{1 - \sin x} + C$