

The Method of Cylindrical Shells (Shell Method)

The shell method is a way of finding an exact value of the area of a solid of revolution. A solid of revolution is formed when a cross sectional strip (Figure 1) of a graph is rotated around the xy-plane. In the shell method, a strip that is parallel to the axis of rotation is taken so that when it is rotated about the xy-plane it forms a cylindrical shell (Figure 2).

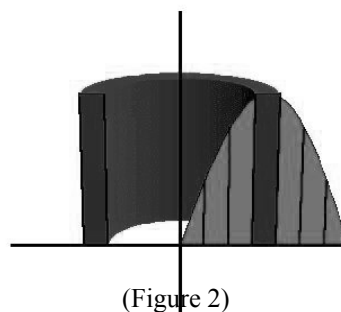
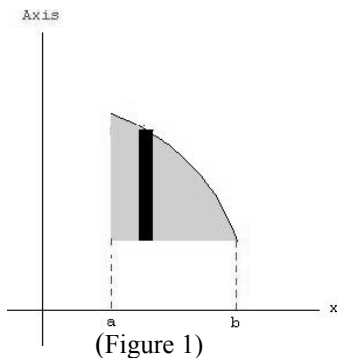


Figure 2 shows the graph of a parabola. The area bounded by the parabola and the x-axis is divided into cross sections. A cross section is then rotated around the y-axis. When setting up problems, the cross section should be parallel to the axis of rotation.

There are two general formulas for finding the volume by the shell method. One is used when the axis of rotation is horizontal, and the other is used when the axis of rotation is vertical. The formulas are shown below.

Axis of rotation is vertical $V = 2\pi \int_a^b x \cdot f(x) dx$	Axis of rotation is horizontal $V = 2\pi \int_a^b y \cdot g(y) dy$
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V is the volume,

a and **b** are the limits of integration/bounds on the graph such that $0 \leq a \leq b$,

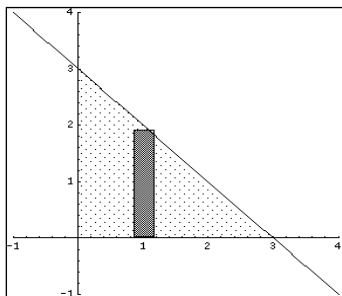
x/y is the distance of the cross section to the axis of rotation,

f(x)/g(y) is your function/equation.

The main difference between the two is which variable to integrate with respect to.

Example 1

The equation $f(x) = 3 - x$ revolved about the y-axis. $0 \leq x \leq 3$.



Note: Sketching a graph of the function helps illustrate.

The problem asks to revolve about the y-axis, the strip is parallel to the y-axis.

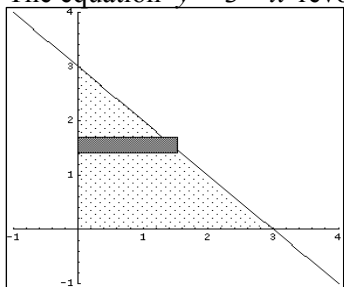
The limits of integration are given by the information; $0 \leq x \leq 3$. So **a** = 0 and **b** = 3.

The distance from the strip to the y-axis is some unknown distance, **x**. **f(x)** is the given equation.

$$V = 2\pi \int_0^3 x \cdot (3 - x) dx$$

Example 2

The equation $y = 3 - x$ revolved about the x-axis. $0 \leq x \leq 3$.



The strip is parallel to the axis of revolution. Writing the equation in terms of y , solve for x .

$$y = 3 - x$$

First x is added to both sides to get:

$$y + x = 3$$

Now y is subtracted from both sides to get:

$$x = 3 - y \text{ or}$$

$$g(y) = 3 - y$$

The bounds are obtained by plugging in the given x values and solving for y .

$$x = 3 - y$$

$0 \leq x \leq 3$ is given so 0 and 3 will be plugged into x .

$$0 = 3 - y$$

y is added to both sides to get:

$$y = 3$$

The same is done for $x = 3$.

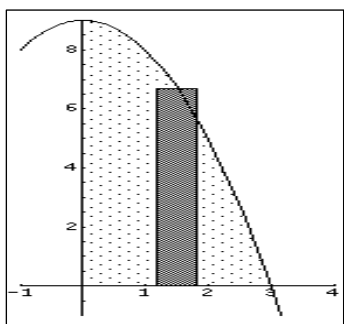
$y = 0, y = 3$ are the bounds.

These will be the limits of integration (**a** and **b**).

$$v = 2\pi \int_0^3 y \cdot (3 - y) dy$$

Example 3

The equation $y = 9 - x^2$ is revolved around the y-axis. $0 \leq x \leq 3$



Since the graph is being revolved around the y-axis the vertical equation is needed. The bounds are **a** = 0 and **b** = 3.

$$v = 2\pi \int_0^3 x \cdot (9 - x^2) dx$$

the x is distributed;

$$v = 2\pi \int_0^3 (9x - x^3) dx$$

the equation is then integrated;

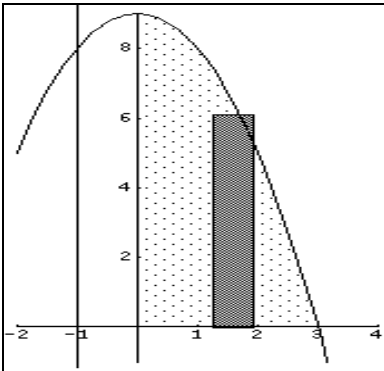
$$V = 2\pi \cdot \left(\frac{9x^2}{2} - \frac{x^4}{4} \right) \Big|_0^3$$

Plug in the limits of integration,

$$V = \frac{81\pi}{2}$$

Example 4

The equation $y = 9 - x^2$ is revolved around the line $x = -1$. $0 \leq x \leq 3$



In this situation, it no longer asks to revolve about the x or y axis, but rather a line. To set these up, it should be noted that the x in the formula is the distance from the strip to the axis of rotation. In this case, the axis of rotation is moved one more unit away from the strip. When the problem is set up, the x will turn into $x + 1$.

$$V = 2\pi \int_0^3 (x+1) \cdot (9-x^2) dx$$

Foiling the equations yields;

$$V = 2\pi \int_0^3 (-x^3 - x^2 + 9x + 9) dx$$

Integrating the equation gives;

$$V = 2\pi \left(\frac{-x^4}{4} - \frac{x^3}{3} + \frac{9x^2}{2} + 9x \right) \Big|_0^3$$

Plug in the limits,

$$V = \frac{153\pi}{2}$$