

BU Department of Mathematics

Math 101 Calculus I

Fall 2001 Second Midterm

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1. Evaluate the following integrals:

(a) $\int \sec^4 x \tan^2 x dx$

Solution:

$$I = \int \sec^2 x \tan^2 x \sec^2 x dx = \int (1 + \tan^2 x) \tan^2 x \sec^2 x dx = \int (\tan^2 x + \tan^4 x) \sec^2 x dx$$

Letting $u = \tan x$, we have $du = \sec^2 x dx$.

$$\text{Hence } I = \int (u^2 + u^4) du = \frac{1}{3}u^3 + \frac{1}{5}u^5 + c = \frac{1}{3} \tan^3 x + \frac{1}{5} \tan^5 x + c.$$

(b) $\int \frac{\arcsin(\ln x)}{x} dx$

Solution:

$$\text{Let } t = \ln x, \text{ then } dt = \frac{dx}{x}.$$

$$\Rightarrow I = \int \arcsin t dt.$$

$$\text{Now let } u = \arcsin t, dv = dt. \text{ Then } du = \frac{dt}{\sqrt{1-t^2}}, v = t.$$

$$\text{Hence by the method of integration by parts, } I = t \arcsin t - \int \frac{t}{\sqrt{1-t^2}} dt.$$

$$\text{If we let } z = 1 - t^2, \text{ we get } dz = -2t dt \text{ and so } I = \ln x \cdot \arcsin(\ln x) + \frac{1}{2} \int \frac{dz}{\sqrt{z}}.$$

$$\text{Therefore, } I = \ln x \cdot \arcsin(\ln x) + \sqrt{1 - \ln^2 x} + c.$$

2. Let R be the region in the first quadrant bounded by the x -axis, the y -axis, the curve $y = 1 + \sqrt{x}$ and the line $y = 3 - x$.

(a) Sketch the region R .

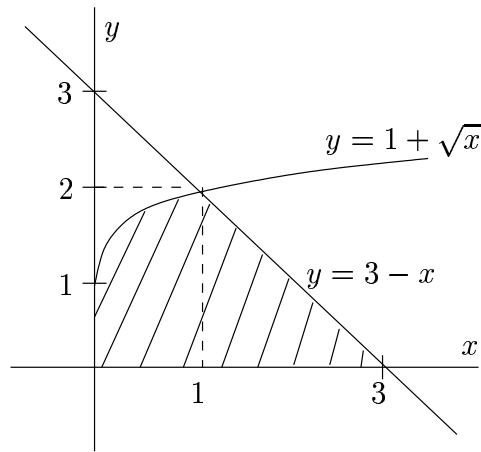
Solution:

Intersection point(s):

$$1 + \sqrt{x} = 3 - x \Rightarrow (\sqrt{x})^2 = (2 - x)^2 \Rightarrow x = 4 - 4x + x^2$$

$$\Rightarrow 0 = x^2 - 5x + 4 \Rightarrow 0 = (x - 4)(x - 1) \Rightarrow x = 1$$

$$\text{Also note that } y = 1 + \sqrt{x} \Rightarrow x = (y - 1)^2$$



(b) Using integration methods, set up (DO NOT EVALUATE) an expression for the area of the region R .

Solution:

$$A = \int_0^1 (1 + \sqrt{x}) dx + \int_1^3 (3 - x) dx$$

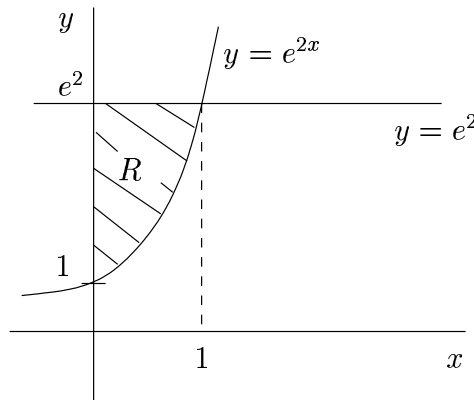
OR

$$A = \int_0^1 (3 - y) dy + \int_1^2 [(3 - y) - (y - 1)^2] dy.$$

3. Let R be the region bounded by the curve $y = e^{2x}$ and the lines $y = e^2$ and $x = 0$.

(a) Sketch the region R .

Solution:



$$y = e^{2x} \Leftrightarrow x = \frac{\ln y}{2}.$$

(b) Using the disk method, set up (DO NOT EVALUATE) an integral for the volume of the solid of revolution obtained by rotating the region R about the y -axis.

Solution:

$$V = \pi \int_1^{e^2} \frac{\ln^2 y}{4} dy.$$

(c) Using the shell method, set up (DO NOT EVALUATE) an integral for the volume of the same solid.

Solution:

$$V = 2\pi \int_0^1 x(e^2 - e^{2x}) dx.$$

(d) Evaluate ONE of the integrals above.

Solution:

$$\begin{aligned} V &= 2\pi \int_0^1 (xe^2 - xe^{2x}) dx = 2\pi \left[e^2 \cdot \frac{x^2}{2} - \int xe^{2x} dx \right] \\ \text{Let } u &= x, dv = e^{2x} dx. \text{ Then we have } du = dx, v = \frac{1}{2}e^{2x}. \\ \Rightarrow V &= 2\pi \left[\frac{e^2}{2}x^2 - \frac{1}{2}xe^{2x} + \frac{1}{2} \int e^{2x} dx \right] = 2\pi \left[\frac{e^2}{2}x^2 - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} \right]_0^1 \\ &= 2\pi \left[\frac{e^2}{2} - \frac{e^2}{2} + \frac{1}{4}e^2 - \frac{1}{4} \right] = \frac{\pi}{2}(e^2 - 1). \end{aligned}$$

4. Evaluate the following limits:

(a) $\lim_{x \rightarrow 1} \frac{1}{\ln x} \int_1^x e^{\sin \frac{\pi t}{2}} dt$

Solution:

$$\begin{aligned} &= \lim_{x \rightarrow 1} \frac{\int_1^x e^{\sin \frac{\pi t}{2}} dt}{\ln x} \left[\frac{0}{0} \right] \\ &= \lim_{x \rightarrow 1} \frac{\frac{d}{dx} \int_1^x e^{\sin \frac{\pi t}{2}} dt}{\frac{1}{x}} \text{ (by L'Hopital's Rule)} \\ &= \lim_{x \rightarrow 1} \frac{e^{\sin \frac{\pi x}{2}}}{\frac{1}{x}} = \frac{e^{\sin \frac{\pi}{2}}}{1} = e. \end{aligned}$$

(b) $\lim_{x \rightarrow 0} (e^x + x)^{\frac{1}{x}} \quad [1^\infty]$

Solution:

For $y = (e^x + x)^{\frac{1}{x}}$, $\ln y = \frac{1}{x} \ln(e^x + x)$.

So $\lim_{x \rightarrow 0} (\ln y) = \lim_{x \rightarrow 0} \frac{\ln(e^x + x)}{x} \quad \left[\frac{0}{0} \right]$

$= \lim_{x \rightarrow 0} \frac{\frac{e^x + 1}{e^x + x}}{1} = \frac{2}{1} = 2$ (by L'Hopital's Rule).

Since $\ln(\lim_{x \rightarrow 0} y) = \lim_{x \rightarrow 0} (\ln y) = 2$, we have $\lim_{x \rightarrow 0} y = \lim_{x \rightarrow 0} (e^x + x)^{\frac{1}{x}} = e^2$.

BU Department of Mathematics

Math 101 Calculus I

Fall 2002 Second Midterm

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1. Evaluate the following integrals:

(a) $\int \frac{x + \arcsin x}{\sqrt{1-x^2}} dx$, where $|x| < 1$.

Solution:

Let I denote the integral given. Then $I = \int \frac{x}{\sqrt{1-x^2}} dx + \int \frac{\arcsin x}{\sqrt{1-x^2}} dx = I_1 + I_2$.

Letting $u = 1 - x^2$ implies $du = -2x dx$:

$$I_1 = -\frac{1}{2} \int \frac{du}{\sqrt{u}} = -\sqrt{u} + c_1.$$

Letting $v = \arcsin x$ gives $dv = \frac{dx}{\sqrt{1-x^2}}$:

$$I_2 = \int v dv = \frac{v^2}{2} + c_2.$$

Writing I in the original variable:

$$I = I_1 + I_2 = -\sqrt{1-x^2} + \frac{1}{2}(\arcsin x)^2 + c.$$

(b) $\int \sqrt{x} \ln x dx$, where $x > 0$.

Solution:

We use integration by parts with the setting:

$$u = \ln x \Rightarrow du = \frac{dx}{x} \quad \text{and} \quad dv = \sqrt{x} \Rightarrow v = \frac{2}{3}x^{3/2}.$$

Then the original integral I becomes:

$$I = \frac{2}{3}x^{3/2} \ln x - \frac{2}{3} \int \frac{x^{3/2}}{x} dx = \frac{2}{3}x^{3/2} \ln x - \frac{4}{9}x^{3/2} + c.$$

(c) $\int_0^1 \frac{5x dx}{(x+2)(x^2+1)}$.

Solution:

We use partial fractions:

$$\frac{5x}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

which entails $A = -2$, $B = 2$ and $C = 1$.

$$\begin{aligned} \int_0^1 \frac{5x dx}{(x+2)(x^2+1)} &= -2 \int_0^1 \frac{dx}{x+2} + \int_0^1 \frac{2x+1}{x^2+1} dx \\ &= -2 \ln|x+2| \Big|_0^1 + \ln(x^2+1) \Big|_0^1 + \arctan x \Big|_0^1 \\ &= 3 \ln 2 - 2 \ln 3 + \frac{\pi}{4}. \end{aligned}$$

2. Let f be a twice differentiable function which never takes the value -1 and satisfies:

$$x = \int_0^{f(x)} \frac{dt}{(1+t^3)^{\frac{1}{3}}}$$

for all x . Show that $f'f'' = f^2$. (Hint: You may differentiate both sides with respect to x .)

Solution:

Differentiating both sides as suggested:

$$1 = \frac{1}{(1+f^3(x))^{\frac{1}{3}}} f'(x) \implies f'(x) = (1+f^3(x))^{\frac{1}{3}}.$$

Once more differentiating:

$$f''(x) = (1+f^3(x))^{-\frac{2}{3}} f^2(x) f'(x) = (1+f^3(x))^{-\frac{1}{3}} f^2(x).$$

Then clearly:

$$f'(x)f''(x) = f^2(x).$$

3. Find the area of the region bounded by the curves $y = \frac{x^3}{\sqrt{1+x^2}}$, $y = 0$, $x = -1$ and $x = 1$.

Solution:

Note that the function y is symmetric with respect to the origin, i.e. it is odd. Moreover $y < 0$ if $x < 0$, and $y > 0$ if $x > 0$. Hence the region bounded by the curves given is symmetric with respect to the origin. This means the area A is given by:

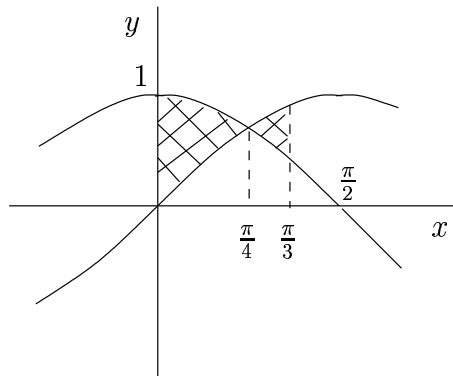
$$A = \int_{-1}^0 -\frac{x^3}{\sqrt{1+x^2}} dx + \int_0^1 \frac{x^3}{\sqrt{1+x^2}} dx = 2 \int_0^1 \frac{x^3}{\sqrt{1+x^2}} dx.$$

Letting $u = 1+x^2$ we have $du = 2x dx$. The boundaries become: $x = 0 \rightarrow u = 1$ and $x = 1 \rightarrow u = 2$. Then:

$$A = 2 \frac{1}{2} \int_1^2 \frac{u-1}{\sqrt{u}} du = \left[\frac{2}{3} u^{3/2} - 2\sqrt{u} \right]_1^2 = \frac{2}{3} (2 - \sqrt{2}).$$

4. Find the volume of the solid generated when the region enclosed by $y = \cos x$, $y = \sin x$, $x = 0$ and $x = \frac{\pi}{3}$ is revolved about the x -axis.

Solution:



The volume of the solid is given by:

$$\begin{aligned} V &= \pi \int_0^{\pi/4} (\cos^2 x - \sin^2 x) dx + \pi \int_{\pi/4}^{\pi/3} (\sin^2 x - \cos^2 x) dx \\ &= \pi \int_0^{\pi/4} \cos 2x dx + \pi \int_{\pi/4}^{\pi/3} -\cos 2x dx = \frac{\pi}{2} \sin 2x \Big|_0^{\pi/4} - \frac{\pi}{2} \sin 2x \Big|_{\pi/4}^{\pi/3} \\ &= \left[1 - \frac{\sqrt{3}}{4} \right] \pi. \end{aligned}$$

MATH 101 Second Midterm Examination

December 8, 2003

17:00-18:00

Student number:

Name:

Signature:

1	/40 points
2	/15 points
3	/15 points
4	/30 points
Total	/100 points

In order to get full credit, you have to show your work and explain what you are doing. Calculators are not allowed, nor needed. Good luck.

(1) a) Evaluate $\lim_{x \rightarrow 0} F'(x)$ if $F = \int_1^x \frac{\sin^2 t}{t} dt$. (15 points)

By the fundamental theorem of calculus, we have

$$\frac{d}{dx} F(x) = \frac{d}{dx} \int_1^x \frac{\sin^2 t}{t} dt = \frac{\sin^2 x}{x}$$

and

$$\lim_{x \rightarrow 0} F'(x) = \lim_{x \rightarrow 0} \frac{\sin^2 x}{x} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \sin x = 1 \cdot 0 = 0.$$

b) Find $f'(x)$ if $f(x) = x^{\ln x}$ ($x > 0$). (15 points)

We have

$$\ln f(x) = (\ln x)(\ln x) = (\ln x)^2.$$

Differentiating we get

$$\frac{1}{f(x)} f'(x) = 2(\ln x)^1 \cdot \frac{1}{x}$$

and so

$$f'(x) = f(x) 2 \frac{\ln x}{x} = 2x^{\ln x} \frac{\ln x}{x}.$$

c) Evaluate $\lim_{x \rightarrow 0} (1 + 2x)^{\frac{1}{3x}}$. (10 points)

Let $y = (1 + 2x)^{\frac{1}{3x}}$. Then

$$\ln y = \frac{1}{3x} \ln(1 + 2x)$$

and

$$\lim_{x \rightarrow 0} \ln y = \lim_{x \rightarrow 0} \frac{\ln(1 + 2x)}{3x} = \lim_{x \rightarrow 0} \frac{\frac{1}{1+2x} \cdot 2}{3} = \frac{2}{3}$$

and

$$\ln \lim_{x \rightarrow 0} y = \lim_{x \rightarrow 0} \ln y = \frac{2}{3},$$

therefore

$$\lim_{x \rightarrow 0} (1 + 2x)^{\frac{1}{3x}} = \lim_{x \rightarrow 0} y = e^{2/3}.$$

(2) Find the arc length of the curve $y = \frac{e^x + e^{-x}}{2}$ from $x = 0$ to $x = 1$.

(15 points)

We have

$$y = \frac{1}{2}(e^x + e^{-x})$$

$$y' = \frac{1}{2}(e^x + e^{-x}(-1)) = \frac{1}{2}(e^x - e^{-x})$$

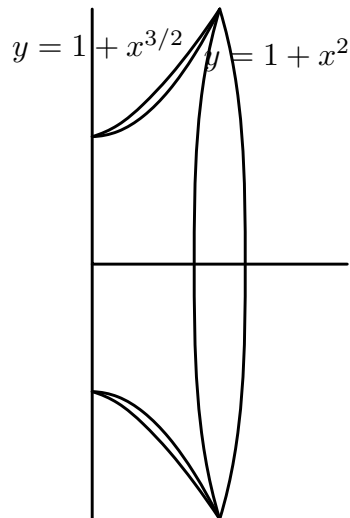
$$1 + (y')^2 = 1 + \frac{1}{4}(e^x - e^{-x})^2 = \frac{4}{4} + \frac{1}{4}(e^{2x} - 2 + e^{-2x}) = \frac{1}{4}(e^x + e^{-x})^2$$

$$\sqrt{1 + (y')^2} = \left| \frac{1}{2}(e^x + e^{-x}) \right| = \frac{1}{2}(e^x + e^{-x})$$

and therefore the required arclength is

$$\begin{aligned} \int_0^1 \sqrt{1 + (y')^2} dx &= \int_0^1 \frac{1}{2}(e^x + e^{-x}) dx \\ &= \frac{1}{2} [e^x - e^{-x}]_0^1 = \frac{1}{2}(e - e^{-1}) - \frac{1}{2}(1 - 1) \\ &= \frac{1}{2}(e - e^{-1}). \end{aligned}$$

(3) Find the volume of the solid generated when the region enclosed by the curves $y = 1 + x^{3/2}$ and $y = 1 + x^2$ is revolved about the x -axis. (15 points)



$$\begin{aligned}
 \int_0^1 \pi[(1 + x^{3/2})^2 - (1 + x^2)^2] dx &= \pi \int_0^1 [(1 + 2x^{3/2} + x^3) - (1 + 2x^2 + x^4)] dx \\
 &= \pi \left[2\frac{x^{5/2}}{5/2} + \frac{x^4}{4} - 2\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 \\
 &= \pi \left[\frac{4}{5} + \frac{1}{4} - \frac{2}{3} - \frac{1}{5} \right] \\
 &= \frac{11}{60} \pi.
 \end{aligned}$$

(4) Evaluate the following integrals:

a) $\int \frac{\sin x \, dx}{\sqrt{2 + \cos x}}$

(15 points)

Put $u = 2 + \cos x$; then $du = -\sin x \, dx$ and

$$\begin{aligned} \int \frac{\sin x \, dx}{\sqrt{2 + \cos x}} &= \int \frac{-du}{\sqrt{u}} \\ &= -\int u^{-1/2} \, du \\ &= -\frac{u^{1/2}}{1/2} + c \\ &= -2\sqrt{u} + c \\ &= -2\sqrt{2 + \cos x} + c. \end{aligned}$$

b) $\int_0^{1/4} \frac{dx}{\sqrt{1 - 4x^2}}$

(15 points)

$$\begin{aligned} \int_0^{1/4} \frac{dx}{\sqrt{1 - 4x^2}} &= \frac{1}{2} \int_0^{1/4} \frac{2 \, dx}{\sqrt{1 - (2x)^2}} \\ &= \frac{1}{2} \int_0^{1/2} \frac{du}{\sqrt{1 - u^2}} \\ &= \frac{1}{2} \left[\sin^{-1} u \right]_0^{1/2} \\ &= \frac{1}{2} \left[\sin^{-1} \frac{1}{2} - \sin^{-1} 0 \right] \\ &= \frac{1}{2} \left[\frac{\pi}{6} - 0 \right] = \frac{1}{12} \pi. \end{aligned}$$

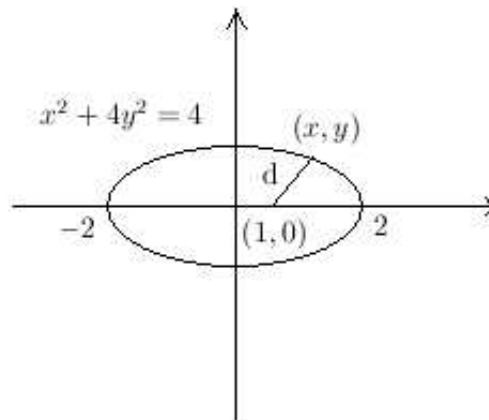
B U Department of Mathematics
Math 101 Calculus I

Fall 2004 Second Midterm

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1.) Find the point(s) on the ellipse $x^2 + 4y^2 = 4$ nearest the point $(1, 0)$. (Hint: Minimize the square of the distance.)

Solution:



$d = \sqrt{(x-1)^2 + (y-0)^2}$. Let L be the square of the distance d .

$D = d^2 = (x-1)^2 + y^2$ where $-2 \leq x \leq 2$

Now, $x^2 + 4y^2 = 4 \Rightarrow y^2 = 1 - \frac{x^2}{4}$

$D = (x-1)^2 + 1 - \frac{x^2}{4}$ and $\frac{dD}{dx} = 2(x-1) - \frac{2x}{4} = 0$

$x - 1 - \frac{x}{4} = 0 \Rightarrow x = \frac{4}{3}$

We must show that $x = \frac{4}{3}$ is a min. point.

$x = 2 \Rightarrow D = 9$, $x = \frac{4}{3} \Rightarrow D = \frac{2}{3}$, $x = -2 \Rightarrow D = 1$

Hence $x = \frac{4}{3}$ minimizes D and minimum occurs at $\left(\frac{4}{3}, \pm \frac{\sqrt{5}}{3}\right)$

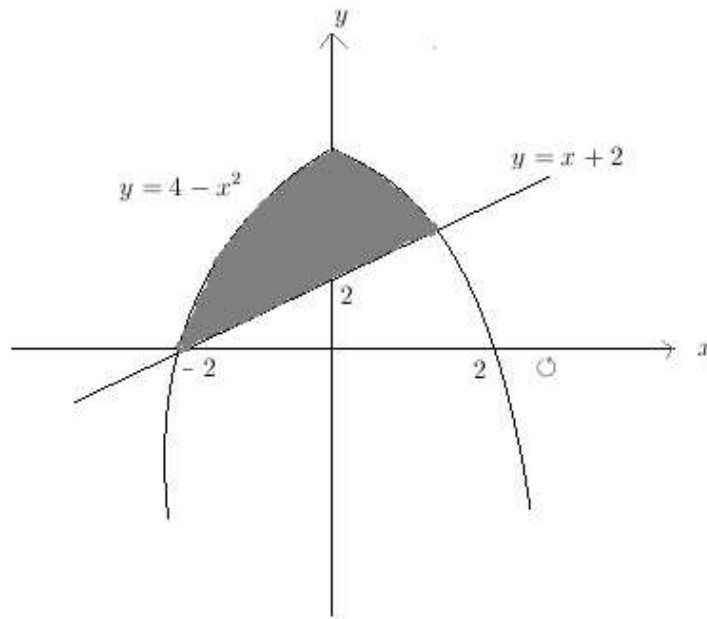
2.) Find the volume of the solid that results when the area of the region enclosed by $y = 4 - x^2$ and $y = x + 2$ is revolved about the x-axis

Solution:

Intersection points :

$$4 - x^2 = x + 2$$

$$x^2 + x - 2 = 0 \Rightarrow (x+2)(x-1) = 0 \Rightarrow x = -2, x = 1$$



$$V = \int_{-2}^1 \pi [(4 - x^2)^2 - (x + 2)^2] dx$$

$$V = \int_{-2}^1 \pi [16 - 8x^2 + x^4 - x^2 - 4x - 4] dx$$

$$V = \int_{-2}^1 \pi [12 - 4x - 9x^2 + x^4] dx = \pi \left[12x - 2x^2 - 3x^3 + \frac{x^5}{5} \right]_{-2}^1 = \frac{108\pi}{5}$$

3.) a) Given $f'(x) = x\sqrt{x^2 + 1}$, find $f(x)$ if $f(2) = 12$.

Solution:

$$f'(x) = x\sqrt{x^2 + 5}$$

$$f(x) = \int x\sqrt{x^2 + 5} dx$$

$$u = x^2 + 5 \quad du = 2xdx$$

$$\frac{du}{2} = xdx$$

$$f(x) = \frac{1}{2} \int u^{1/2} du = \frac{u^{3/2}}{3} + C = \frac{(x^2 + 5)^{3/2}}{3} + C$$

$$f(2) = 12 \Rightarrow 12 = \frac{(2^2 + 5)^{3/2}}{3}$$

$$12 = 9 + C$$

$$C = 3$$

$$f(x) = \frac{(x^2 + 5)^{3/2}}{3} + 3$$

b) Evaluate $\int \frac{dx}{x(1 - \ln x)}$

Solution:

$$u = 1 - \ln x$$

$$du = -\frac{dx}{x}$$

$$\int \frac{dx}{x(1 - \ln x)} = -\int \frac{du}{u} = -\ln|u| + C = -\ln|1 - \ln x| + C$$

4.) Sketch the curve $f(x) = \frac{x^2}{x^2 - 9}$ by examining, a) the domain, b) the x and y-intercepts, c) all asymptotes, d) symmetry, e) maximum, minimum and inflection points, f) intervals of increase and decrease, g) concavity and by making a table of your data.

Solution:

Domain: $\mathbb{R} - \{-3, +3\}$

$x = 0 \Rightarrow y = 0$ (0,0) is the x and y intercept.

Vertical asymptotes: $x = -3, x = +3$

$$\lim_{x \rightarrow -3^+} \frac{x^2}{x^2 - 9} = +\infty \quad \lim_{x \rightarrow -3^-} \frac{x^2}{x^2 - 9} = -\infty$$

$$\lim_{x \rightarrow -3^+} \frac{x^2}{x^2 - 9} = -\infty \quad \lim_{x \rightarrow -3^-} \frac{x^2}{x^2 - 9} = +\infty$$

Horizontal asymptote: $y = 1$

$$\lim_{x \rightarrow \pm\infty} \frac{x^2}{x^2 - 9} = \lim_{x \rightarrow \pm\infty} \frac{x^2}{x^2(1 - \frac{9}{x^2})} = 1.$$

$f(-x) = f(x)$. The graph is symmetric with respect to y-axis.

$$f'(x) = -\frac{18x}{(x-9)^2} \quad f''(x) = \frac{54(x^2+3)}{(x^2-9)^3} \neq 0$$

$$f'(x) = 0 \Rightarrow x = 0.$$

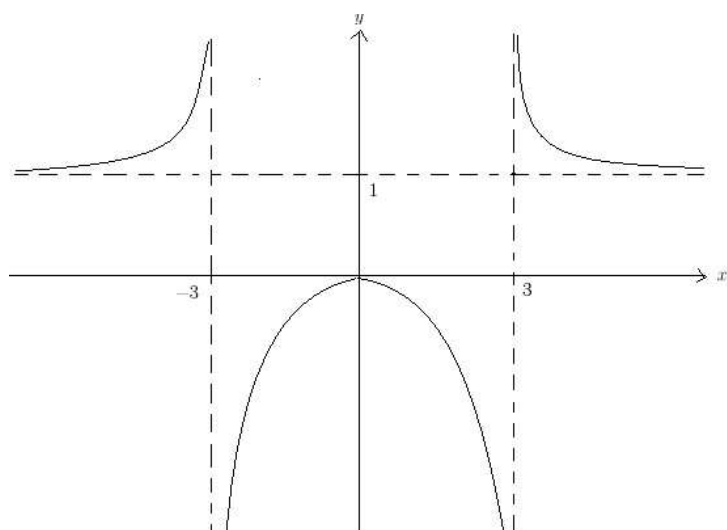
$f'(x)$ is undefined at $x = \pm 3$ (but not in the domain of $f(x)$)

Critical number: $x=0$.

	$-\infty$	-3	0	$+3$	$+\infty$
$f'(x)$	+	+	o	-	-
$f''(x)$	+	-	-	-	+
$f(x)$	$\nearrow \cup$	$\nearrow \cap$	$\searrow \cap$	$\searrow \cup$	

$f''(0) < 0$, so there is a rel.max at (0,0).

The graph of the function is shown below.



B U Department of Mathematics

Math 101 Calculus I

Fall 2005 Second Midterm

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1. (a) If f is continuous on $[2, 5]$ and

$$1 \leq f'(x) \leq 4, \forall x \in (2, 5)$$

then show that

$$3 \leq f(5) - f(2) \leq 12$$

Solution:

By mean value theorem there is some $c \in (2, 5)$ such that $f'(c) = \frac{f(5) - f(2)}{5 - 2}$

but $1 \leq f'(c) \leq 4$ so $1 \leq \frac{f(5) - f(2)}{5 - 2} \leq 4$ hence we have $3 \leq f(5) - f(2) \leq 12$

- (b) Evaluate the integral

$$\int x \sin x^2 \cos x^2 dx$$

Solution:

Let $u = \sin(x^2)$ then $du = 2x \cos(x^2)$

$$\mathbb{I} = \int x \sin(x^2) \cos(x^2) = \int \frac{u du}{2} = \frac{u^2}{4} + c = \frac{\sin^2(x^2)}{4} + c$$

2. Consider the function

$$f(x) = 2 - \frac{3}{x} - \frac{3}{x^2}$$

- (a) Determine the domain of f .
- (b) Find all horizontal, vertical and oblique asymptotes of f .
- (c) Find the intervals on which f is increasing or decreasing.
- (d) Find all local extrema of f , if any.
- (e) Find the intervals on which the graph of f is concave up or down.
- (f) Find the points of inflection, if any.
- (g) Sketch the graph of f .

Solution:

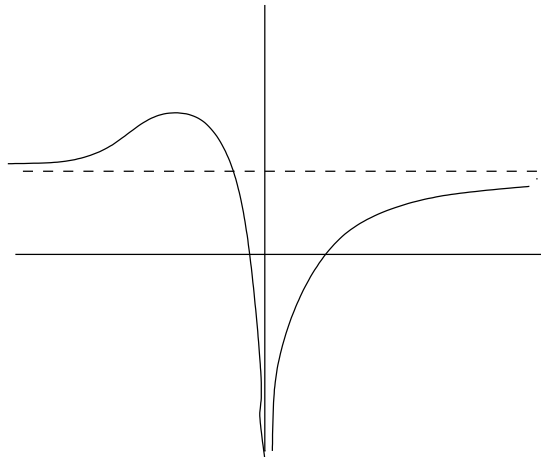
- (a) Domain of f is the real line except the point zero i.e. $\mathbb{R} \setminus \{0\}$.
- (b) $f(x) = \frac{2x^2 - 3x - 3}{x^2}$ then we have $\lim_{x \rightarrow \pm\infty} 2 - \frac{3}{x} - \frac{3}{x^2}$ and also $\lim_{x \rightarrow 0^+} f(x) = -\infty = \lim_{x \rightarrow 0^-}$ so $y=2$ is a horizontal asymptote and $x=0$ is a vertical asymptote and no oblique asymptotes.

(c)

x	$-\infty$	-3	-2	0	$+\infty$
f'	+	+	-	+	+
f''	+	-	-	-	-
f	inc.	inc.	dec.	inc.	inc.
f	∪	∩	∩	∩	∩

Where $f'(x) = \frac{3}{x^2} + \frac{6}{x^3}$, $f'(x) = 0 \Rightarrow x = -2$ and $f''(x) = -\frac{6}{x^3} + -\frac{18}{x^4}$,
 $f''(x) = 0 \Rightarrow x = -3$

- (d) $f(-3) = \frac{8}{3}$ and $f(-2) = \frac{11}{4}$
- (e) Done in the above table!
- (f) Done in the above table!
- (g)



3. Evaluate the following limits

$$a) \lim_{x \rightarrow 0} \frac{1}{\sin x} \int_0^{\sin 2x} \cos 5t dt$$

Solution:

$$\frac{\int_0^{\sin(2x)} \cos(5t) dt}{\sin(x)} = \frac{0}{0} \quad \text{By applying L'Hospital's rule we get;}$$

$$\lim_{x \rightarrow 0} \frac{\cos(5(\sin(2x)))2 \cos(2x)}{\cos(x)} = 2$$

$$b) \lim_{n \rightarrow \infty} \left(\frac{1}{1+n} + \frac{1}{2+n} + \frac{1}{3+n} + \dots + \frac{1}{n+n} \right)$$

Solution:

$$\frac{1}{1+n} + \frac{1}{2+n} + \frac{1}{3+n} + \dots + \frac{1}{n+n} = \frac{\frac{1}{n}}{\frac{1}{n}+1} + \frac{\frac{1}{n}}{\frac{2}{n}+1} + \dots + \frac{\frac{1}{n}}{\frac{n}{n}+1} = \sum_{k=1}^n \frac{1}{n} \frac{1}{\frac{k}{n}+1}$$

this is a Riemann sum corresponding to the function $\frac{1}{1+x}$ in $[0,1]$

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} \frac{1}{\frac{k}{n}+1} = \int_0^1 \frac{dx}{1+x} = \ln|1+x| \Big|_0^1 = \ln 2 - \ln 1 = \ln 2$$

4. Let R be the region bounded by the curves $y = -x$ and $y^2 = 2 - x$ and the x -axis.

(a) Sketch the region R and find its area.

(b) Let S_1 be the solid generated by rotating the region R about the line $x = -2$. Set up an intergral to calculate the volume of S_1 . (Do not evaluate the integral).

(c) Let S_2 be the solid generated by rotating the region R about the x -axis. Set up an intergral to calculate the volume of S_2 . (Do not evaluate the integral).

Solution:

$$\begin{aligned} 2 - x &= x^2 \\ (a) \quad 0 &= (x - 1)(x + 2) \\ x &= -2 \quad x = 1 \end{aligned}$$

$$A = \int_{-2}^0 \sqrt{2-x} - (-x) dx + \int_0^2 \sqrt{2-x} dx \text{ or}$$

$$A = \int_0^2 2 - y^2 - (-y) dy = 2y - \frac{y^3}{3} + \frac{y^2}{2} \Big|_0^2 = \frac{10}{3}$$

$$x = 2 - y^2 \quad x = -y$$

$$(b) V_1 = \pi \int_0^2 [((2 - y^2) + 2)^2 - (-y + 2)^2] dy$$

$$(c) V_2 = \pi \left[\int_{-2}^0 (2 - x) - (x^2) dx + \int_0^2 (2 - x) dx \right]$$

BU Department of Mathematics

Math 101 Calculus I

Fall 1999 Second Midterm

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1.) Find the area of the region bounded by the graph of $y = x \ln x$ and the x -axis between $x = 1$ and $x = e$.

Solution:

Since the function $y = x \ln x$ is positive on the interval $[1, e]$, the area of this region, A , is equal to the integral: $A = \int_1^e x \ln x dx$.

(Integration by parts) Put $u = \ln x$, $dv = x$. We get

$$A = \frac{x^2 \ln x}{2} - \int_1^e \frac{x}{2} dx = \frac{1}{4}(e^2 + 1)$$

2.) (a) Given $F(x) = \int_{x^2}^x \cos^3(t+1)dt$, find $\frac{d}{dx} F(x)$.

Solution:

$$\frac{d}{dx} F(x) = \cos^3(x+1) - 2x \cos^3(x^2+1)$$

(b) Find $\lim_{x \rightarrow +\infty} (1 + e^{-x})e^x$.

Solution:

$$\text{Substitute } u = e^x: \lim_{x \rightarrow +\infty} (1 + e^{-x})e^x = \lim_{u \rightarrow +\infty} (1 + \frac{1}{u})^u = e.$$

3.) Integrate $\int \sqrt{\frac{\sin x}{\cos^5 x}} dx$.

Solution:

$$\int \sqrt{\frac{\sin x}{\cos^5 x}} dx = \int \frac{1}{\cos^2 x} \sqrt{\frac{\sin x}{\cos x}} dx = \int \sec^2 x \sqrt{\tan x} dx$$

Substitute $u = \tan x$, then we get

$$= \int \sqrt{u} du = \frac{2}{3} u^{\frac{3}{2}} + c = \frac{2}{3} \tan^{\frac{3}{2}} x + c$$

4.) (a) Show that $\int_0^a f(x) dx = \int_0^a f(a-x) dx$.

Solution:

$$\text{Substitute } u = a - x: \int_0^a f(x) dx = \int_a^0 -f(a-u) du = \int_0^a f(a-u) du.$$

(b) Deduce that $\int_0^\pi \frac{x \sin^{100} x}{1 + \cos^2 x} dx = \frac{\pi}{2} \int_0^\pi \frac{\sin^{100} x}{1 + \cos^2 x} dx$.

Solution:

$$\begin{aligned} \text{By part (a), } \int_0^\pi \frac{x \sin^{100} x}{1 + \cos^2 x} dx &= \int_0^\pi \frac{(\pi - x) \sin^{100}(\pi - x)}{1 + \cos^2(\pi - x)} dx \\ &= \pi \int_0^\pi \frac{\sin^{100} x}{1 + \cos^2 x} dx - \int_0^\pi \frac{x \sin^{100} x}{1 + \cos^2 x} dx. \end{aligned}$$

$$\text{Hence } 2 \int_0^\pi \frac{x \sin^{100} x}{1 + \cos^2 x} dx = \pi \int_0^\pi \frac{\sin^{100} x}{1 + \cos^2 x} dx$$

5.) Determine whether the series $\sum_{n=2}^{+\infty} \frac{1}{n(\ln n)^3}$ is convergent.

Solution:

By Integral Test, consider the integral $\int_1^{+\infty} \frac{1}{x(\ln x)^3} dx$.

By substitution $u = \ln x$, the improper integral $\int_1^{+\infty} \frac{1}{u^3} du$ is finite, hence the series is convergent.

BU Department of Mathematics

Math 101 Calculus I

Spring 2000 Second Midterm

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1. Find $\int x(\ln x)^2 dx$.

Solution:

Using integration by parts: $(\ln x)^2 = u$ and $x dx = dv$

Then $2\frac{1}{x} \ln x dx = du$ and $\frac{x^2}{2} = v$

$$\int x(\ln x)^2 dx = \frac{x^2}{2}(\ln x)^2 - \int \frac{x^2}{2} 2\frac{1}{x} \ln x dx = \frac{x^2}{2}(\ln x)^2 - \int x \ln x dx.$$

Using integration by parts once again: $\ln x = u$ and $x dx = dv$ we get:

$$\int x(\ln x)^2 dx = \frac{x^2}{2}(\ln x)^2 - \left(\frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{1}{x} dx \right).$$

Now combining everything, we obtain the final result to be:

$$\int x(\ln x)^2 dx = \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2} \ln x + \frac{x^2}{4} + C.$$

2. Verify that the average value of a linear function $f(x) = ax + b$ over the interval $[x_1, x_2]$ is equal to the value of the function at the midpoint of the interval.

Solution:

$$\begin{aligned} \text{Average value} &= \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} (ax + b) dx = \frac{1}{x_2 - x_1} \left(\frac{ax^2}{2} + bx \right) \Big|_{x_1}^{x_2} \\ &= \frac{1}{x_2 - x_1} \left(\frac{ax_2^2}{2} + bx_2 - \frac{ax_1^2}{2} - bx_1 \right) \\ &= \frac{1}{x_2 - x_1} \left(\frac{a}{2}(x_2^2 - x_1^2) + b(x_2 - x_1) \right) \\ &= \frac{a}{2}(x_2 + x_1) + b = f\left(\frac{x_1 + x_2}{2}\right) \end{aligned}$$

where $\frac{x_1 + x_2}{2}$ is indeed the midpoint of interval $[x_1, x_2]$.

3. Find arc-length of $y = \frac{\sqrt{x}(3-x)}{3}$ for $0 \leq x \leq 3$.

Solution:

Recall that arc-length of $y = f(x)$ from $x = a$ to $x = b$ is given by the integral:

$$\int_a^b \sqrt{1 + (y')^2} dx.$$

Let us now compute the necessary quantities for the given curve:

$$y' = \frac{1}{3} \frac{1}{2\sqrt{x}}(3-x) + \frac{1}{3}\sqrt{x}(-1) = \frac{1}{2\sqrt{x}} - \frac{\sqrt{x}}{6} - \frac{\sqrt{x}}{3} = \frac{1}{2\sqrt{x}} - \frac{\sqrt{x}}{2},$$

$$(y')^2 = \frac{1}{4x} - 2 \frac{1}{2\sqrt{x}} \frac{\sqrt{x}}{2} + \frac{x}{4} = \frac{1}{4x} - \frac{1}{2} + \frac{x}{4},$$

$$1 + (y')^2 = \frac{1}{4x} + \frac{1}{2} + \frac{x}{4} = \left(\frac{1}{2\sqrt{x}} + \frac{\sqrt{x}}{2} \right)^2 \text{ then } \sqrt{1 + (y')^2} = \frac{1}{2\sqrt{x}} + \frac{\sqrt{x}}{2}.$$

Now we are ready to compute the required arc-length L :

$$L = \int_0^3 \sqrt{1 + (y')^2} dx = \int_0^3 \left(\frac{1}{2\sqrt{x}} + \frac{\sqrt{x}}{2} \right) dx = \left. \frac{x^{\frac{1}{2}}}{2} + \frac{x^{\frac{3}{2}}}{\frac{2}{3}} \right|_0^3 = 2\sqrt{3}.$$

4. (a) Show that $(\arctan x)' = \frac{1}{1+x^2}$.

Solution:

$$(\tan(\arctan x))' = x' \text{ (taking the derivative of both sides)}$$

$$[1 + (\tan(\arctan x))^2](\arctan x)' = 1,$$

$$\text{Thus } (\arctan x)' = \frac{1}{1+x^2}.$$

(b) Find $((\tan x)^{\arctan x})'$.

Solution:

We will use logarithmic differentiation:

$$y = (\tan x)^{\arctan x} \Rightarrow \ln y = \arctan x \ln(\tan x).$$

Now take derivatives of both sides:

$$\frac{y'}{y} = \frac{1}{1+x^2} \ln(\tan x) + \arctan x \frac{1}{\tan x} (1 + (\tan x)^2),$$

$$y' = (\tan x)^{\arctan x} \left(\frac{1}{1+x^2} \ln(\tan x) + \arctan x \frac{1}{\tan x} (1 + (\tan x)^2) \right).$$

(c) Find $\lim_{x \rightarrow \infty} \left(\cos \frac{1}{x} \right)^x$.

Solution:

Instead of the function itself, it advantageous to consider its logarithm, we then convert back via exponentiation.

$$y = \left(\cos \frac{1}{x} \right)^x \Rightarrow \ln y = x \ln \cos \frac{1}{x}$$

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} x \ln \cos \frac{1}{x} = \lim_{h \rightarrow 0^+} \frac{\ln \cos h}{h} \text{ (where } h = 1/x)$$

$$\Rightarrow \lim_{h \rightarrow 0^+} \frac{-\frac{\sin h}{\cos h}}{1} = 0 \Rightarrow \lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} e^{\ln y} = e^0 = 1.$$

BU Department of Mathematics

Math 101 Calculus I

Spring 2001 Second Midterm

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1. Find the integral $\int_1^3 \sqrt{x} \tan^{-1} \sqrt{x} dx$.

Solution:

We use integration by parts:

Let $u = \tan^{-1} \sqrt{x}$ and $dv = \sqrt{x}$.

Then $\frac{du}{dx} = \frac{1}{1+x} \frac{1}{2\sqrt{x}}$ and $v = \frac{x^{3/2}}{3/2}$.

$$\begin{aligned} \int_1^3 \sqrt{x} \tan^{-1} \sqrt{x} dx &= \tan^{-1} \sqrt{x} \frac{x^{3/2}}{3/2} \Big|_1^3 - \frac{2}{3} \int_1^3 x^{3/2} \frac{1}{1+x} \frac{1}{2\sqrt{x}} dx \\ &= \tan^{-1} \sqrt{x} \frac{2}{3} x^{3/2} \Big|_1^3 - \frac{1}{3} \int_1^3 \left(1 - \frac{1}{1+x}\right) dx \\ &= \left[\tan^{-1} \sqrt{x} \frac{2}{3} x^{3/2} - \frac{1}{3} (x - \ln |1+x|) \right]_1^3 \\ &= 2\sqrt{3} \tan^{-1} \sqrt{3} - \frac{2}{3} \tan^{-1} 1 - \frac{1}{3} (3 - \ln |4| - 1 + \ln |2|) \\ &= 2\sqrt{3} \frac{\pi}{3} - \frac{\pi}{6} - \frac{2}{3} + \frac{\ln 2}{3}. \end{aligned}$$

2. Find the integral $\int \frac{dx}{(9x^2 - 1)^{3/2}}$.

Solution:

Using trigonometric substitution: $x = \frac{1}{3} \sec \theta \Rightarrow \frac{dx}{d\theta} = \frac{1}{3} \sec \theta \tan \theta$. Hence:

$$\begin{aligned} I &= \int \frac{dx}{(9x^2 - 1)^{3/2}} = \int \frac{dx}{27(x^2 - \frac{1}{9})^{3/2}} = \int \frac{\frac{1}{3} \sec \theta \tan \theta d\theta}{\tan^3 \theta} \\ &= \frac{1}{3} \int \frac{\sec \theta}{\tan^2 \theta} d\theta = \frac{1}{3} \int \frac{\cos \theta}{\sin^2 \theta} d\theta. \end{aligned}$$

Now let $u = \sin \theta$, then $\frac{du}{d\theta} = \cos \theta$. Hence, the integral becomes:

$$I = \frac{1}{3} \int \frac{du}{u^2} = -\frac{1}{3u} + C = -\frac{1}{3 \sin \theta} + C = -\frac{x}{3\sqrt{x^2 - \frac{1}{9}}} + C.$$

3. Find $f(x)$, given that $f(1) = 0$ and that $y = f(x)$ satisfies the differential equation $\frac{dy}{dx} = xe^{x-y}$.

Solution:

We can separate the variables:

$$\frac{dy}{dx} = x \frac{e^x}{e^y} \Rightarrow \int e^y dy = \int xe^x dx = xe^x - \int e^x dx$$

integrating the right hand side by parts letting $u = x$ and $dv = e^x dx$. Finishing the integration we get:

$$e^y = xe^x - e^x + C \Rightarrow y = \ln(xe^x - e^x + C)$$

where C is an integration constant and to be determined by the given condition:

$f(1) = \ln(e - e + C) = 0 \Rightarrow C = 1$ so that the unique solution becomes:

$$y = \ln(xe^x - e^x + 1).$$

4. Evaluate $\frac{d}{dx} \int_{3x}^{x^2} \frac{t-1}{t+1} dt$.

Solution:

$$\begin{aligned} \frac{d}{dx} \int_{3x}^{x^2} \frac{t-1}{t+1} dt &= \frac{d}{dx} \left[\int_{3x}^a \frac{t-1}{t+1} dt + \int_a^{x^2} \frac{t-1}{t+1} dt \right] \\ &= \frac{d}{dx} \left[- \int_a^{3x} \frac{t-1}{t+1} dt + \int_a^{x^2} \frac{t-1}{t+1} dt \right] = -3 \left(\frac{3x-1}{3x+1} \right) + \left(\frac{x^2-1}{x^2+1} \right) 2x, \end{aligned}$$

by using the fundamental theorem of calculus.

5. Evaluate the following limits, if they exist:

(a) $\lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2}{1 - \cos 2x}$.

(b) $\lim_{x \rightarrow \infty} \left(\cos \frac{1}{x} \right)^x$.

Solution:

(a) This is indeterminate of the form $\frac{0}{0}$. Applying L'Hopital's Rule we get:

$$\lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2}{1 - \cos 2x} = \lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{2 \sin 2x} = \lim_{x \rightarrow 0} \frac{e^x + e^{-x}}{4 \cos 2x} = \frac{1}{2}.$$

(b) This is of the form 1^∞ , so it is good to consider the logarithm of the given function:

$$L = \lim_{x \rightarrow \infty} \ln \left(\cos \frac{1}{x} \right)^x = \lim_{x \rightarrow \infty} x \ln \left(\cos \frac{1}{x} \right) = \lim_{x \rightarrow \infty} \frac{\ln(\cos 1/x)}{1/x}.$$

Apply L'Hopital's rule since the limit has the indeterminate form $\frac{0}{0}$:

$$L = \lim_{x \rightarrow \infty} \frac{\frac{1}{\cos 1/x} (-\sin 1/x) (-1/x^2)}{-1/x^2} = \lim_{x \rightarrow \infty} \frac{-\sin 1/x}{\cos 1/x} = 0.$$

Hence the desired limit is:

$$\lim_{x \rightarrow \infty} \left(\cos \frac{1}{x} \right)^x = e^0 = 1.$$

BU Department of Mathematics

Math 101 Calculus I

Spring 2002 Second Midterm

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1. Given $f(1) = 3$, $f(4) = 7$, $f(14) = 23$, evaluate $\int_1^2 (x^2 + 1)f'(x^3 + 3x)dx$.

Solution:

Let $u = x^3 + 3x$ then $du = 3(x^2 + 1)dx$. So the integral becomes:

$$\int_1^2 (x^2 + 1)f'(x^3 + 3x)dx = \frac{1}{3} \int_4^{14} f'(u)du = \frac{1}{3}[f(14) - f(4)] = \frac{1}{3}[23 - 7] = \frac{16}{3}.$$

2. Evaluate the following limits.

(a) $\lim_{x \rightarrow 0} (e^x + x)^{\frac{1}{x}}$.

Solution:

$\lim_{x \rightarrow 0} (e^x + x)^{\frac{1}{x}} = [1^\infty]$ exponential type indeterminacy. Instead of $y = (e^x + x)^{\frac{1}{x}}$ let us consider its logarithm $\ln y = \frac{\ln(e^x + x)}{x}$:

$$\lim_{x \rightarrow 0} (\ln y) = \lim_{x \rightarrow 0} \frac{\ln(e^x + x)}{x} = \left[\frac{0}{0} \right] = \lim_{x \rightarrow 0} \frac{\frac{e^x + 1}{e^x + x}}{1} = \frac{2}{1} = 2.$$

Now exponentiate this to find the desired limit:

$$\lim_{x \rightarrow 0} y = \lim_{x \rightarrow 0} (e^x + x)^{\frac{1}{x}} = e^2.$$

(b) $\lim_{x \rightarrow 0^+} \frac{1}{x} \int_0^x e^{t^2} dt$.

Solution:

We use the fundamental theorem of calculus when applying L'Hopital's rule:

$$\lim_{x \rightarrow 0^+} \frac{1}{x} \int_0^x e^{t^2} dt = \lim_{x \rightarrow 0^+} \frac{\int_0^x e^{t^2} dt}{x} = \left[\frac{0}{0} \right] = \lim_{x \rightarrow 0^+} \frac{e^{x^2}}{1} = 1.$$

(c) $\lim_{n \rightarrow \infty} \sum_{k=1}^n \ln \sqrt[n]{2 + \frac{k}{n}}$ (Hint: Recognize it as a Riemann sum).

Solution:

Observe $\lim_{n \rightarrow \infty} \sum_{k=1}^n \ln \sqrt[n]{2 + \frac{k}{n}} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} \ln \left(2 + \frac{k}{n} \right)$. This is the Riemann sum for $f(x) = \ln x$ over $[2, 3]$. Then

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \ln \sqrt[n]{2 + \frac{k}{n}} = \int_2^3 \ln x dx.$$

We use integration by parts: let $\ln x = u$ and $dx = dv$ then $du = \frac{1}{x} dx$ and $v = x$. Then:

$$\int_2^3 \ln x dx = x \ln x \Big|_2^3 - \int_2^3 dx = 3 \ln 3 - 2 \ln 2 - (3 - 2) = \ln \frac{27}{4} - 1.$$

3. Find the integrals:

(a) $\int \frac{e^{\frac{x}{2}}}{1 + e^{\frac{x}{3}}} dx$.

Solution:

$$\text{Let } u = e^{\frac{x}{6}} \Rightarrow \begin{cases} u^6 = e^x \\ u^3 = e^{\frac{x}{2}} \\ u^2 = e^{\frac{x}{3}} \end{cases}.$$

Computing the differential: $6u^5 = e^x \frac{du}{dx} \implies 6u^5 du = u^6 dx$ which means $\frac{6}{u} du = dx$.
Plugging into the integral:

$$\begin{aligned} \int \frac{e^{\frac{x}{2}}}{1 + e^{\frac{x}{3}}} dx &= 6 \int \frac{u^2}{u^2 + 1} du = 6 \int \left(1 - \frac{1}{u^2 + 1} \right) \\ &= 6[u - \arctan u] + c = 6[e^{\frac{x}{6}} - \arctan e^{\frac{x}{6}}] + c. \end{aligned}$$

(b) $\int x \arcsin x dx$.

Solution:

By integration by parts: $\arcsin x = u$ and $x dx = dv \implies du = \frac{dx}{\sqrt{1-x^2}}$ and $v = \frac{x^2}{2}$.
Rewriting the integral:

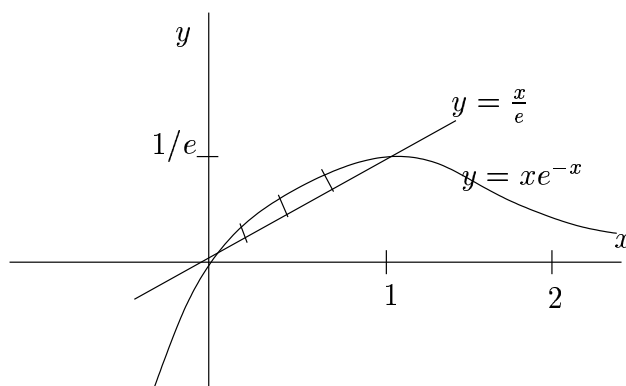
$$I = \int x \arcsin x dx = \frac{1}{2} x^2 \arcsin x - \int \frac{x^2}{\sqrt{1-x^2}} dx.$$

Setting $x = \sin \theta$ we get:

$$\begin{aligned}
 I &= \frac{1}{2}x^2 \arcsin x - \frac{1}{2} \int \frac{\sin^2 \theta}{\cos \theta} \cos \theta d\theta \\
 &= \frac{1}{2}x^2 \arcsin x - \frac{1}{4} \int (1 - \cos 2\theta) d\theta \\
 &= \frac{1}{2}x^2 \arcsin x - \frac{1}{4} \left[\theta - \frac{1}{2} \sin 2\theta \right] + c \\
 &= \frac{1}{2}x^2 \arcsin x - \frac{1}{4} [\arcsin x - x\sqrt{1-x^2}] + c.
 \end{aligned}$$

4. Find the area bounded by the graphs of $y = xe^{-x}$ and $y = \frac{x}{e}$.

Solution:



We make a picture in order to see the situation. The second one is line through the origin. For $y = xe^{-x}$, we look at derivatives and asymptotes: $y' = e^{-x} - xe^{-x} = e^{-x}(1-x) \implies y'' = e^{-x}(x-2)$. Moreover $\lim_{x \rightarrow -\infty} y = -\infty$, and $\lim_{x \rightarrow \infty} xe^{-x} = \lim_{x \rightarrow \infty} \frac{x}{e^x} = 0$.

We now check the intersection points of these two curves:

$$\frac{x}{e} = xe^{-x} \implies x(1 - e^{1-x}) = 0 \implies x = 0, x = 1.$$

Then the area A bounded by both curves reads:

$$\begin{aligned}
 A &= \int_0^1 \left(xe^{-x} - \frac{x}{e} \right) dx = \int_0^1 xe^{-x} dx - \frac{x^2}{2e} \Big|_0^1 \\
 &= -xe^{-x} \Big|_0^1 + \int_0^1 e^{-x} dx - \frac{x^2}{2e} \Big|_0^1 \\
 &= -e^{-x}(x+1) \Big|_0^1 - \frac{x^2}{2e} \Big|_0^1 = -\frac{2}{e} + 1 - \frac{1}{2e} = \frac{2e-5}{2e}.
 \end{aligned}$$

5. The region between the graphs of $y = x^2$ and $y = \frac{1}{2} - x^2$ is revolved about the y -axis to form a lens. Calculate the volume of the lens.

Solution:

We first make a picture to see this area and find the intersection points that we need:

$$x^2 = \frac{1}{2} - x^2 \implies x^2 = \frac{1}{4} \implies x = \pm \frac{1}{2}.$$

We now apply the shell method:

$$\begin{aligned} V &= 2\pi \int_0^{\frac{1}{2}} x \left(\frac{1}{2} - x^2 - x^2 \right) dx = 2\pi \int_0^{\frac{1}{2}} \left(\frac{x}{2} - 2x^3 \right) dx \\ &= 2\pi \left[\frac{x^2}{4} - 2\frac{x^4}{4} \right] \Big|_0^{\frac{1}{2}} = 2\pi \left[\frac{1}{16} - \frac{1}{32} \right] = \frac{\pi}{16}. \end{aligned}$$

BU Department of Mathematics

Math 101 Calculus I

Spring 2004 Second Midterm

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1. Evaluate the following limits, if they exist (justify your answer).

a) $\lim_{x \rightarrow \infty} x \tan^{-1}\left(\frac{2}{x}\right)$.

Solution:

$$\lim_{x \rightarrow \infty} x \tan^{-1}\left(\frac{2}{x}\right) = [\infty \cdot 0]$$

$$= \lim_{x \rightarrow \infty} \frac{\tan^{-1}\left(\frac{2}{x}\right)}{\frac{1}{x}} = \left[\frac{0}{0}\right]$$

$$\text{By L'Hospital's} \quad = \lim_{x \rightarrow \infty} \frac{1}{1 + \left(\frac{4}{x^2}\right)} \cdot \left(-\frac{2}{x^2}\right) \cdot \frac{1}{\left(-\frac{1}{x^2}\right)} = \lim_{x \rightarrow \infty} \frac{2}{1 + \frac{4}{x^2}} = 2$$

b) $\lim_{x \rightarrow \infty} (x + e^x)^{\frac{2}{x}}$

Solution:

$$\lim_{x \rightarrow \infty} (x + e^x)^{\frac{2}{x}} = [\infty^0]$$

$$\text{For } y = (x + e^x)^{\frac{2}{x}} \Rightarrow \ln y = \frac{2 \ln(x + e^x)}{x} \rightarrow \left[\frac{\infty}{\infty}\right] \quad (x \rightarrow \infty)$$

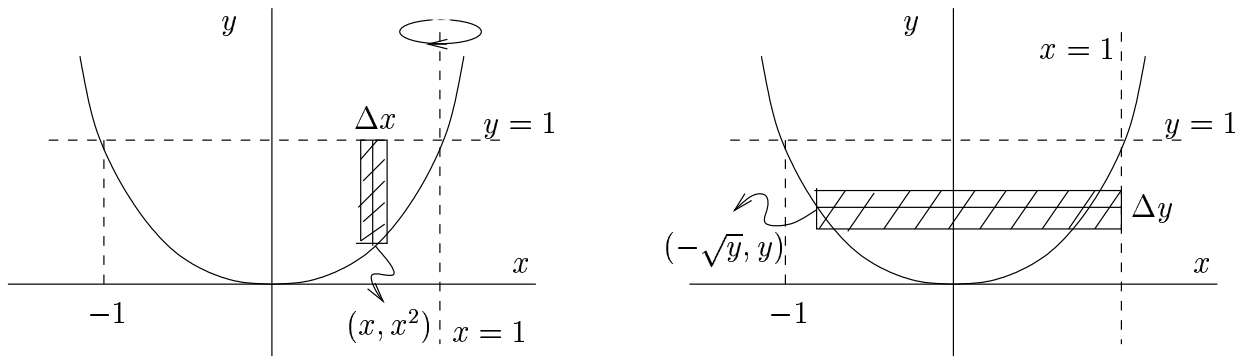
$$\text{By L'H} \quad \lim_{x \rightarrow \infty} (\ln y) = \lim_{x \rightarrow \infty} \frac{2 \left(\frac{1 + e^x}{x + e^x}\right)}{1} = \left[\frac{\infty}{\infty}\right]$$

$$\text{By L'H} \quad \lim_{x \rightarrow \infty} 2 \cdot \frac{e^x}{1 + e^x} = 2$$

$$\text{So } \lim_{x \rightarrow \infty} (x + e^x)^{\frac{2}{x}} = e^2$$

2. Find the volume of the solid generated by revolving the region bounded by the parabola $y = x^2$ and the line $y = 1$ about the line $x = 1$.

Solution:



By Shell Method: $V = 2\pi \int_{-1}^1 (1-x)(1-x^2)dx = \frac{8\pi}{3}$

By Disc Method: $V = \pi \int_0^1 [(1+\sqrt{y})^2 - (1-\sqrt{y})^2] dy = \frac{8\pi}{3}$

3. Find the length of the curve $y = \ln(\cos x)$ $0 \leq x \leq \frac{\pi}{3}$.

Solution:

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2 x = \sec^2 x$$

$$l = \int_0^{\frac{\pi}{3}} \sqrt{\sec^2 x} dx = \ln |\sec x + \tan x| \Big|_0^{\frac{\pi}{3}} = \ln(2 + \sqrt{3}).$$

4. Evaluate the integrals;

a) $\int \cos(\sqrt{x}) dx$

Solution:

Put $t = \sqrt{x}$. We get $t^2 = x$, $2tdt = dx$ and

$$I = \int \cos(\sqrt{x}) dx = 2 \int t \cdot \cos t \cdot dt.$$

Now let $t = u$ and $\cos t \cdot dt = dv$ so that $du = dt$ and $v = \sin t$. Then,

$$\begin{aligned} I &= 2 \left[t \sin t - \int \sin t dt \right] \\ &= 2(t \sin t + \cos t) + C \\ &= 2(\sqrt{x} \sin(\sqrt{x}) + \cos(\sqrt{x})) + C. \end{aligned}$$

b) $\int_0^2 \frac{dx}{|1-x^2|}$

Solution:

$$\int_0^2 \frac{dx}{|1-x^2|} = \int_0^1 \frac{dx}{1-x^2} - \int_1^2 \frac{dx}{1-x^2}$$

$$f(x) = \frac{1}{1-x^2} = \frac{1/2}{1-x} + \frac{1/2}{1+x}$$

$$\text{and } \int \frac{dx}{1-x^2} = -\frac{1}{2} \ln |1-x| + \frac{1}{2} \ln |1+x| + C = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right|.$$

$$\int_0^1 \frac{dx}{1-x^2} = \lim_{a \rightarrow 1^-} \int_0^a \frac{dx}{1-x^2} \quad (\text{improper integral})$$

$$= \lim_{a \rightarrow 1^-} \frac{1}{2} \ln \left| \frac{1+a}{1-a} \right| = -\infty.$$

$$- \int_1^2 \frac{dx}{1-x^2} = \lim_{b \rightarrow 1^+} \int_b^2 \frac{-dx}{1-x^2} = \lim_{b \rightarrow 1^+} \left[\frac{1}{2} \ln \left| \frac{1+b}{1-b} \right| - \frac{1}{2} \ln 3 \right] = -\infty.$$

So the improper integrals both diverge. Hence the given improper integral diverges.

B U Department of Mathematics
Math 101 Calculus I

Spring 2005 Second Midterm

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1. Evaluate the integrals below:

(a)[10] $\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

Solution:

$$\text{Let } I = \int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$$

$$\ln x = u, \frac{dx}{x} = du$$

$$x = e \Rightarrow u = 1, x = e^4 \Rightarrow u = 4$$

$$\Rightarrow I = \int_1^4 \frac{du}{\sqrt{u}} = 2\sqrt{u} \Big|_1^4 = 2.$$

(b)[10] $\int \frac{\sin 2\theta}{\sin^2 \theta + 1} d\theta$

Solution:

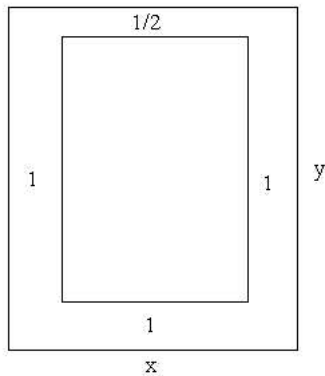
$$\sin^2 \theta + 1 = u \Rightarrow 2 \sin \theta \cos \theta d\theta = \sin 2\theta d\theta = du$$

$$\Rightarrow \int \frac{\sin 2\theta}{\sin^2 \theta + 1} d\theta = \int \frac{du}{u} = \ln |u| + C = \ln |\sin^2 \theta + 1| + C = \ln(\sin^2 \theta + 1) + C$$

$$\text{since } \sin^2 \theta + 1 > 0$$

2.[25] A page has an area of 90 cm^2 . A text is to be printed onto this page with 1 cm margins at the bottom and both sides and a $\frac{1}{2}$ cm margin at the top. Find the dimensions of the page that will allow the largest printed area.

Solution:



$90 = xy$ fixed. Area to be fixed maximized is

$$A = (x - 2)(y - 3/2), \text{ writing } y = 90/x, \quad x \neq 0$$

$$\Rightarrow A = (x - 2)\left(\frac{90}{x} - \frac{3}{2}\right)$$

$$A = 90 - \frac{3}{2}x - \frac{180}{x} + 3 = 93 - \frac{3}{2}x - \frac{180}{x}$$

$$y \geq \frac{3}{2}, x \geq 2,$$

$$\text{If } y = 3/2 \Rightarrow x = 90 \cdot \frac{2}{3}. \text{ Hence, } 2 \leq x \leq 60$$

$$A' = -\frac{3}{2} + \frac{180}{x^2} = 0 \Rightarrow x^2 = 120.$$

$$x = \sqrt{120} = 2\sqrt{30} \text{ since } x > 0$$

There are three values to check:

Critical point:

$$x = 2\sqrt{30} \Rightarrow y = \frac{90}{2\sqrt{30}} = \frac{45}{\sqrt{30}}.$$

Boundary points:

$$x = 2 \Rightarrow y = 45 \text{ and } A = 0 \left[\lim_{x \rightarrow 2^+} A = 0 \right]$$

$$x = 60 \Rightarrow y = 3/2 \text{ and } A = 0 \left[\lim_{x \rightarrow 60^-} A = 0 \right]$$

Therefore, dimensions $x = 2\sqrt{30}, y = \frac{45}{\sqrt{30}}$ give maximum printed area.

$$A = 93 - \frac{3}{2} \cdot 2\sqrt{30} - \frac{180}{2\sqrt{30}} = 93 - 6\sqrt{30}$$

3. Evaluate the following limits:

(a)[13] $\lim_{x \rightarrow 0^+} (\sin x)^x$

Solution:

Let $y = (\sin x)^x$

$\lim_{x \rightarrow 0^+} y$ gives $[0^0]$. Take log of y :

$\ln y = x \ln \sin(x) \rightarrow 0 \cdot \infty$ [$x \ln \sin(x) \geq 0$ because $\cos x \rightarrow 0^+$]

$$\lim_{x \rightarrow 0^+} \frac{\ln \sin x}{1/x} \left[\frac{\infty}{\infty} \right] = \lim_{x \rightarrow 0^+} \frac{\frac{\cos x}{\sin x}}{\frac{-1}{x^2}} = - \lim_{x \rightarrow 0^+} \frac{x^2 \cos x}{\sin x} = - \lim_{x \rightarrow 0^+} \frac{x}{\sin x} \cdot x \cdot \cos x$$

$$\Rightarrow \lim_{x \rightarrow 0^+} \ln y = - \lim_{x \rightarrow 0^+} \frac{x}{\sin x} \cdot \lim_{x \rightarrow 0^+} x \cdot \lim_{x \rightarrow 0^+} \cos x = 1 \cdot 0 \cdot 1 = 0.$$

$$\Rightarrow \lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} (\sin x)^x = e^0 = 1.$$

(b)[12] $\lim_{x \rightarrow 3} \frac{x^3 - 27}{\int_9^{3x} \cos \pi t^2 dt}$

Solution:

If we put $x = 3$, we get $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

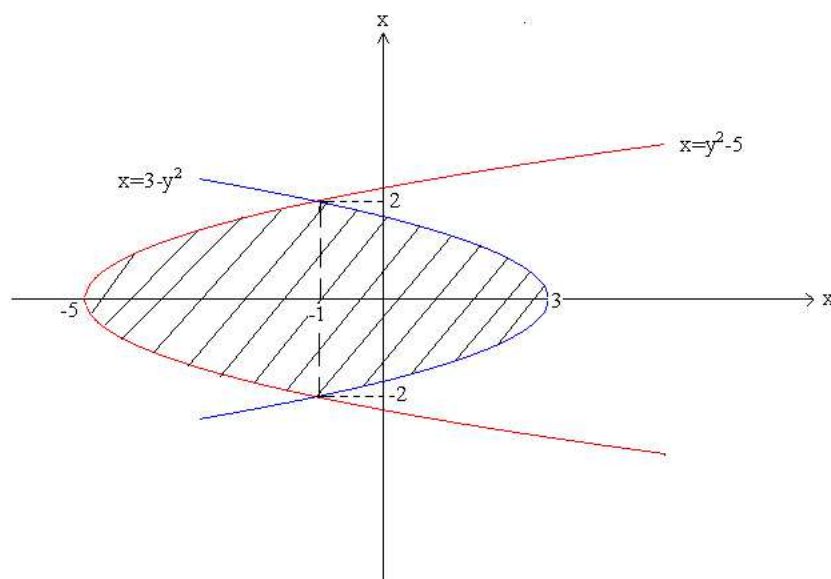
F.T.C must be used.

$$\lim_{x \rightarrow 3} \frac{x^3 - 27}{\int_9^{3x} \cos \pi t^2 dt} = \lim_{x \rightarrow 3} \frac{3x^2}{3 \cos(\pi 9x^2)} = \frac{9}{\cos(\pi \cdot 81)} = -9.$$

4. Sketch the region enclosed by the graphs of $x = y^2 - 5$ and $x = 3 - y^2$.

(a)[10] Express the area of this region by setting up an integral with respect to x . (Do not evaluate.)

Solution:



Intersection points: $y^2 - 5 = 3 - y^2 \Rightarrow 2y^2 = 8 \Rightarrow y = \pm 2 \Rightarrow x = -1$.

$$\begin{aligned}
 A &= \int_{-5}^{-1} \left(\sqrt{x+5} - \left(-\sqrt{x+5} \right) \right) dx + \int_{-1}^3 \left(\sqrt{3-x} - \left(-\sqrt{3-x} \right) \right) dx \\
 &= \int_{-5}^{-1} \left(2\sqrt{x+5} \right) dx + \int_{-1}^3 \left(2\sqrt{3+x} \right) dx
 \end{aligned}$$

(b)[10] Express the area of this region by setting up an integral with respect to y . (Do not evaluate.)

Solution:

$$A = \int_{-2}^2 (3 - y^2) - (y^2 - 5) dy = \int_{-2}^2 (8 - 2y^2) dy$$

(c)[10] Find the area of this region by evaluating one of the definite integrals found above. (Which one would you prefer? Of course, the simpler.)

Solution:

$$\begin{aligned}
 A &= \int_{-2}^2 (8 - y^2) dy = 8y - \frac{2}{3}y^3 \Big|_{-2}^2 \\
 &= \left(16 - \frac{2}{3}8 \right) - \left(-16 + \frac{2}{3}8 \right) = \frac{64}{3}
 \end{aligned}$$

Spring 2006 Second Midterm

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1. Evaluate

(a) $\lim_{x \rightarrow 0^+} [\sin(x^2)]^{1/\ln x}$

Solution:

$$y = (\sin x^2)^{1/\ln x}$$

$$\ln y = \frac{1}{\ln x} \ln \sin x^2 \Rightarrow \lim_{x \rightarrow 0^+} \ln y = \left[\frac{\infty}{\infty} \right] = \lim_{x \rightarrow 0^+} \frac{2x \cos x^2}{1/x}$$

$$\ln y = \lim_{x \rightarrow 0^+} \frac{2x^2 \cos x^2}{\sin x^2} = \lim_{x \rightarrow 0^+} \frac{x^2}{\sin x^2} \cdot \lim_{x \rightarrow 0^+} 2 \cos x^2 = 2$$

$$\lim_{x \rightarrow 0^+} \ln y = 2 \Rightarrow \ln \lim_{x \rightarrow 0^+} y = 2 \Rightarrow \lim_{x \rightarrow 0^+} y = e^2$$

$$\Rightarrow \lim_{x \rightarrow 0^+} (\sin x^2)^{1/\ln x} = e^2$$

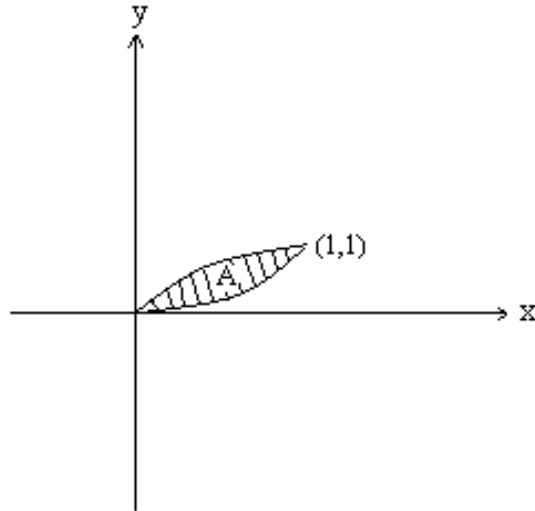
(b) $\lim_{n \rightarrow \infty} \sum_{j=1}^n \frac{1}{n + jc}$ for $c > 1$

Solution:

$$\frac{1}{nj + c} = \frac{1/n}{1 + \frac{jc}{n}} \Rightarrow \lim_{n \rightarrow \infty} \sum_{j=1}^n \frac{1}{n} \cdot \frac{1}{1 + \frac{jc}{n}} = \int_0^1 \frac{dx}{1 + xc}$$

$$\int_0^1 \frac{dx}{1 + xc} = \frac{1}{c} \int_0^1 \frac{cdx}{1 + xc} = \frac{1}{c} \ln |1 + xc|_0^1 = \frac{1}{c} \ln(1 + c)$$

2. (a) Find the area of the region R bounded by the curves of $y = x^2$ and $x = y^2$. Solution:



$$y = x^2 = y^4 \Rightarrow y - y^4 = 0 \Rightarrow y(1 - y^3) = 0, \quad y = 0, \quad y = 1$$

$$A = \int_0^1 (\sqrt{x} - x^2) dx = \frac{2}{3} x^{3/2} - \frac{x^3}{3} \Big|_0^1 = 2/3 - 1/3 = 1/3$$

- (b) Find the value of $a > 0$ such that when the region bounded by the curve $y = 1 + \sqrt{x}e^{x^2}$, the line $y = 1$ and the line $x = a$ is rotated about the line $y = 1$, a volume of 2π is generated.

Solution:

$$V = \pi \int_0^a (\sqrt{x}e^{x^2})^2 dx = 2\pi$$

$$\pi \int_0^a xe^{2x^2} dx = 2\pi$$

$$2x^2 = u \Rightarrow 4xdx = du$$

$$\pi \int_0^{2a^2} \frac{e^u du}{4} = 2\pi$$

$$\frac{\pi}{4}(e^{2a^2} - 1) = 2\pi$$

$$e^{2a^2} - 1 = 8$$

$$e^{2a^2} = 9 \Rightarrow 2a^2 = \ln 9 = \ln 3^2 = 2 \ln 3$$

$$a = \sqrt{\ln 3}$$

3. Integrate

$$(a) \int \frac{5x}{(x+2)(x^2+1)} dx$$

Solution:

$$I = \int \frac{5x}{(x+2)(x^2+1)} dx$$

$$\frac{5x}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

$$A(x^2+1) + (Bx+C)(x+2) = 5x$$

$$x = -2 \Rightarrow 5A = -10, A = -2$$

$$A + B = 0, B = 2$$

$$x = 0 \Rightarrow A + 2C = 0, C = 1$$

$$I = \int \left(\frac{-2}{x+2} + \frac{2x}{x^2+1} + \frac{1}{x^2+1} \right) dx = -2 \ln|x+2| + \ln(x^2+1) + \arctan x + C$$

$$I = \ln \frac{x^2+1}{(x+2)^2} + \arctan x + C$$

$$(b) \int x \ln \sqrt{x} dx$$

Solution:

$$I = \int x \ln \sqrt{x} dx$$

$$\sqrt{x} = u \Rightarrow \frac{1}{2\sqrt{x}} dx = du$$

$$x = u^2$$

$$I = \int u^2 \ln u \cdot 2u du = 2 \int u^3 \ln u du = \left[\ln u \cdot \frac{u^4}{4} - \int \frac{u^4 du}{4u} \right]$$

$$\ln u = U, \frac{1}{u} du = dU$$

$$u^3 du = dV, \frac{u^4}{4} = V$$

$$\Rightarrow I = \frac{u^4 \ln u}{2} - \frac{u^4}{8} + C = \frac{x^2 \ln \sqrt{x}}{2} - \frac{x^2}{8} + C$$

4. (a) If f and g are continuous functions on $[a, b]$ satisfying $\int_a^b f(x)dx = \int_a^b g(x)dx$, then show that there exists $c \in (a, b)$ such that $f(c) = g(c)$.

Solution:

Let $h(x) = f(x) - g(x)$, h is continuous so by Mean Value Theorem

$$\exists c \in (a, b) \quad h(c) = \frac{1}{b-a} \int_a^b (f(t) - g(t))dt = \frac{1}{b-a} \left[\int_a^b f(t)dt - \int_a^b g(t)dt \right]$$

$$h(c) = \frac{1}{b-a} \cdot 0 = 0$$

So, $\exists c \in (a, b) \quad h(c) = f(c) - g(c) = 0 \Rightarrow \exists c \in (a, b) \quad f(c) = g(c)$

- (b) Evaluate $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{\sin^2 x} dx$

Solution:

$$I = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{\sin^2 x} dx$$

This is an improper integral.

$$I = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{\sin^2 x} dx = \lim_{c \rightarrow 0^-} \int_{-\pi/2}^c \frac{\cos x}{\sin^2 x} dx + \lim_{b \rightarrow 0^+} \int_b^{\pi/2} \frac{\cos x}{\sin^2 x} dx$$

$$u = \sin x \Rightarrow du = \cos x dx, \quad \int \frac{\cos x}{\sin^2 x} dx = \int \frac{du}{u^2} = -\frac{1}{u} + C$$

$$I = \lim_{c \rightarrow 0^-} \left(-\frac{1}{\sin x} \Big|_{-\pi/2}^c \right) + \lim_{b \rightarrow 0^+} \left(-\frac{1}{\sin x} \Big|_b^{\pi/2} \right)$$

$$\Rightarrow I = \lim_{c \rightarrow 0^-} -\frac{1}{\sin c} + (-1) + \lim_{b \rightarrow 0^+} -1 + \frac{1}{\sin b} = \infty, \text{ divergent.}$$

BU Department of Mathematics

Math 101 Calculus I

Spring 1999 Second Midterm

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1. Evaluate $\lim_{h \rightarrow 0^+} \left(\frac{1}{h} \int_2^{2+h} \sqrt{t^2 + 2} dt \right)$.

Solution:

$$\lim_{h \rightarrow 0^+} \left(\frac{1}{h} \int_2^{2+h} \sqrt{t^2 + 2} dt \right) = \lim_{h \rightarrow 0^+} \frac{f(h)}{g(h)}$$

$$\text{where } f(h) = \frac{1}{h} \int_2^{2+h} \sqrt{t^2 + 2} dt \text{ and } g(h) = h.$$

$\lim_{h \rightarrow 0^+} \frac{f(h)}{g(h)}$ has $\frac{0}{0}$ indeterminacy. So L'Hôpital's rule can be applied:

$$\lim_{h \rightarrow 0^+} \left(\frac{1}{h} \int_2^{2+h} \sqrt{t^2 + 2} dt \right) = \lim_{h \rightarrow 0^+} \frac{\sqrt{(2+h)^2 + 2} \cdot 1 - \sqrt{2^2 + 2} \cdot 0}{1} = \sqrt{6}.$$

2. Evaluate

a) $\lim_{n \rightarrow \infty} \frac{\pi}{n} \left[\sin \frac{\pi}{n} + \sin \frac{2\pi}{n} + \sin \frac{3\pi}{n} + \dots + \sin \frac{n\pi}{n} \right]$

b) $\lim_{x \rightarrow \infty} \frac{x^2}{(\ln x)^3}$

Solution:

a) $\lim_{n \rightarrow \infty} \frac{\pi}{n} \left[\sin \frac{\pi}{n} + \sin \frac{2\pi}{n} + \sin \frac{3\pi}{n} + \dots + \sin \frac{n\pi}{n} \right] = \lim_{n \rightarrow \infty} R_n$

where $R_n = \frac{\pi}{n} \left[\sin \frac{\pi}{n} + \sin \frac{2\pi}{n} + \sin \frac{3\pi}{n} + \dots + \sin \frac{n\pi}{n} \right]$ is a Riemann sum for $f(x) = \sin x$ $x \in [0, \pi]$ with $\Delta x = \frac{\pi}{n}$. So

$$\lim_{n \rightarrow \infty} R_n = \int_0^\pi \sin x dx = -\cos x \Big|_0^\pi = \cos 0 - \cos \pi = 2.$$

b) $\lim_{x \rightarrow \infty} \frac{x^2}{(\ln x)^3}$ has $\left[\frac{\infty}{\infty} \right]$ indeterminacy. Hence we can apply L'Hôpital's rule:

$$\lim_{x \rightarrow \infty} \frac{x^2}{(\ln x)^3} = \lim_{x \rightarrow \infty} \frac{2x}{3(\ln x)^2} = \lim_{x \rightarrow \infty} \frac{2x^2}{3(\ln x)^2} \text{ which has } \left[\frac{\infty}{\infty} \right] \text{ indeterminacy. Reap-}$$

plying L'Hôpital's rule gives:

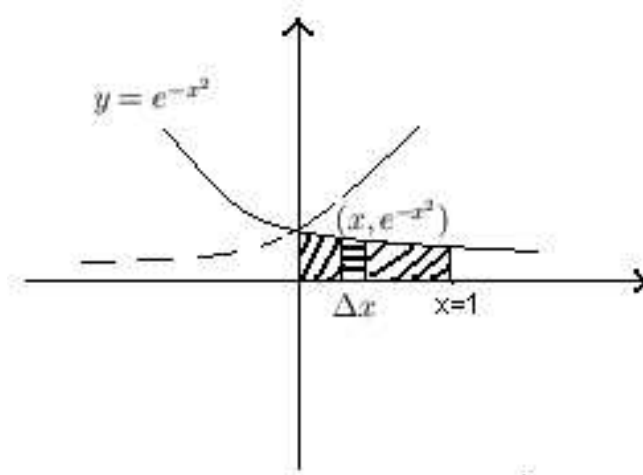
$$\lim_{x \rightarrow \infty} \frac{2x^2}{3(\ln x)^2} = \lim_{x \rightarrow \infty} \frac{4x}{6 \ln x} = \lim_{x \rightarrow \infty} \frac{2x^2}{3 \ln x} \text{ has } \left[\frac{\infty}{\infty} \right] \text{ indeterminacy again. We apply}$$

L'Hôpital's rule for the last time to get:

$$\lim_{x \rightarrow \infty} \frac{2x^2}{3 \ln x} = \lim_{x \rightarrow \infty} \frac{4x}{\frac{3}{x}} = \lim_{x \rightarrow \infty} \frac{4x^2}{3} = \infty.$$

3. Find the volume of the solid generated if the region bounded by the graphs of $y = e^{-x^2}$, $x = 0$, $x = 1$, $y = 0$ is revolved about y -axis.

Solution:



By Shell Method:

$$V = 2\pi \int_0^1 x e^{-x^2} dx. \text{ Put } u = -x^2 \text{ so that } du = -2x dx.$$

$$\Rightarrow V = -\pi \int_0^{-1} e^u du = \pi(e^u)|_{-1}^0 = \pi(1 - \frac{1}{e}).$$

4. Evaluate

a) $\int \frac{dx}{e^x + e^{x/2}}$

b) $\int_0^1 x \ln x dx$

Solution:

a) $\int \frac{dx}{e^x + e^{x/2}} \quad u = e^{x/2} \quad 2 \ln u = x$

$$\int \frac{2du}{u(u^2 + u)} = \int \frac{2du}{u^2(u + 1)}$$

$$\frac{2}{u^2(u + 1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u + 1}$$

$$2 = Au(u + 1) + B(u + 1) + Cu^2$$

$$u = 0 \Rightarrow B = 2$$

$$u = -1 \Rightarrow C = 2$$

$$\text{and } A + C = 0 \Rightarrow A = -2.$$

$$\begin{aligned} \int \frac{2du}{u(u^2+u)} &= -\int \frac{2}{u} du + \int \frac{2}{u^2} du + \int \frac{2}{u+1} du \\ &= -2 \ln |u| - \frac{2}{u} + 2 \ln |u+1| + C = 2 \ln \left| \frac{e^{x/2} + 1}{e^{x/2}} \right| - \frac{2}{e^{x/2}} + C \end{aligned}$$

b) $\int_0^1 x \ln x dx$ is an improper integral.

$$\text{Hence, } \int_0^1 x \ln x dx = \lim_{a \rightarrow 0^+} \int_a^1 x \ln x dx, \quad (1 > a > 0).$$

Using integration by parts: $u = \ln x$, $x dx = dv \Rightarrow du = \frac{dx}{x}$, $v = \frac{x^2}{2}$

$$\begin{aligned} \int_0^1 x \ln x dx &= \lim_{a \rightarrow 0^+} \left[\frac{x^2 \ln x}{2} - \int \frac{1}{2} x dx \right]_a^1 \\ &= \lim_{a \rightarrow 0^+} \left[-\frac{1}{4} - \frac{a^2 \ln a}{2} + \frac{a^2}{1} \right] \\ &= -\frac{1}{4}. \end{aligned}$$

Note that

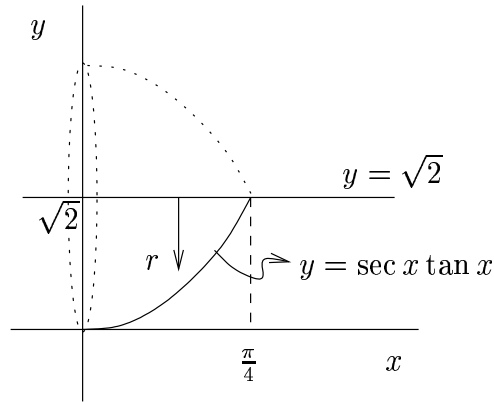
$$\begin{aligned} \lim_{a \rightarrow 0^+} a^2 \ln a &= [0 \cdot \infty] = \lim_{a \rightarrow 0^+} \left[\frac{\ln a}{a^{-2}} \right] = \left[\frac{\infty}{\infty} \right] \\ &= \lim_{a \rightarrow 0^+} \frac{1/a}{-2a^{-3}} = -\frac{1}{2} a^2 = 0. \end{aligned}$$

BU Department of Mathematics
Math 101 Calculus I Summer 2004 Second Midterm

1. Find the volume of solid generated by revolving the region in the first quadrant bounded above by the line $y = \sqrt{2}$, below by the curve $y = \sec x \tan x$, and on the left by the y-axis, about the line $y = \sqrt{2}$.

Solution:

$$f(x) = \sec x \tan x, \quad f(0) = 0, \quad f\left(\frac{\pi}{4}\right) = \sqrt{2}.$$



$$\begin{aligned} V &= \pi \int_0^{\pi/4} r^2 dx = \pi \int_0^{\pi/4} (\sqrt{2} - \sec x \tan x)^2 dx \\ &= \pi \int_0^{\pi/4} (2 - 2\sqrt{2} \sec x \tan x + \sec^2 x \tan^2 x) dx \\ &= \pi \left[2x - 2\sqrt{2} \sec x \tan x + \frac{\tan^3 x}{3} \right]_0^{\pi/4} \\ &= \pi \left(\frac{\pi}{2} - 2\sqrt{2} \cdot \sqrt{2} + \frac{1}{3} - (0 - 2\sqrt{2} + 0) \right) \\ &= \frac{\pi^2}{2} + \frac{\pi}{3}(6\sqrt{2} - 11) \end{aligned}$$

2. Evaluate:

$$(a) \int_0^4 x e^{-x} dx \quad (b) \int_4^8 \frac{y}{y^2 - 2y - 3} dy \quad (c) \int_0^{\pi/2} \cot x dx \quad (d) \int \frac{x^3}{\sqrt{9 - x^2}} dx$$

Solution:

(a) Take $u=x$ and $e^{-x} dx = dv$. Then:

$$\begin{aligned} \int x e^{-x} dx &= -x e^{-x} - \int -e^{-x} dx \\ \int_0^4 x e^{-x} dx &= \left. -x e^{-x} - e^{-x} \right|_0^4 \\ &= (-4e^{-4} - e^{-4}) - (0 - 1) \\ &= 1 - 5e^{-4} \end{aligned}$$

(b)

$$\begin{aligned}\frac{y}{y^2 - 2y - 3} &= \frac{A}{y + 1} + \frac{B}{y - 3} \\ y &= \frac{A}{y + 1} + \frac{B}{y - 3} \\ 3 &= 4B \implies B = \frac{3}{4} \\ -1 &= -4A \implies A = \frac{1}{4}\end{aligned}$$

So,

$$\begin{aligned}\int_4^8 \frac{y}{y^2 - 2y - 3} dy &= \int_4^8 \left(\frac{1}{4} \frac{1}{y + 1} + \frac{3}{4} \frac{1}{y - 3} \right) dy \\ &= \left[\frac{1}{4} \ln |y + 1| + \frac{3}{4} \ln |y - 3| \right]_4^8 \\ &= \frac{1}{4} \ln 9 + \frac{3}{4} \ln 5 - \frac{1}{4} \ln 5 - \frac{3}{4} \ln 1 \\ &= \frac{\ln 9}{4} + \frac{\ln 5}{2}\end{aligned}$$

(c)

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cot x dx &= \lim_{a \rightarrow 0^+} \int_a^{\frac{\pi}{2}} \frac{\cos x}{\sin x} dx \\ &= \lim_{a \rightarrow 0^+} \ln |\sin x| \Big|_a^{\frac{\pi}{2}} \\ &= \lim_{a \rightarrow 0^+} (\ln a) \\ &= +\infty, \text{ diverges.}\end{aligned}$$

(d) Take $x = 3 \sin \theta$. Then,

$$\begin{aligned}\int \frac{x^3}{\sqrt{9 - x^2}} dx &= \int \frac{27 \sin^3 \theta}{\sqrt{9 - 9 \sin^2 \theta}} 3 \cos \theta d\theta \\ &= \int \frac{27 \sin^3 \theta}{3 \cos \theta} 3 \cos \theta d\theta \\ &= \int 27 \sin^3 \theta d\theta \\ &= 27 \int \sin \theta (1 - \cos^2 \theta) d\theta \\ &= 27 \left(\int \sin \theta d\theta + \int \cos^2 \theta (-\sin \theta) d\theta \right) \\ &= 27 \left(-\cos \theta + \frac{\cos^3 \theta}{3} \right) \\ &= 27 \left(-\frac{(9 - x^2)^{\frac{1}{2}}}{3} + \frac{(9 - x^2)^{\frac{3}{2}}}{3 \cdot 27} \right) \\ &= -9(9 - x^2)^{\frac{1}{2}} + \frac{(9 - x^2)^{\frac{3}{2}}}{3} + C\end{aligned}$$

3. Determine whether the following series converge.

$$(a) \sum_{n=1}^{\infty} \frac{1}{n(1 + \ln^2 n)} \quad (b) \sum_{n=1}^{\infty} e^{-n} n^3 \quad (c) \sum_{n=1}^{\infty} \frac{n^n}{(2^n)^2} \quad (d) \sum_{n=1}^{\infty} \frac{2n^2 + 3}{5n^3 + 4n + 6}$$

Solution:

(a) Integral test :

$$f(x) = \frac{1}{x(1 + \ln^2 x)}, \quad x > 0. \quad \text{clearly decreasing}$$

Take $u = \ln x$. Then:

$$\int_1^{\infty} \frac{1}{x(1 + \ln^2 x)} dx = \int_0^{\infty} \frac{du}{1 + u^2} = \tan^{-1} u \Big|_0^{\infty} = \frac{\pi}{2} \implies \text{series converges}$$

(b) Ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{e^{-(n+1)}(n+1)^3}{e^{-n}n^3} = \frac{(n+1)^3}{n^3} \frac{1}{e}$$
$$\implies \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \frac{1}{e} < 1 \implies \text{series converges}$$

(c) Root test:

$$\left(\frac{n^n}{(2^n)^2} \right)^{\frac{1}{n}} = \frac{n}{2^2} = \frac{n}{4} \rightarrow +\infty \implies \text{series diverges}$$

(d) Limit comparison test:

$$\text{Compare to } \sum_{n=1}^{\infty} \frac{1}{n}. \text{ Observe that } \frac{(2n^2 + 3)}{5n^3 + 4n + 6} n \rightarrow \frac{2}{5}.$$

$$0 < \frac{2}{5} < 1 \implies \text{series diverges since } \sum \frac{1}{n} \text{ diverges.}$$