DIFFERENTIATION II

In this article we shall investigate some mathematical applications of differentiation. We shall be concerned with a "rate of change" problem; we shall discuss the **Mean Value Theorem** and its application to finding relative extrema; and finally we shall look at the **L' Hospital's Rule** and the **Taylor's Theorem**, both of which are very useful in evaluating limits.

1. Rate of Change Problems

Recall that the derivative of a function *f* is defined by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

if it exists. If *f* is a function of time *t*, we may write the above equation in the form

$$f'(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

and hence we may interpret $f'(t)$ as the (instantaneous) rate of change of the quantity *f* at time *t*. This allows us to investigate rate of change problems with the techniques in differentiation. We illustrate with a few examples below.

Example 1.1.

A particle moves along the *x*-axis. Its displacement at time *t* is given by

$$x(t) = t^3 + 2t^2 - 4t + 1.$$

Find its velocity and acceleration as functions of time *t*. Prove that the particle is traveling away from the origin when $t \geq 1$.

Solution. The velocity of the particle is defined as the rate of change of the displacement of the particle. So the velocity of the particle at time *t* is given by

$$x'(t) = 3t^2 + 4t - 4\ .$$

Similarly the acceleration of the particle at time *t*, being the rate of change of the velocity of the particle, is given by

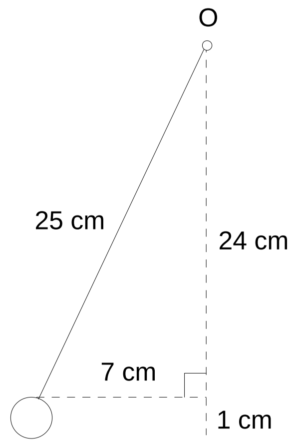
$$x''(t) = 6t + 4\ .$$

Since

$$x'(t) = 3t^2 + 4t - 4 = 3(t-1)^2 + 10t - 1 > 0$$

for $t \geq 1$, we see that the velocity of the particle is positive for $t \geq 1$. Together with $x(1) = 0$ we conclude that the particle is travelling away from the origin when $t \geq 1$.

Example 1.2.



A pendulum swings in a vertical plane, keeping its inextensible string at a length of 25 cm. When the bob is 1 cm above its equilibrium position, the speed of the bob in the *x*-direction is 2 cm/s. Find the speed of the bob in the *y*-direction at that instant. (Assume that the size of the bob is negligible.)

Solution. Let $x(t)$ and $y(t)$ be the x and y displacement of the bob at time t relative to the fixed end of the string O. Then

$$[x(t)]^2 + [y(t)]^2 = 25^2$$

for all t, because the string is of length 25 cm at all times. Differentiate with respect to t, we get $2x(t)x'(t) + 2y(t)y'(t) = 0$, so

$$|x(t)x'(t)| = |y(t)y'(t)| \tag{1.1}$$

for all t. When the bob is 1 cm above its equilibrium position (say at time t_0), we have $|y(t_0)| = 24$, $|x(t_0)| = 7$ and $|x'(t_0)| = 2$ (See figure). This gives, according to (1.1), that $|y'(t)| = 7/12$, i.e. the speed of the bob in the *y*-direction at that instant is 7/12 cm/s. (Note: speed is a scalar quantity which does not take into account of sign consideration.)

Example 1.3.

Sand is leaking through a small hole at the bottom of a conical funnel at the rate of 12 cm^3/s. If the radius of the funnel is 8 cm and the altitude of the funnel is 16 cm, find the rate at which the depth of the sand falls when the sand is 10 cm deep inside the cone.

Solution. Let $V(t)$, $r(t)$ and $h(t)$ be the volume, radius and depth of the sand inside the cone at time t respectively. Then

$$V(t) = \frac{\pi}{3}[r(t)]^2 h(t) \tag{1.2}$$

for all t. Now by an argument using similar triangles we see that $\frac{r(t)}{h(t)} = \frac{8}{16}$ for all t. Hence $r(t) = h(t)/2$ for all t, and so (1.2) reads

$$V(t) = \frac{\pi}{12}[h(t)]^3 . \tag{1.3}$$

Differentiate with respect to t, we have

$$V'(t) = \frac{\pi}{4}[h(t)]^2 h'(t) \tag{1.4}$$

for all t. Suppose at a particular t we have $h(t) = 10$ and $V'(t) = -12$. Then (1.4) gives $h'(t) = -\frac{12}{25\pi}$. Hence the depth of the sand in the cone decreases at a rate of $\frac{12}{25\pi}$ cm/s when the sand is 10 cm deep inside the cone. (Question: Why can we get (1.4) by differentiating (1.3)? Answer: By assumption $V(t) = V_0 - 12t$ for some constant V_0, so $V(t)$ is a differentiable function of t. Then $h=\left(\frac{12}{\pi}V\right)^{\frac{1}{3}}$ is differentiable at t if t satisfies $V(t)\neq 0$.)

2. The Mean Value Theorem and Local Extrema

We move on to discuss how we can find the local extrema of differentiable functions. We shall establish the *Mean Value Theorem*, which is important and interesting in itself.

Definition 2.1.

A function f is said to attain a *local maximum* at a point c if there exists an open interval (a, b) containing c such that f attains its maximum over (a, b) at c, i.e. $f(x)\leq f(c)$ for all x in (a, b). Similarly f is said to attain a *local minimum* at c if there exists an open interval (a, b) containing c such that f attains its minimum over (a, b) at c, i.e. $f(x)\geq f(c)$ for all x in (a, b).

We first recall a simple lemma from the chapter about limits.

Lemma 2.1.

Let f be a function and M be a constant.

(a) If $f(x)\geq M$ for all $x>a$, then $\lim_{x\to a} f(x)\geq M$ if the limit exists.

(b) If $f(x)\leq M$ for all $x<a$, then $\lim_{x\to a} f(x)\leq M$ if the limit exists.

We now state and prove the following **Interior Extremum Theorem**.

Theorem 2.2. (Interior Extremum Theorem)

Let f be a differentiable function on (a, b). Then if f attains an extremum at a point c in (a, b), we have $f'(c)=0$.

Proof. We prove the case when f attains a minimum at c. The case where f attains a maximum at c is similar and left to the reader.

Suppose f is differentiable on (a, b) and attains a minimum at c in (a, b). Then $f(c+h)-f(c)\geq 0$ for all h, so

$$\frac{f(c+h)-f(c)}{h}\geq 0 \quad \text{for} \quad h>0\text{, and}$$

$$\frac{f(c+h)-f(c)}{h}\leq 0 \quad \text{for} \quad h<0\text{.}$$

By Lemma 2.1 (applied to the function $h \mapsto \dfrac{f(c+h)-f(c)}{h}$) we see that $f'(c) \ge 0$ and $f'(c) \le 0$, so $f'(c) = 0$. (Note the existence of the limit $f'(c)$ is guaranteed by the differentiability of f over (a, b) which contains c.) We are done.

Q.E.D.

The interior extremum theorem asserts that for a differentiable function f, we only need to look for the points where the derivative of f vanish when we look for an interior extremum. But be careful: a general extremum may occur on the boundary, not in the interior. (e.g. the function $f(x) = x$, as a function defined on [0, 1], has its maximum occurring at x = 1, which is not an interior point. (Note in this case $f'(1) \ne 0$.))

Also note that the interior extremum theorem only tells us where the possible local extremum are. It does NOT tell us how to determine whether a point is an interior extremum. Indeed the converse of the interior extremum theorem is not true. (For example the function $f(x) = x^3$ (defined on the whole real line) has $f'(0) = 0$ but 0 is not an interior extremum.) To decide whether a point is an interior extremum we shall need to make use of the *first derivative test*, which we derive from the Mean Value Theorem. We first use the following lemma (from the chapter of continuity) to prove the **Rolle's Theorem**, a special case of the Mean Value Theorem.

Lemma 2.3.

Let f be a continuous function defined on $[a, b]$. Then there is a point c in $[a, b]$ such that f attains its maximum over $[a, b]$ at c, i.e. $f(x) \le f(c)$ for all x in $[a, b]$. There is also a point d in $[a, b]$ such that f attains its minimum over $[a, b]$ at d, i.e. $f(x) \ge f(d)$ for all x in $[a, b]$.

Theorem 2.4. (Rolle's Theorem)

Let f be a function defined on $[a, b]$ with $f(a) = f(b) = 0$. If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a point c in (a, b) such that $f'(c) = 0$.

Proof. If f is constant zero on $[a, b]$, then we are done, because $f' = 0$ everywhere in $[a, b]$.

Suppose f is not constant zero on $[a, b]$, say f assumes a positive maximum in $[a, b]$ (The case where f assumes a negative minimum is similar and left to the reader.) Then by Lemma 2.3 we can assume there is a point c in $[a, b]$ such that f attains its positive maximum at c. Now $c \ne a$ and $c \ne b$ (because $f(c) > 0$ while $f(a) = f(b) = 0$). So c is in (a, b). By the Interior Extremum Theorem 2.2 we have $f'(c) = 0$, completing the proof.

Q.E.D.

Theorem 2.5. (Mean Value Theorem)

Let f be a function defined on $[a, b]$. If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a point c in (a, b) such that $f'(c)=\frac{f(b)-f(a)}{b-a}$.

Proof. Define, for all x in $[a, b]$,

$$g(x)=\frac{f(b)-f(a)}{b-a}(x-a)-f(x)+f(a).$$

Then g is a function defined on $[a, b]$ with $g(a) = g(b) = 0$, and g is continuous on $[a, b]$ and differentiable on (a, b). So by Rolle's Theorem we see that there is a point c in (a, b) such that $g'(c)=0$. Then $f'(c)=\frac{f(b)-f(a)}{b-a}$ and we are done.

Q.E.D.

Graphically, the Mean Value Theorem says that if f is continuous on $[a, b]$ and differentiable on (a, b), then there must be at least one point c between a and b such that the slope of f at c is equal to the slope of the straight line joining the points $(a, f(a))$ and $(b, f(b))$.

We now prove the *first derivative test* for local extrema. We need the notion of an *increasing* (*decreasing*) function and the following lemma concerning increasing (decreasing) differentiable functions, which we establish from the Mean Value Theorem.

Definition 2.2.

A function f is said to be *increasing* on the interval (a, b) if $f(x)\le f(y)$ for all x and y satisfying $a<x<y<b$. Similarly f is said to be *decreasing* on the interval (a, b) if $f(x)\ge f(y)$ for all x and y satisfying $a<x<y<b$.

Lemma 2.6.

If f is a function differentiable on an interval (a, b) and $f'(x)\ge 0$ for all x in (a, b), then f is increasing on (a, b). Similarly if f is differentiable on (a, b) and $f'(x)\le 0$ for all x in (a, b), then f is decreasing on (a, b).

Proof. We prove the case where $f'(x)\ge 0$ for all x in (a, b). The other case is similar.

If x and y satisfies $a < x < y < b$, then f is differentiable on $[x, y]$, so f is continuous on $[x, y]$ and differentiable on (x, y). By Mean Value Theorem, there exists a point c in (x, y) such that

$$\frac{f(y)-f(x)}{y-x}=f'(c)\geq 0.$$

Then $f(x)\leq f(y)$ and the conclusion follows.

Q.E.D.

Theorem 2.7. (First Derivative Test 1)

Let f be a function differentiable on (a, b) except possibly at a point c in (a, b), and suppose that f is continuous at c. If $f'(x)\geq 0$ on (a,c) and $f'(x)\leq 0$ on (c,b), then f attains its maximum over (a, b) at the point c, i.e. $f(x)\leq f(c)$ for all x in (a, b).

Proof. Since f is continuous at $c\in(a,b)$, we have $f(c)=\lim_{y\to c^-} f(y)=\lim_{y\to c^+} f(y)$. Now if c is such that $f'(x)\geq 0$ on (a, c) and $f'(x)\leq 0$ on (c, b), then by Lemma 2.6 we have

(i) f is increasing on (a, c); and

(ii) f is decreasing on (c, b).

By (i), for all x, y satisfying $a < x < y < c$, we have $f(x)\leq f(y)$; letting y tend to c we see that $f(x)\leq \lim_{y\to c^-} f(y)=f(c)$ holds for all x in (a, c). Similarly, by (ii), for all x, y satisfying $c < y < x < b$, we have $f(x)\leq f(y)$; letting y tend to c we see that $f(x)\leq \lim_{y\to c^+} f(y)=f(c)$ holds for all x in (c, b). Hence $f(x)\leq f(c)$ holds for all x in (a, b) (the inequality is trivially satisfied when $x = c$) and we are done.

Q.E.D.

Corollary 2.8. (First Derivative Test 2)

Let f be a function differentiable on (a, b) except possibly at a point c in (a, b), and suppose that f is continuous at c. If $f'(x)\leq 0$ on (a,c) and $f'(x)\geq 0$ on (c,b), then f attains its minimum over (a, b) at the point c, i.e. $f(x)\geq f(c)$ for all x in (a, b).

Proof. Apply Theorem 2.7 to the function $-f$.

Q.E.D.

We illustrate below how the first derivative test can be used together with the interior extremum theorem to locate the local extrema of a function.

Example 2.1.

Locate the local extrema of the function $f(x)=3x^5-5x^3$.

Solution. Note that f is a differentiable function on the whole real line and $f'(x)=15x^2(x-1)(x+1)$ for all real values of x, so $f'(x)=0$ if and only if $x = -1$, 0 or 1. Now the interior extremum theorem implies that any local extremum of f must occur at points where f' is equal to zero (because f is differentiable everywhere), so the interior extremum of f can only occur at $x = -1$, 0 or 1. We check whether each point is an interior extremum using the following table:

	$x<-1$	$x=-1$	$-1<x<0$	$x=0$	$0<x<1$	$x=1$	$x>1$
$f'(x)$	+ve	0	–ve	0	–ve	0	+ve

From the table, together with the first derivative test, we conclude that f attains a local maximum at $x = -1$ and f attains a local minimum at $x = 1$. Note that $x = 0$ is neither a local maximum nor a local minimum despite that $f'(0)=0$, because f is decreasing throughout the interval $(-1,1)$. Hence we conclude that there is only one local maximum of f, namely $x = -1$, and there is only one local minimum of f, namely $x = 1$.

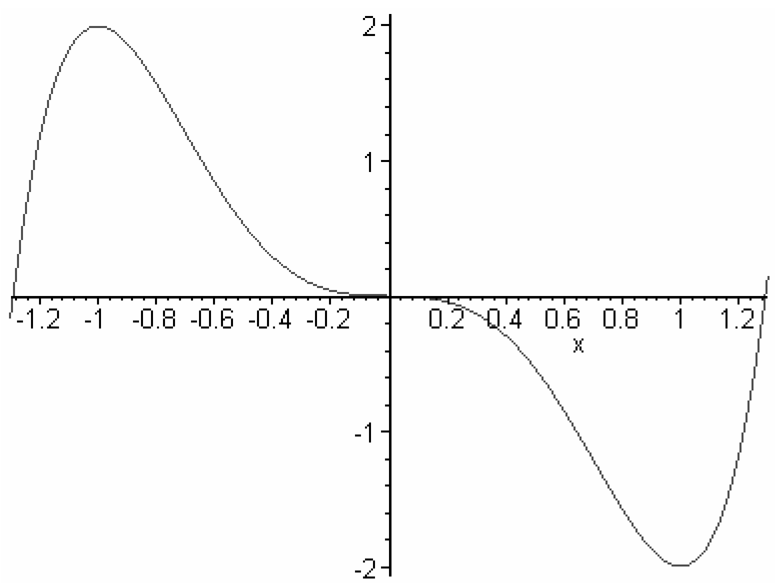


Figure 1: A graph of $f(x)=3x^5-5x^3$

Example 2.2.

Locate the local extrema of the function $f(x)=|x|-|x-1|+|x-2|$.

Solution. Note that f is a function defined on the whole real line and f is differentiable on the whole real line except at $x = 0$, 1, 2. Now

$$f'(x)=\begin{cases}-1 & \text{if } x<0 \text{ or } 1<x<2 \\ 1 & \text{if } 0<x<1 \text{ or } x>2\end{cases}$$

so $f'(x)\neq 0$ whenever $f'(x)$ exists. Now the interior extremum theorem implies that any local extremum of f must occur at points where f' does not exist (because $f'(x)\neq 0$ whenever $f'(x)$ exists), so the interior extremum of f can only occur at $x = 0$, 1 or 2. We check whether each point is an interior extremum using the following table:

	$x<0$	$x=0$	$0<x<1$	$x=1$	$1<x<2$	$x=2$	$x>2$
$f'(x)$	–ve	undefined	+ve	undefined	–ve	undefined	+ve

From the table, together with the first derivative test, we conclude that f attains a local maximum at $x = 1$ and f attains a local minimum at $x = 0$ and $x = 2$. Thus we conclude that there is only one local maximum of f, namely $x = 1$, and there are exactly two local minima of f, namely $x = 0$ and $x = 2$.

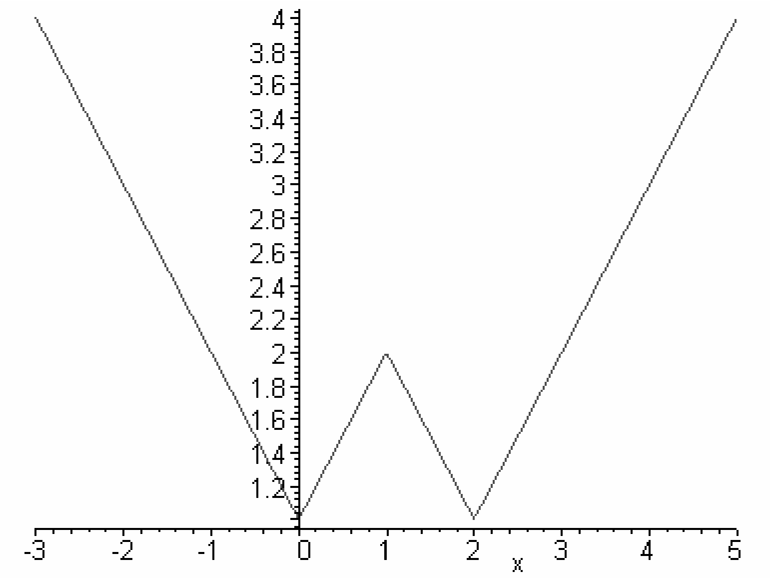


Figure 2: A graph of $f(x)=|x|-|x-1|+|x-2|$

Let us conclude this section by illustrating how the Mean Value Theorem can be useful in proving inequalities.

Example 2.3.

Prove that $|\sin x-\sin y|\le|x-y|$ for all real values of x and y.

Solution. Note we only need to prove $\left|\frac{\sin x-\sin y}{x-y}\right|\le 1$ for $x\ne y$. But for $x<y$, we have the sine function continuous on $[x, y]$ and differentiable on (x, y) so by Mean Value Theorem we have

$$\frac{\sin x-\sin y}{x-y}=\cos c$$

for some c in (x, y). Hence $\left|\frac{\sin x-\sin y}{x-y}\right|=|\cos c|\le 1$. Similarly, $\left|\frac{\sin x-\sin y}{x-y}\right|\le 1$ holds when $x > y$ as well. Hence we are done.

The first derivative test can be useful proving inequalities as well.

Example 2.4.

Prove that $e^x \ge 1 + x$ for all real values of *x*.

Solution. Define $f(x) = e^x - 1 - x$. Then *f* is differentiable on the whole real line. We find, further, that, $f'(x) = e^x - 1$ so $f'(x) > 0$ for $x > 0$ and $f'(x) < 0$ for $x < 0$. By the first derivative test, we find that *f* attains a minimum at the point 0. So $f(x) \ge f(0) = 0$ for all real values of *x*.

3. L' Hospital's Rule

Our aim in this section is to state the L' Hospital's Rule which helps us to evaluate some limits which are difficult to obtain using other methods. We shall be concerned with a precise statement of the theorem rather than the proof.

We provide some motivation first: consider

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

We know that the limit exists and is equal to 1. However we cannot evaluate this by the most elementary methods of calculating limits: there are no common factors in the denominator and numerator already, and yet we cannot evaluate the limit by direct substitution (which would yield 0/0, something meaningless: note we cannot cancel the zeros and say this limit is equal to 1, because such argument applied to the limit

$$\lim_{x \to 0} \frac{2\sin x}{x}$$

would yield that this limit is equal to 1, which is untrue). So we need some other ways to calculate the limit. One may use a geometric approach to argue that this limit exists and is equal to 1; alternatively one may use the L' Hospital's Rule, which we state as follows.

Theorem 3.1. (L' Hospital's Rule 1)

Suppose *f* and *g* are two functions defined on an interval (a, b) (where $-\infty \le a < b \le \infty$) such that

1. both *f* and *g* are differentiable in (a, b) with $g'(x) \ne 0$ for all *x* in (a, b);

2. $\lim_{x \to a+} f(x)$ and $\lim_{x \to a+} g(x)$ both exists and equals 0; and

3. $\lim_{x \to a+} \frac{f'}{g'}(x)$ exists and equals *L*, where *L* is a real number or $L = \pm\infty$.

Then $\lim_{x \to a^+} \frac{f(x)}{g(x)}$ exists and $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$. The case for left hand limit is similar.

Example 3.1.

Evaluate the limit $\lim_{x\to 0}\frac{\sin x}{x}$.

Solution. By L' Hospital's Rule 1 we have $\lim_{x\to 0}\frac{\sin x}{x}=1$, because $\lim_{x\to 0}\frac{\cos x}{1}$ exists and equals 1. (Note $\sin x$ and x are both differentiable everywhere, and they have a common limit of 0 as x tends to 0. Also the function x has a derivative of constant 1 which never equals 0.)

Remark. This may not be a good approach to evaluate $\lim_{x\to 0}\frac{\sin x}{x}$ because we need to know this limit before we could differentiate $\sin x$ if we define $\sin x$ geometrically.

To see why L' Hospital's Rule should work, let us assume a is a (finite) real number and take f and g to be straight lines passing through the point $(a, 0)$, with slopes m and n respectively, with $n \neq 0$. In other words, let us take $f(x)=m(x-a)$ and $g(x)=n(x-a)$. Then we see that although

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

cannot be evaluated by independently taking limits of the numerator and the denominator, we have

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{m}{n}=\frac{f'(a)}{g'(a)},$$

which is the ratio of the slopes of f and g at a. Hence in general if f and g can be approximated by straight lines near the point a, we should expect something like

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f'(a)}{g'(a)}$$

to hold. L' Hospital's Rule is just one way of formulating this observation precisely.

Note that in the L' Hospital's Rule stated above (and also in the one stated below), we only need both f and g to be differentiable in a deleted neighbourhood of the point a which x tends to. We do not require f and g to be differentiable at the point a. This is one of the most powerful features of the L' Hospital's Rule we state here. (c.f. Example 3.4 below.)

We now give another form of L' Hospital's Rule, which deals with limits of the form $\frac{\infty}{\infty}$.

Theorem 3.2. (L' Hospital's Rule 2)

Suppose *f* and *g* are two functions defined on an interval (a, b) (where $-\infty \le a < b \le \infty$) such that

1. both *f* and *g* are differentiable in (a, b) with $g'(x) \ne 0$ for all *x* in (a, b);

2. $\lim_{x \to a+} g(x)$ exists (in the extended sense) and equals $\pm\infty$; and

3. $\lim_{x \to a+} \frac{f'}{g'}(x)$ exists and equals *L*, where *L* is a real number or $L = \pm\infty$.

Then $\lim_{x \to a^+} \frac{f(x)}{g(x)}$ exists and $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$. The case for left hand limit is similar.

Notice that in L' Hospital's Rule 2 we only require the denominator to limit to infinity, but we need no assumption on the limit of the numerator.

Example 3.2.

Evaluate the limit $\lim_{x \to \infty} \frac{x}{e^x}$.

Solution. By L' Hospital's Rule 2 we have $\lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = 0$. (Check the conditions of L' Hospital's Rule 2 for the functions *x* and e^x !)

Example 3.3.

Evaluate the limit $\lim_{x \to \infty} \frac{x + \cos x}{x}$.

Solution. We are tempted to use L' Hospital's Rule 2, because $\lim_{x \to \infty} x$ exists and is equal to ∞; but we know that the limit $\lim_{x \to \infty} \frac{1 - \sin x}{1}$, which comes from a formal application of L' Hospital's Rule 2, does not exist. So, we *cannot* apply the L' Hospital's rule to draw any conclusion. To find the limit, recall that $\lim_{x \to \infty} \frac{\cos x}{x} = 0$, so

$$\lim_{x \to \infty} \frac{x + \cos x}{x} = \lim_{x \to \infty} \left(1 + \frac{\cos x}{x} \right) = 1 + 0 = 1.$$

Example 3.4.

Evaluate the limit $\lim_{x\to 0^+} x\ln x$.

Solution. Rewrite the limit as $\lim_{x\to 0^+}\frac{\ln x}{1/x}$. Then by L' Hospital's Rule 2 we have

$$\lim_{x\to 0^+}\frac{\ln x}{1/x}=\lim_{x\to 0^+}\frac{1/x}{-1/x^2}=0 .$$

(Remember to check the conditions of L' Hospital's Rule 2 for the functions ln x and $1/x$).

We have stated without proof various forms of L' Hospital's Rule because the proofs are too involved. However, we can prove a theorem similar to L' Hospital's Rule here, where we assumed that f and g are actually differentiable at a and $g'(a)\neq 0$.

Proposition 3.3.

Suppose the functions f and g are both differentiable at the point a and $f(a) = g(a) = 0$. Then if $g'(a)\neq 0$ and $\frac{f'(a)}{g'(a)}=L$ we have $\lim_{x\to a}\frac{f(x)}{g(x)}=L$.

Proof. Simply observe that

$$\frac{f(x)}{g(x)}=\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f(x)-f(a)}{x-a}\bigg/\frac{g(x)-g(a)}{x-a}$$

so letting x tend to a we get (because both the limits in the numerator and denominator exist and the limit in the denominator is non-zero) that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=L .$$

Q.E.D.

Note how the conditions of Proposition 3.3 are different from the ones in the various versions of the L' Hospital's Rule. Example 3.1 can be computed using both Proposition 3.3 and the L' Hospital's Rule, while Example 3.4 can only be computed using the L' Hospital's Rule but not Proposition 3.3. Here we present an example when one must use Proposition 3.3 instead of the L' Hospital's Rule.

Example 3.5.

Evaluate the limit $\lim_{x\to 0}\frac{f(x)}{x}$, where f is defined by $f(x)=\begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational}\end{cases}$.

Solution. It is easy to check f is differentiable at 0 and $f'(0)=0$. The rest of the conditions of Proposition 3.3 are easily checked. Hence by Proposition 3.3 we get $\lim_{x\to 0}\frac{f(x)}{x}=\frac{f'(0)}{1}=0$.

Remark. It is easy to see 0 is the only point at which f is differentiable, so the L' Hospital's Rule does not apply here. But of course apart from using Proposition 3.3 one should also be able to derive $\lim_{x\to 0}\frac{f(x)}{x}=0$ directly using the definition of limit (just note that $\left|\frac{f(x)}{x}\right|\le |x|$ for $x\neq 0$).

Example 3.6.

Evaluate the limit $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$.

Solution. Rewrite $\frac{1}{x}-\frac{1}{\sin x}$ as $\frac{\sin x - x}{x\sin x}$. We are then tempted to use L' Hospital's Rule 1. This means that we need to show

$$\lim_{x\to 0}\frac{(\sin x - x)'}{(x\sin x)'}=\lim_{x\to 0}\frac{\cos x - 1}{\sin x + x\cos x}$$

exists and find the limit. (Check that the differentiability conditions, and that the non-zero requirement on the derivative, are fulfilled.) We cannot do so by direct substitution, because both the numerator and the denominator are zero when $x = 0$. However, we can apply the L' Hospital's Rule 1 here again, and we see that because

$$\lim_{x\to 0}\frac{(\cos x - 1)'}{(\sin x + x\cos x)'}=\lim_{x\to 0}\frac{-\sin x}{2\cos x - x\sin x}$$

exists and is equal to 0, we have

$$\lim_{x\to 0}\frac{\cos x - 1}{\sin x + x\cos x}$$

exists and is equal to 0. (Again check the differentiability conditions and the non-zero requirement on the derivative, are satisfied.) So by our preceding discussion

$$\lim_{x\to 0}\frac{\sin x - x}{x\sin x}$$

exists and is equal to 0.

Remark. This kind of inductive argument is very important. It tells us that it is possible sometimes to apply L' Hospital's Rule several times.

Example 3.7.

Evaluate the limit $\lim_{x\to 0^+} x^x$.

Solution. Let $y = x^x$. Our strategy is to compute

$$\lim_{x\to 0} \ln y .$$

If the limit exists, by taking exponents, we get our original limit. (Note we need to make use of the continuity of the exponential function here.) If the limit is positive or negative infinity, by simple limiting arguments (some comparisons, maybe), we find that the limit is positive infinity or zero respectively. Now

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} x \ln x = 0$$

by Example 3.4, so by continuity of the exponential function we have

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\lim_{x\to 0^+} \ln y} = 1 .$$

Remark. We only compute the right-hand limit because for all $\varepsilon > 0$ there are infinitely many x satisfying $-\varepsilon < x < 0$ such that x^x is not real.

Let us remark here that we do not need L' Hospital's Rule for limits of the form like 0^∞. One can easily see this by following the routine in the previous example, which tells us that after taking logarithm, the limit is in the form $-\infty \cdot \infty = -\infty$, which then implies that the original limit is zero.

We conclude this section with a discrete version of the L' Hospital's Rule.

Theorem 3.4. (Discrete L' Hospital's Rule; Stolz Theorem)

Let $\{u_n\}$ and $\{v_n\}$ be two sequences such that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = \infty$ and $0 < u_n < u_{n+1}$ for all n. If $\lim_{n\to\infty} \frac{v_{n+1} - v_n}{u_{n+1} - u_n} = L$ (for some real number L), then $\lim_{n\to\infty} \frac{v_n}{u_n} = L$.

Proof. We assume the reader is familiar with the notion of limsup and liminf. One may prefer to omit the proof if one does not know the notion of limsup and liminf.

Let $\varepsilon > 0$ be given. Then there exists a positive integer N such that

$$L-\varepsilon \le \frac{v_{n+1}-v_n}{u_{n+1}-u_n} \le L+\varepsilon \tag{3.1}$$

for all $n \ge N$. Since $u_n < u_{n+1}$ for all n, rearranging (3.1) gives

$$(L-\varepsilon)(u_{n+1}-u_n) \le v_{n+1}-v_n \le (L+\varepsilon)(u_{n+1}-u_n) \tag{3.2}$$

for all $n \ge N$. Let k satisfy $k \ge N$. Then by (3.2) we have

$$(L-\varepsilon)\sum_{n=N}^{k}(u_{n+1}-u_n) \le \sum_{n=N}^{k}(v_{n+1}-v_n) \le (L+\varepsilon)\sum_{n=N}^{k}(u_{n+1}-u_n)$$

so

$$(L-\varepsilon)(u_{k+1}-u_N) \le v_{k+1}-v_N \le (L+\varepsilon)(u_{k+1}-u_N)\,.$$

Since $u_{k+1} > 0$ for all k, we have

$$(L-\varepsilon)\left(1-\frac{u_N}{u_{k+1}}\right) \le \frac{v_{k+1}}{u_{k+1}} - \frac{v_N}{u_{k+1}} \le (L+\varepsilon)\left(1-\frac{u_N}{u_{k+1}}\right)\,.$$

Letting k go to infinity we get

$$L-\varepsilon \le \liminf_{k\to\infty}\frac{v_{k+1}}{u_{k+1}} \le \limsup_{k\to\infty}\frac{v_{k+1}}{u_{k+1}} \le L+\varepsilon\,.$$

Letting ε tend to 0 we get $\lim_{n\to\infty}\frac{v_n}{u_n} = L$.

Q.E.D.

Remark. The above proof can be generalized to give a proof of the L' Hospital's Rules 1 and 2 that we have given. The main tool that we miss is something called the *Cauchy Mean Value Theorem,* which is a generalization of our Mean Value Theorem.

Example 3.8.

Evaluate the limit $\lim_{n\to\infty}\frac{\ln n}{n}$.

Solution. By Theorem 3.4 we have, because $\lim_{n\to\infty}\frac{\ln(1+1/n)}{n+1-n}$ exists and equals 0, that $\lim_{n\to\infty}\frac{\ln n}{n} = 0$.

4. Taylor's Theorem

We conclude this article with a survey of the Taylor's Theorem, which is often even more helpful than the L' Hospital's Rule in evaluating limits.

Definition 4.1.

Let *f* be a differentiable function. If *f'*, the derivative of *f*, is also differentiable, we say *f* is *twice differentiable* and we denote the derivative of *f'* by $f^{(2)}$. Inductively, suppose a function *f* is $n-1$ times differentiable (i.e. $f^{(n-1)}$ exists). We say *f* is *n times differentiable* if $f^{(n-1)}$ is also differentiable, and denote the derivative of $f^{(n-1)}$ by $f^{(n)}$.

Definition 4.2.

We say a function *f* is of class C^n if it is *n* times differentiable and $f^{(n)}$ is continuous, and we say a function is C^∞ if it is C^n for all positive integers *n*.

Example 4.1.

Polynomials, trigonometric functions and exponential functions are of class C^∞. The function

$$f(x)=\begin{cases} x^2\sin\dfrac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

is differentiable but not of class C^1.

Proposition 4.1.

If *f* is C^1, then at any point *a* we have

$$\lim_{x\to a}\frac{f(x)-f(a)-f'(a)(x-a)}{x-a}$$

exists with a value equal to 0.

Proof. Trivial.

For example, if we take $a = 0$, then the above proposition says that for small values of *x*, we have $f(x)\approx f(0)+f'(0)x$, with an error which tends to zero at a rate faster than that of *x*. In this sense the degree 1 polynomial $P(x)=f(0)+f'(0)x$ provides a good approximation of *f* near the point 0. But why should this be the case?

Observe that the polynomial $P(x)=f(0)+f'(0)x$ satisfies

$$\begin{cases} P(0) = f(0) \\ P'(0) = f'(0) \end{cases}.$$

Hence as x moves away from the point 0, both $f(x)$ and $P(x)$ escapes the value $f(0) = P(0)$ at the same initial rate. Thus to well approximate a general function f by a degree n polynomial near the point 0, we are led to consider a polynomial $P_{n;0}$ whose derivatives up to (and including) order n agree with those of f. In other words, we are led to consider a polynomial which satisfies

$$\begin{cases} P_{n;0}(0) = f(0) \\ P_{n;0}'(0) = f'(0) \\ P_{n;0}^{(2)}(0) = f^{(2)}(0) \\ \vdots \\ P_{n;0}^{(n)}(0) = f^{(n)}(0) \end{cases}. \qquad (4.1)$$

There is only one polynomial of degree n which satisfies (4.1): it is given by

$$P_{n;0}(x) = f(0) + f'(0)\frac{x}{1!} + f^{(2)}(0)\frac{x^2}{2!} + \ldots + f^{(n)}(0)\frac{x^n}{n!}.$$

(Verify!) More generally, when we want to approximate a function f by a degree n polynomial near a point a, we usually consider the following *nth order Taylor polynomial*, which we denote by $P_{n;a}(x)$.

Definition 4.3.

For any n times differentiable function f, we define the *nth order Taylor polynomial* of f at a point a to be the following polynomial in x:

$$f(a) + f'(a)\frac{x-a}{1!} + f^{(2)}(a)\frac{(x-a)^2}{2!} + \ldots + f^{(n)}(a)\frac{(x-a)^n}{n!}.$$

We denote this polynomial by $P_{n;a}(x)$.

Note that $P_{n;a}(x)$ is the unique degree n polynomial which satisfies

$$\begin{cases} P_{n;a}(a) = f(a) \\ P_{n;a}'(a) = f'(a) \\ P_{n;a}^{(2)}(a) = f^{(2)}(a) \\ \vdots \\ P_{n;a}^{(n)}(a) = f^{(n)}(a) \end{cases}.$$

(Here the derivative is with respect to the variable x; when we talk about the Taylor polynomial at the point a, we always assume that x is the variable and a is usually kept fixed.) In our new notation, Proposition 4.1 now reads

$$\lim_{x \to a} \frac{f(x) - P_{1;a}(x)}{x - a} = 0$$

where $P_{1;a}(x)$ is the first order Taylor polynomial of f at the point a. We expect a corresponding result for the nth order Taylor polynomial. But let us look at some examples first.

Example 4.2.

The nth order Taylor polynomial of a polynomial P at any point a is equal to P itself. The nth order Taylor polynomial of some common functions at the point 0 are listed in the following table:

$f(x)$	$P_{n;0}(x)$
$\frac{1}{1-x}$	$1 + x + x^2 + x^3 + \ldots + x^n$
$\cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots +$ term of degree n
$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots +$ term of degree n
e^x	$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$
$\ln(1-x)$	$-x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots - \frac{x^n}{n}$
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{(-1)^{n-1} x^n}{n}$

For instance, the 9^{th} and 10^{th} Taylor polynomial of the function $\sin x$ at the point 0 are both equal to

$$P_{9;0}(x) = P_{10;0}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}.$$

Also, the nth Taylor polynomial of the function $\cos x$ at the point π is given by

$$P_{n;\pi}(x) = -1 + \frac{(x-\pi)^2}{2!} - \frac{(x-\pi)^4}{4!} + \frac{(x-\pi)^6}{6!} - \ldots + \text{ term of degree } n.$$

The following theorem generalizes Proposition 4.1, as was expected, and illustrates how the nth Taylor polynomial of a function at a point a serve as a good degree n polynomial estimate of f near a.

Theorem 4.2. (Taylor's Theorem)

If *f* is C^n for some positive integer *n*, then at any point *a* we have

$$\lim_{x \to a} \frac{f(x) - P_{n;a}(x)}{(x-a)^n}$$

exists with a value equal to 0, where $P_{n;a}(x)$ is the *n*th Taylor polynomial of *f* at the point *a*.

Proof. Let f be C^n and $P_{n;a}(x)$ be the *n*th Taylor polynomial of f at a. Define $g(x) = f(x) - P_{n;a}(x)$. Then $g(a) = g'(a) = g^{(2)}(a) = ... = g^{(n)}(a) = 0$, where the derivatives are taken with respect to *x*. Now by Mean Value Theorem, given any *x* we have

$$|g(x)| = |g(a) + g'(c_1)(x-a)| = |g'(c_1)||x-a|$$

for some c_1 between *a* and *x*. Using the fact that $g'(a) = 0$ and repeating, we have

$$|g'(c_1)| = |g^{(2)}(c_2)||c_1 - a| \le |g^{(2)}(c_2)||x-a|$$

for some c_2 between *a* and *x*, so $|g(x)| \le |g^{(2)}(c_2)||x-a|^2$. Proceeding inductively, we have

$$|g(x)| \le |g^{(n)}(c_n)||x-a|^n$$

for some c_n between *a* and *x*. Thus

$$\left|\frac{g(x)}{(x-a)^n}\right| \le |g^{(n)}(c_n)|$$

where c_n is between *a* and *x*. Letting *x* tend to *a* we see that

$$\lim_{x \to a} \frac{g(x)}{(x-a)^n}$$

exists with a value equal to 0, using the continuity of $g^{(n)}$ and the fact that $g^{(n)}(a) = 0$. Substituting *g*(*x*) we get our desired result.

Q.E.D.

Taylor's Theorem is very useful when we evaluate limits. We illustrate this by some examples.

Example 4.3.

Evaluate the limit $\lim_{x \to 0} \frac{2\cos x - 2 + x^2}{x^4}$.

Solution. Write

$$\frac{2\cos x-2+x^2}{x^4}=\frac{2\left[\cos x-\left(1-\frac{x^2}{2}\right)\right]}{x^4}=\frac{2\left[\cos x-\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)\right]}{x^4}+\frac{1}{12}.$$

Then as x tends to 0, $\dfrac{\cos x-\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)}{x^4}$ tends to 0 (by Taylor's Theorem; note that the 4th order Taylor polynomial of cos x at 0 is $1-\frac{x^2}{2}+\frac{x^4}{24}$), so $\lim\limits_{x\to 0}\dfrac{2\cos x-2+x^2}{x^4}$ exists and is equal to $\dfrac{1}{12}$.

Remark. Of course we could have evaluated the above limit by L' Hospital's Rule. But this would then involve four differentiations (to get rid of the x^4 in the denominator). Our method above is computationally simpler (provided that you remember the Taylor polynomial of cos x).

Before we look at some further examples, let us introduce the small *o* notation, which is often a convenient shorthand.

Definition 4.4.

If *f*, *g* are functions such that $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=0$, we write "$f(x)=o(g(x))$ as $x\to a$".

In other words, if it is understood that we are concerned about the behaviour as *x* tends to *a*, then $o(g(x))$ represents a quantity which satisfies $\lim\limits_{x\to a}\dfrac{o(g(x))}{g(x)}=0$. We shall also write "$f(x)=P(x)+o(g(x))$ as $x\to a$" as a shorthand for $\lim\limits_{x\to a}\dfrac{f(x)-P(x)}{g(x)}=0$. In this small *o* notation, our Taylor's Theorem now reads:

Corollary 4.3. (Taylor's Theorem)

If *f* is C^n for some positive integer *n*, then at any point *a* we have

$$f(x)=P_{n;a}(x)+o\left((x-a)^n\right) \quad \text{as} \quad x\to a,$$

where $P_{n;a}(x)$ is the *n*th Taylor polynomial of *f* at the point *a*. In particular, we have

$$f(x)=P_{n;0}(x)+o(x^n) \quad \text{as} \quad x\to 0.$$

The solution of Example 4.3 can now be written as follows:

$$\frac{2\cos x-2+x^2}{x^4}=\frac{2\left(1-\frac{x^2}{2}+\frac{x^4}{24}+o\left(x^4\right)\right)-2+x^2}{x^4}=\frac{\frac{x^4}{12}+o\left(x^4\right)}{x^4}=\frac{1}{12}+\frac{o\left(x^4\right)}{x^4}$$

so as x tends to 0 we get $\lim_{x\to 0}\frac{2\cos x-2+x^2}{x^4}=\frac{1}{12}$. (Note we have used the fact that $2o\left(x^4\right)=o\left(x^4\right)$. This is simply another way of saying that $\lim_{x\to 0}\frac{2f(x)}{x^4}=0$ if and only if $\lim_{x\to 0}\frac{f(x)}{x^4}=0$.)

Example 4.4.

Evaluate $\lim_{x\to 0}\frac{x^2}{e^x-1-x}$.

Solution. Since $e^x=1+x+\frac{x^2}{2}+o(x^2)$ as x tends to 0, we have

$$\frac{x^2}{e^x-1-x}=\frac{x^2}{\frac{1}{2}x^2+o(x^2)}=\frac{1}{\frac{1}{2}+\frac{o(x^2)}{x^2}}\to 2 \quad \text{as } x \text{ tends to 0.}$$

Example 4.5.

Evaluate $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$.

Solution. Since the 2nd order Taylor polynomial of sin x at the point 0 is x, we have $\sin x=x+o(x^2)$ as x tends to 0, so

$$\frac{1}{x}-\frac{1}{\sin x}=\frac{1}{x}-\frac{1}{x+o(x^2)}=\frac{o(x^2)}{x(x+o(x^2))}=\frac{o(x^2)}{x^2+o(x^3)}=\frac{\frac{o(x^2)}{x^2}}{1+\frac{o(x^3)}{x^2}}\to\frac{0}{1+0}=0 \quad \text{as } x \text{ tends to 0.}$$

Remark. Compare with Example 3.6.

Throughout our discussion of Taylor polynomials, we have only been concerned about the approximation of a given function near a given point. We only focused on how fast the error term of a Taylor approximation improves as the variable x tends to a given point a. We have not discussed whether the error term of the nth Taylor approximation of a function will improve as n tends to infinity. One way to do so is to write down an explicit expression for the error term. There are a number of ways to do so; below is one. (It is also commonly known as Taylor's Theorem.)

Theorem 4.4. (Taylor's Theorem)

If f is n + 1 times differentiable for some non-negative integer n, then for any distinct points a and x there exists a point c between a and x such that

$$f(x) = P_{n;a}(x) + f^{(n+1)}(c)\frac{(x-a)^{n+1}}{(n+1)!},$$

where $P_{n;a}(x)$ is the nth Taylor polynomial of f at the point a.

Proof. Let f be n + 1 times differentiable for some non-negative integer n and let a, x be two given points. Then let M to be the constant satisfying

$$f(x) = P_{n;a}(x) + M\frac{(x-a)^{n+1}}{(n+1)!}, \tag{4.2}$$

we define a new function $g(t)$ by

$$g(t) = f(t) - P_{n;a}(t) - M\frac{(t-a)^{n+1}}{(n+1)!}.$$

Now the function g is also n + 1 times differentiable, and it satisfies

$$g(a) = g'(a) = g^{(2)}(a) = \ldots = g^{(n)}(a) = 0,$$

so from $g(x) = 0$ we get, by the Mean Value Theorem, that

$$g'(c_1) = 0$$

for some c_1 between a and x. Using the fact that $g'(a) = 0$ as well and repeating, we have by Mean Value Theorem again that

$$g^{(2)}(c_2) = 0$$

for some c_2 between a and x. Proceeding inductively, we have

$$g^{(n+1)}(c_{n+1}) = 0$$

for some c_{n+1} between a and x. But $g^{(n+1)}(t) = f^{(n+1)}(t) - M$, so c_{n+1} satisfies $f^{(n+1)}(c_{n+1}) = M$.

Substituting this into (4.2), we get

$$f(x) = P_{n;a}(x) + f^{(n+1)}(c_{n+1})\frac{(x-a)^{n+1}}{(n+1)!}$$

where c_{n+1} is between a and x. This proves our theorem if we take $c = c_{n+1}$.

Q.E.D.

Remark. This is just more or less a refinement of the proof of Theorem 4.2.

With the help of the theorem, we see that for any point x and for any positive integer n we have

$$\left|\cos x-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots+\frac{(-1)^n x^{2n}}{(2n)!}\right)\right|=\left|\frac{x^{2n+1}}{(2n+1)!}\sin c\right|\le\left|\frac{x^{2n+1}}{(2n+1)!}\right|$$

for some c between 0 and x. If we keep x fixed for a moment and let n tend to infinity, we see from the Sandwich theorem that

$$\lim_{n\to\infty}\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots+\frac{(-1)^n x^{2n}}{(2n)!}\right)$$

exists and is equal to cos x. In other words, we have

$$\begin{aligned}\cos x&=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\\&=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\end{aligned}.$$

This holds for all real x, and thus we have got an infinite series expansion of the function cos x. This is usually referred to as the *power series expansion* of the function cos x about the point 0. Similarly, we have the following power series expansions of the functions sin x and e^x:

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!},$$

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

which are valid for all real numbers x. But one must not be too enthusiastic at this point and conclude that $f(x)=\lim_{n\to\infty}P_{n;0}(x)$ holds for all C^∞ functions and all real x. Of course, for $f(x)=\lim_{n\to\infty}P_{n;0}(x)$ to make sense, we need $P_{n;0}(x)$ to be defined for all positive integers n, so we need f to be C^∞; but the mere assumption that f is C^∞ does not guarantee the expression $f(x)=\lim_{n\to\infty}P_{n;0}(x)$ to hold for all real x. The following example shows that in fact even f is C^∞, $P_{n;0}(x)$ may fail to converge to $f(x)$, even when x is very close to 0.

Example 4.6.

Compute the nth Taylor polynomial of the function $f(x)=\begin{cases}e^{-\frac{1}{x^2}} & \text{if } x\ne 0\\ 0 & \text{if } x=0\end{cases}$ at the point 0, and conclude that $f(x)=\lim_{n\to\infty}P_{n;0}(x)$ does not hold unless $x = 0$.

Solution. Note that $f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{e^{-\frac{1}{x^2}}}{x}=0$, and it is a straight forward exercise to show that $f^{(n)}(x)=\frac{\text{polynomial in } x}{\text{polynomial in } x}e^{-\frac{1}{x^2}}$ for all positive integers *n* at all $x\neq 0$. Hence inductively, we get

$$f^{(n)}(0)=\lim_{x\to 0}\frac{f^{(n-1)}(x)-f^{(n-1)}(0)}{x-0}=\lim_{x\to 0}\frac{f^{(n-1)}(x)}{x}=\lim_{x\to 0}\frac{\text{polynomial in } x}{\text{polynomial in } x}e^{-\frac{1}{x^2}}=0$$

for all positive integers *n*. Thus $P_{n;0}(x)=0$ for all positive integers *n* and for all real *x*, from which we conclude that $f(x)=\lim_{n\to\infty}P_{n;0}(x)$ does not hold unless $f(x)=0$, i.e. $x = 0$.

The above example shows that the theory of power series expansion is deeper than one might suspect at first sight. We leave this hard problem till a later chapter.

We conclude this article by mentioning that one can define what is meant by C^n or C^∞ on an open interval (α,β) instead of on the whole real line, and then base upon the whole theory of Taylor series on C^n or C^∞ functions on the interval. One only needs to be careful when applying the Mean Value Theorem. We leave it to the reader to work this out.

5. Exercise

1. A particle moves on the real line. Its position at time *t* is given by $x(t)=\frac{2}{t+1}+\ln(t+1)$ for $t\geq 0$.
   (a) Find the velocity of the particle at time *t*.
   (b) Find the maximum speed of the particle for $t\geq 0$.
   (c) Does the particle change its direction in the time interval $t\geq 0$? If yes, when?

2. Find all local minimum(s) of the function $f(x)=\begin{cases}x^4-6x^2-8x+1 & \text{if } x\neq 0\\ 3 & \text{if } x=0\end{cases}$.

3. Prove that $\frac{x-1}{x}\leq \ln x\leq x-1$ for $x\geq 1$.

4. Show that $\cos x\geq 1-\frac{x^2}{2}$ for $x\geq 0$.

5. Let

$$f(x)=\begin{cases}3x+4x^2\sin\dfrac{1}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}.$$

Show that $f'(0)>0$, but f is not increasing near 0, in the sense that f is not increasing in any open interval containing 0. Compare with Theorem 2.6.

6. Let f be a differentiable function and assume $\lim_{x\to\infty} f'(x)=22$. Show that $\lim_{x\to\infty}\dfrac{f(x)}{x}$ exists and has a value of 22.

7. Show that if $f'(x)=0$ for all real numbers x then f is a constant function.

8. Suppose f is differentiable and $\lim_{x\to\infty}\left(f(x)+f'(x)\right)=2$. Show that $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f'(x)$ both exist and find their values. (Hint: Write $f(x)=\dfrac{e^x f(x)}{e^x}$ and apply L' Hospital's Rule.)

9. Suppose f is differentiable in an open interval containing a real number a and $f^{(2)}(a)$ exists for this a. Show that

$$\lim_{h\to 0}\frac{f(a+h)-2f(a)+f(a-h)}{h^2}$$

exists and is equal to $f^{(2)}(a)$.

10. Evaluate the following limits using both L' Hospital's Rule and Taylor's Theorem. Compare the two methods.

(a) $\lim_{x\to 0}\dfrac{x-\sin x}{x^3+x^4}$

(b) $\lim_{x\to 1}\dfrac{x-1-\ln x}{x(x-1)^2}$