B U Department of Mathematics

Math 101 Calculus I

Fall 2001 Second Midterm

1. Evaluate the following integrals:

(a) $\displaystyle\int \sec^4 x \tan^2 x dx$

Solution:

$$I = \int \sec^2 x \tan^2 x \sec^2 x dx = \int (1+\tan^2 x) \tan^2 x \sec^2 x dx = \int (\tan^2 x + \tan^4 x) \sec^2 x dx$$

Letting $u = \tan x$, we have $du = \sec^2 x dx$.

Hence $I = \displaystyle\int (u^2 + u^4) du = \frac{1}{3}u^3 + \frac{1}{5}u^5 + c = \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + c.$

(b) $\displaystyle\int \frac{\arcsin(\ln x)}{x} dx$

Solution:

Let $t = \ln x$, then $dt = \dfrac{dx}{x}$.

$\Rightarrow I = \displaystyle\int \arcsin t dt.$

Now let $u = \arcsin t, dv = dt$. Then $du = \dfrac{dt}{\sqrt{1-t^2}}, v = t.$

Hence by the method of integration by parts, $I = t \arcsin t - \displaystyle\int \frac{t}{\sqrt{1-t^2}} dt.$

If we let $z = 1 - t^2$, we get $dz = -2t dt$ and so $I = \ln x \cdot \arcsin(\ln x) + \dfrac{1}{2}\displaystyle\int \frac{dz}{\sqrt{z}}.$

Therefore, $I = \ln x \cdot \arcsin(\ln x) + \sqrt{1 - \ln^2 x} + c.$

2. Let R be the region in the first quadrant bounded by the x-axis, the y-axis, the curve $y = 1+\sqrt{x}$ and the line $y = 3 - x$.

(a) Sketch the region R.

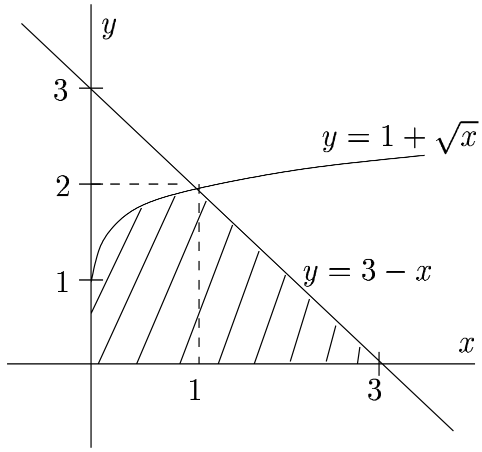
Solution:

Intersection point(s):

$1 + \sqrt{x} = 3 - x \Rightarrow (\sqrt{x})^2 = (2-x)^2 \Rightarrow x = 4 - 4x + x^2$

$\Rightarrow 0 = x^2 - 5x + 4 \Rightarrow 0 = (x-4)(x-1) \Rightarrow x = 1$

Also note that $y = 1 + \sqrt{x} \Rightarrow x = (y-1)^2$



(b) Using integration methods, set up (DO NOT EVALUATE) an expression for the area of the region R.

Solution:

$$A = \int_0^1 (1 + \sqrt{x})dx + \int_1^3 (3 - x)dx$$

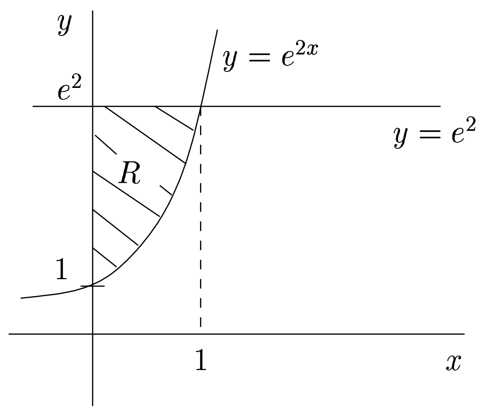
OR

$$A = \int_0^1 (3 - y)dy + \int_1^2 [(3 - y) - (y - 1)^2]dy.$$

3. Let R be the region bounded by the curve $y = e^{2x}$ and the lines $y = e^2$ and $x = 0$.

(a) Sketch the region R.

Solution:



$$y = e^{2x} \Leftrightarrow x = \frac{\ln y}{2}.$$

(b) Using the disk method, set up (DO NOT EVALUATE) an integral for the volume of the solid of revolution obtained by rotating the region R about the y-axis.

Solution:

$$V = \pi \int_1^{e^2} \frac{\ln^2 y}{4} dy.$$

(c) Using the shell method, set up (DO NOT EVALUATE) an integral for the volume of the same solid.

Solution:

$$V = 2\pi \int_0^1 x(e^2 - e^{2x})dx.$$

(d) Evaluate ONE of the integrals above.

Solution:

$$V = 2\pi \int_0^1 (xe^2 - xe^{2x})dx = 2\pi \left[e^2 \cdot \frac{x^2}{2} - \int xe^{2x} dx \right]$$

Let $u = x, dv = e^{2x}dx$. Then we have $du = dx, v = \frac{1}{2}e^{2x}$.

$$\Rightarrow V = 2\pi \left[\frac{e^2}{2}x^2 - \frac{1}{2}xe^{2x} + \frac{1}{2}\int e^{2x}dx \right] = 2\pi \left[\frac{e^2}{2}x^2 - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} \right]_0^1$$

$$= 2\pi \left[\frac{e^2}{2} - \frac{e^2}{2} + \frac{1}{4}e^2 - \frac{1}{4} \right] = \frac{\pi}{2}(e^2 - 1).$$

4. Evaluate the following limits:

(a) $\lim_{x \to 1} \frac{1}{\ln x} \int_1^x e^{\sin \frac{\pi t}{2}} dt$

Solution:

$$= \lim_{x \to 1} \frac{\int_1^x e^{\sin \frac{\pi t}{2}} dt}{\ln x} \left[\frac{0}{0}\right]$$

$$= \lim_{x \to 1} \frac{\frac{d}{dx}\int_1^x e^{\sin \frac{\pi t}{2}} dt}{\frac{1}{x}} \quad \text{(by L'Hopital's Rule)}$$

$$= \lim_{x \to 1} \frac{e^{\sin \frac{\pi x}{2}}}{\frac{1}{x}} = \frac{e^{\sin \frac{\pi}{2}}}{1} = e.$$

(b) $\lim_{x \to 0}(e^x + x)^{\frac{1}{x}}$ $[1^\infty]$

Solution:

For $y = (e^x + x)^{\frac{1}{x}}$, $\ln y = \frac{1}{x}\ln(e^x + x)$.

So $\lim_{x \to 0}(\ln y) = \lim_{x \to 0} \frac{\ln(e^x + x)}{x}$ $\left[\frac{0}{0}\right]$

$= \lim_{x \to 0} \frac{\frac{e^x+1}{e^x+x}}{1} = \frac{2}{1} = 2$ (by L'Hopital's Rule).

Since $\ln\left(\lim_{x \to 0} y\right) = \lim_{x \to 0}(\ln y) = 2$, we have $\lim_{x \to 0} y = \lim_{x \to 0}(e^x + x)^{\frac{1}{x}} = e^2$.

B U Department of Mathematics

Math 101 Calculus I

Fall 2002 Second Midterm

1. Evaluate the following integrals:

(a) $\displaystyle\int \frac{x+\arcsin x}{\sqrt{1-x^2}}dx$, where $|x|<1$.

Solution:

Let I denote the integral given. Then $I = \displaystyle\int \frac{x}{\sqrt{1-x^2}}dx + \int \frac{\arcsin x}{\sqrt{1-x^2}}dx = I_1 + I_2$.

Letting $u = 1 - x^2$ implies $du = -2xdx$:

$$I_1 = -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\sqrt{u} + c_1.$$

Letting $v = \arcsin x$ gives $dv = \dfrac{dx}{\sqrt{1-x^2}}$:

$$I_2 = \int v dv = \frac{v^2}{2} + c_2.$$

Writing I in the original variable:

$$I = I_1 + I_2 = -\sqrt{1-x^2} + \frac{1}{2}(\arcsin x)^2 + c.$$

(b) $\displaystyle\int \sqrt{x}\ln x dx$, where $x > 0$.

Solution:

We use integration by parts with the setting:

$$u = \ln x \Rightarrow du = \frac{dx}{x} \quad \text{and} \quad dv = \sqrt{x} \Rightarrow v = \frac{2}{3}x^{3/2}.$$

Then the original integral I becomes:

$$I = \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int \frac{x^{3/2}}{x}dx = \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + c.$$

(c) $\displaystyle\int_0^1 \frac{5xdx}{(x+2)(x^2+1)}$.

Solution:

We use partial fractions:

$$\frac{5x}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

which entails $A=-2,\ B=2$ and $C=1$.

$$\begin{aligned}\int_0^1 \frac{5x\,dx}{(x+2)(x^2+1)} &= -2\int_0^1 \frac{dx}{x+2} + \int_0^1 \frac{2x+1}{x^2+1}dx \\ &= -2\ln|x+2|\Big|_0^1 + \ln(x^2+1)\Big|_0^1 + \arctan x\Big|_0^1 \\ &= 3\ln 2 - 2\ln 3 + \frac{\pi}{4}.\end{aligned}$$

2. Let f be a twice differentiable function which never takes the value -1 and satisfies:

$$x = \int_0^{f(x)} \frac{dt}{(1+t^3)^{\frac{1}{3}}}$$

for all x. Show that $f'f'' = f^2$. (Hint: You may differentiate both sides with respect to x.)

Solution:

Differentiating both sides as suggested:

$$1 = \frac{1}{(1+f^3(x))^{\frac{1}{3}}}\, f'(x) \Longrightarrow f'(x) = (1+f^3(x))^{\frac{1}{3}}.$$

Once more differentiating:

$$f''(x) = (1+f^3(x))^{-\frac{2}{3}}\, f^2(x)\, f'(x) = (1+f^3(x))^{-\frac{1}{3}}\, f^2(x).$$

Then clearly:

$$f'(x)f''(x) = f^2(x).$$

3. Find the area of the region bounded by the curves $y = \dfrac{x^3}{\sqrt{1+x^2}}$, $y=0$, $x=-1$ and $x=1$.

Solution:

Note that the function y is symmetric with respect to the origin, i.e. it is odd. Moreover $y<0$ if $x<0$, and $y>0$ if $x>0$. Hence the region bounded by the curves given is symmetric with respect to the origin. This means the area A is given by:

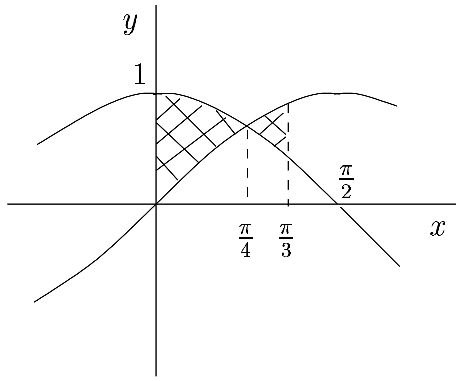
$$A = \int_{-1}^0 -\frac{x^3}{\sqrt{1+x^2}}dx + \int_0^1 \frac{x^3}{\sqrt{1+x^2}}dx = 2\int_0^1 \frac{x^3}{\sqrt{1+x^2}}dx.$$

Letting $u = 1+x^2$ we have $du = 2x dx$. The boundaries become: $x=0 \to u=1$ and $x=1 \to u=2$. Then:

$$A = 2\frac{1}{2}\int_1^2 \frac{u-1}{\sqrt{u}}du = \left[\frac{2}{3}u^{3/2} - 2\sqrt{u}\right]_1^2 = \frac{2}{3}(2-\sqrt{2}).$$

4. Find the volume of the solid generated when the region enclosed by $y = \cos x,\ y = \sin x,\ x = 0$ and $x = \dfrac{\pi}{3}$ is revolved about the $x-$axis.

Solution:



The volume of the solid is given by:

$$
\begin{aligned}
V &= \pi \int_0^{\pi/4} (\cos^2 x - \sin^2 x)dx + \pi \int_{\pi/4}^{\pi/3} (\sin^2 x - \cos^2 x)dx \\
&= \pi \int_0^{\pi/4} \cos 2x dx + \pi \int_{\pi/4}^{\pi/3} - \cos 2x dx = \frac{\pi}{2} \sin 2x \Big|_0^{\pi/4} - \frac{\pi}{2} \sin 2x \Big|_{\pi/4}^{\pi/3} \\
&= \left[1 - \frac{\sqrt{3}}{4}\right] \pi.
\end{aligned}
$$

MATH 101 Second Midterm Examination

December 8, 2003
17:00-18:00
Student number:
Name:
Signature:

1	/40 points
2	/15 points
3	/15 points
4	/30 points
Total	/100 points

In order to get full credit, you have to show your work and explain what you are doing. Calculators are not allowed, nor needed. Good luck.

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(1) a) *Evaluate* $\lim_{x\to 0} F'(x)$ *if* $F = \int_1^x \frac{\sin^2 t}{t} dt.$ (15 points)

By the fundamental theorem of calculus, we have

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_1^x \frac{\sin^2 t}{t}\,dt = \frac{\sin^2 x}{x}$$

and

$$\lim_{x\to 0} F'(x) = \lim_{x\to 0}\frac{\sin^2 x}{x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot \sin x = 1\cdot 0 = 0.$$

b) *Find* $f'(x)$ *if* $f(x) = x^{\ln x}$ $(x > 0)$. (15 points)

We have

$$\ln f(x) = (\ln x)(\ln x) = (\ln x)^2.$$

Differentiating we get

$$\frac{1}{f(x)}f'(x) = 2(\ln x)^1 \cdot \frac{1}{x}$$

and so

$$f'(x) = f(x)2\frac{\ln x}{x} = 2x^{\ln x}\frac{\ln x}{x}.$$

c) *Evaluate* $\lim_{x\to 0}(1+2x)^{\frac{1}{3x}}$. (10 points)

Let $y = (1+2x)^{\frac{1}{3x}}$. Then

$$\ln y = \frac{1}{3x}\ln(1+2x)$$

and

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln(1+2x)}{3x} = \lim_{x\to 0}\frac{\frac{1}{1+2x}\cdot 2}{3} = \frac{2}{3}$$

and

$$\ln \lim_{x\to 0} y = \lim_{x\to 0}\ln y = \frac{2}{3},$$

therefore

$$\lim_{x\to 0}(1+2x)^{\frac{1}{3x}} = \lim_{x\to 0} y = e^{2/3}.$$

(2) *Find the arc length of the curve* $y = \dfrac{e^x + e^{-x}}{2}$ *from* $x = 0$ *to* $x = 1$.

(15 points)

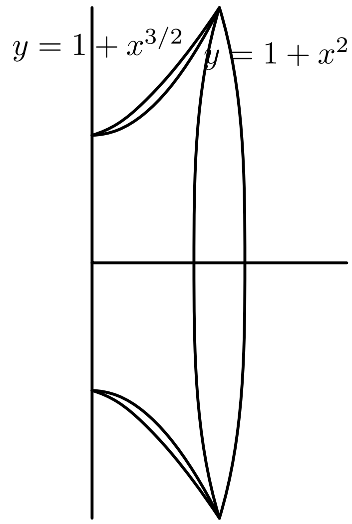
We have

$$\begin{aligned}
y &= \frac{1}{2}(e^x + e^{-x}) \\
y' &= \frac{1}{2}(e^x + e^{-x}(-1)) = \frac{1}{2}(e^x - e^{-x}) \\
1 + (y')^2 &= 1 + \frac{1}{4}(e^x - e^{-x})^2 = \frac{4}{4} + \frac{1}{4}(e^{2x} - 2 + e^{-2x}) = \frac{1}{4}(e^x + e^{-x})^2 \\
\sqrt{1 + (y')^2} &= \left|\frac{1}{2}(e^x + e^{-x})\right| = \frac{1}{2}(e^x + e^{-x})
\end{aligned}$$

and therefore the required arclength is

$$\begin{aligned}
\int_0^1 \sqrt{1 + (y')^2}\, dx &= \int_0^1 \frac{1}{2}(e^x + e^{-x})\, dx \\
&= \frac{1}{2}\left[\, e^x - e^{-x}\right]_0^1 = \frac{1}{2}(e - e^{-1}) - \frac{1}{2}(1 - 1) \\
&= \frac{1}{2}(e - e^{-1}).
\end{aligned}$$

(3) *Find the volume of the solid generated when the region enclosed by the curves* $y = 1 + x^{3/2}$ *and* $y = 1 + x^2$ *is revolved about the* x*-axis.* (15 points)



$$\int_0^1 \pi[(1+x^{3/2})^2 - (1+x^2)^2]\,dx = \pi \int_0^1 [(1+2x^{3/2}+x^3) - (1+2x^2+x^4)]\,dx$$

$$= \pi \left[2\frac{x^{5/2}}{5/2} + \frac{x^4}{4} - 2\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1$$

$$= \pi[\frac{4}{5} + \frac{1}{4} - \frac{2}{3} - \frac{1}{5}]$$

$$= \frac{11}{60}\,\pi.$$

(4) *Evaluate the following integrals:*

a) $\displaystyle\int \frac{\sin x\ dx}{\sqrt{2+\cos x}}$ (15 points)

Put $u = 2 + \cos x$; then $du = -\sin x\, dx$ and

$$\begin{aligned}\int \frac{\sin x\, dx}{\sqrt{2+\cos x}} &= \int \frac{-du}{\sqrt{u}} \\ &= -\int u^{-1/2} du \\ &= -\frac{u^{1/2}}{1/2} + c \\ &= -2\sqrt{u} + c \\ &= -2\sqrt{2+\cos x} + c.\end{aligned}$$

b) $\displaystyle\int_0^{1/4} \frac{dx}{\sqrt{1-4x^2}}$ (15 points)

$$\begin{aligned}\int_0^{1/4} \frac{dx}{\sqrt{1-4x^2}} &= \frac{1}{2}\int_0^{1/4} \frac{2\, dx}{\sqrt{1-(2x)^2}} \\ &= \frac{1}{2}\int_0^{1/2} \frac{du}{\sqrt{1-u^2}} \\ &= \frac{1}{2}\left[\sin^{-1} u\right]_0^{1/2} \\ &= \frac{1}{2}[\sin^{-1}\frac{1}{2} - \sin^{-1} 0] \\ &= \frac{1}{2}[\frac{\pi}{6} - 0] = \frac{1}{12}\,\pi.\end{aligned}$$

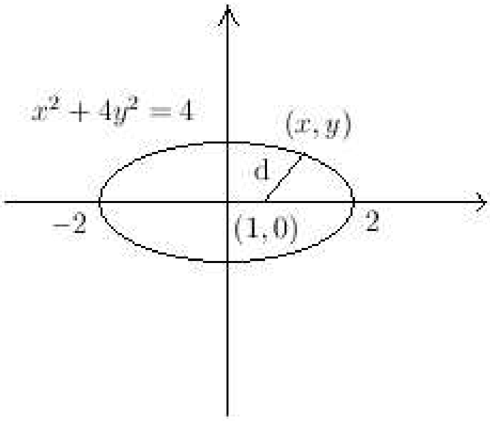
B U Department of Mathematics

Math 101 Calculus I

Fall 2004 Second Midterm

1.) Find the point(s) on the ellipse $x^2 + 4y^2 = 4$ nearest the point $(1, 0)$. (Hint: Minimize the square of the distance.)

Solution:



$d = \sqrt{(x-1)^2 + (y-0)^2}$. Let L be the square of the distance d.

$D = d^2 = (x-1)^2 + y^2$ where $-2 \leq x \leq 2$

Now, $x^2 + 4y^2 = 4 \Rightarrow y^2 = 1 - \dfrac{x^2}{4}$

$D = (x-1)^2 + 1 - \dfrac{x^2}{4}$ and $\dfrac{dD}{dx} = 2(x-1) - \dfrac{2x}{4} = 0$

$x - 1 - \dfrac{x}{4} = 0 \Rightarrow x = \dfrac{4}{3}$

We must show that $x = \dfrac{4}{3}$ is a min. point.

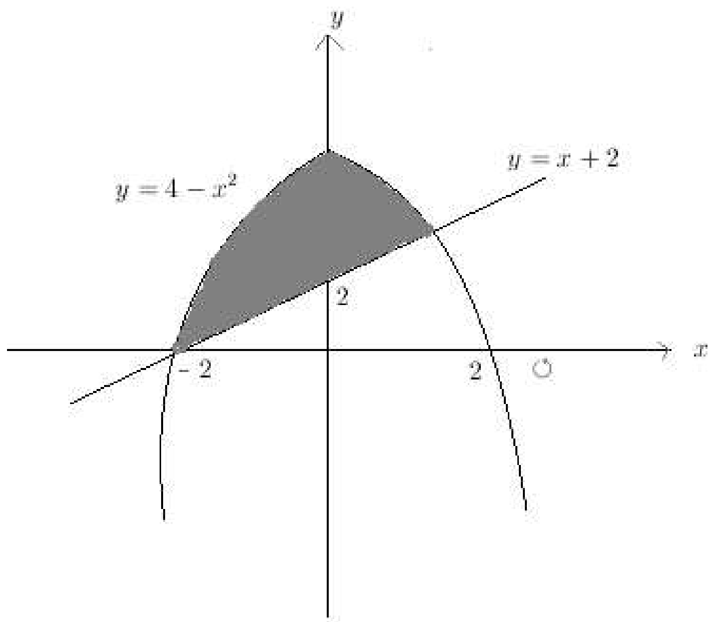
$x = 2 \Rightarrow D = 9,\ \ x = \dfrac{4}{3} \Rightarrow D = \dfrac{2}{3},\ \ x = 2 \Rightarrow D = 1$

Hence $x = \dfrac{4}{3}$ minimizes D and minimum occurs at $\left(\dfrac{4}{3}, \pm\dfrac{\sqrt{5}}{3}\right)$

2.) Find the volume of the solid that results when the area of the region enclosed by $y = 4 - x^2$ and $y = x + 2$ is revolved about the x-axis

Solution:

Intersection points :
$4 - x^2 = x + 2$
$x^2 + x - 2 = 0 \Rightarrow (x+2)(x-1) = 0 \Rightarrow x = -2, x = 1$



$$V = \int_{-2}^{1} \pi \left[(4-x^2)^2 - (x+2)^2\right] dx$$
$$V = \int_{-2}^{1} \pi \left[16 - 8x^2 + x^4 - x^2 - 4x - 4\right] dx$$
$$V = \int_{-2}^{1} \pi \left[12 - 4x - 9x^2 + x^4\right] dx = \pi \left[12x - 2x^2 - 3x^3 + \frac{x^5}{5}\right]_{-2}^{1} = \frac{108\pi}{5}$$

3.) a) Given $f'(x) = x\sqrt{x^2+1}$, find $f(x)$ if $f(2) = 12$.

Solution:

$$f'(x) = x\sqrt{x^2+5}$$

$$f(x) = \int x\sqrt{x^2+5}\, dx$$

$$u = x^2 + 5 \qquad du = 2xdx$$

$$\frac{du}{2} = xdx$$
$$f(x) = \frac{1}{2}\int u^{1/2}du = \frac{u^{3/2}}{3} + C = \frac{(x^2+5)^{3/2}}{3} + C$$

$$f(2) = 12 \Rightarrow 12 = \frac{(2^2+5)^{3/2}}{3}$$

$$12 = 9 + C$$

$$C = 3$$

$$f(x) = \frac{(x^2+5)^{3/2}}{3} + 3$$

b) Evaluate $\displaystyle\int \frac{dx}{x(1-\ln x)}$

Solution:

$$u = 1 - lnx$$
$$du = -\frac{dx}{x}$$
$$\int \frac{dx}{x(1-lnx)} = -\int \frac{du}{u} = -ln|u| + C = -ln|1-lnx| + C$$

4.) Sketch the curve $f(x) = \dfrac{x^2}{x^2-9}$ by examining, a) the domain,b) the x and y-intercepts, c) all asymptotes, d) symmetry, e) maximum,minimum and inflection points, f) intervals of increase and decrease, g) concavity and by making a table of your data.

Solution:

Domain: $\mathbb{R} - \{-3, +3\}$
$x = 0 \Rightarrow y = 0 \quad (0,0)$ is the x and y intercept.
Vertical asymptotes: $x = -3,\ x = +3$

$$\lim_{x\to 3^+} \frac{x^2}{x^2-9} = +\infty \qquad \lim_{x\to 3^-} \frac{x^2}{x^2-9} = -\infty$$
$$\lim_{x\to -3^+} \frac{x^2}{x^2-9} = -\infty \qquad \lim_{x\to -3^-} \frac{x^2}{x^2-9} = +\infty$$

Horizontal asymptote: $y = 1$

$$\lim_{x\to\pm\infty} \frac{x^2}{x^2-9} = \lim_{x\to\pm\infty} \frac{x^2}{x^2(1-\frac{9}{x^2})} = 1.$$

$f(-x) = f(x)$. The graph is symmetric with respect to y-axis.

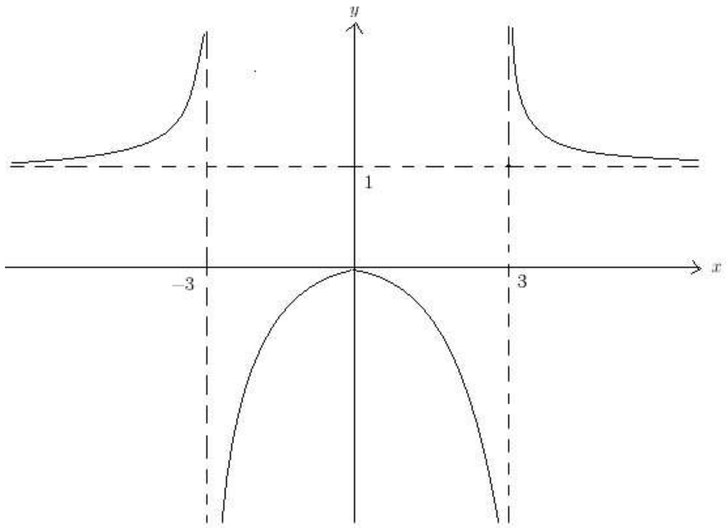
$$f'(x) = -\frac{18x}{(x-9)^2} \qquad f''(x) = \frac{54(x^2+3)}{(x^2-9)^3} \neq 0$$

$f'(x) = 0 \Rightarrow x = 0.$
$f'(x)$ is undefined at $x = \pm 3$ (but not in the domain of f(x)
Critical number: x=0.

	$-\infty$	-3	0		$+3$ $+\infty$
$f'(x)$	+	+	0	-	-
$f''(x)$	+	-		-	+
f(x)	↗ ∪	↗ ∩		↘ ∩	↘ ∪

$f''(0) < 0$, so there is a rel.max at (0,0).

The graph of the function is shown below.



y
1
x
−3
3

B U Department of Mathematics

Math 101 Calculus I

Fall 2005 Second Midterm

1. (a) If f is continuous on $[2,5]$ and

$$1 \leq f'(x) \leq 4,\ \forall x \in (2,5)$$

then show that

$$3 \leq f(5) - f(2) \leq 12$$

Solution:

By mean value theorem there is some $c \in (2,5)$such that $\quad f'(c) = \dfrac{f(5)-f(2)}{5-2}$
but $1 \leq f'(c) \leq 4$ so $1 \leq \dfrac{f(5)-f(2)}{5-2} \leq 4$ hence we have $3 \leq f(5) - f(2) \leq 12$

(b) Evaluate the integral

$$\int x \sin x^2 \cos x^2 dx$$

Solution:

Let $u = \sin(x^2)$ then $du = 2x\cos(x^2)$
$\mathbb{I} = \int x\sin(x^2)\cos(x^2) = \int \dfrac{u\ du}{2} = \dfrac{u^2}{4} + c = \dfrac{\sin^2(x^2)}{4} + c$

2. Consider the function

$$f(x) = 2 - \frac{3}{x} - \frac{3}{x^2}$$

(a) Determine the domain of f.

(b) Find all horizontal, vertical and oblique asymptotes of f.

(c) Find the intervals on which f is increasing or decreasing.

(d) Find all local extrema of f, if any.

(e) Find the intervals on which the graph of f is concave up or down.

(f) Find the points of inflection, if any.

(g) Sketch the graph of f.

Solution:

(a) Domain of f is the real line except the point zero i.e. $\mathbb{R}\backslash\{0\}$.

(b) $f(x) = \dfrac{2x^2 - 3x - 3}{x^2}$ then we have $\lim_{x\to\pm\infty} 2 - \dfrac{3}{x} - \dfrac{3}{x^2}$ and also $\lim_{x\to 0^+} f(x) = -\infty = \lim_{x\to 0^-}$ so y=2 is a horizontal asymptote and x=0 is a vertical asymptote and no oblique asymptotes.

(c)

x	$-\infty$	-3	-2	0	$+\infty$
f'	+	+	-	+	+
f''	+	-	-	-	-
f	inc.	inc.	dec.	inc.	inc.
f	$\smile$	$\frown$	$\frown$	$\frown$	$\frown$

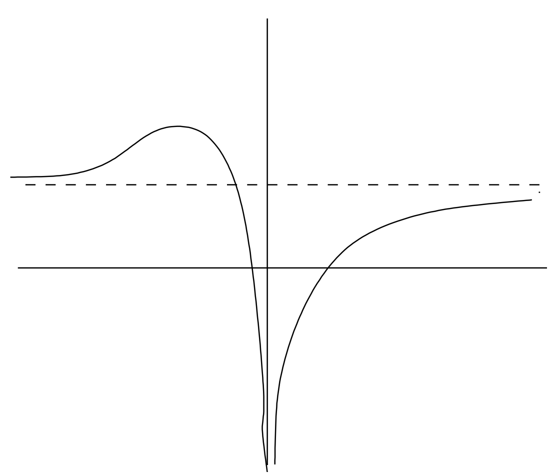
Where $f'(x) = \dfrac{3}{x^2} + \dfrac{6}{x^3}$, $f'(x) = 0 \Rightarrow x = -2$ and $f''(x) = -\dfrac{6}{x^3} + -\dfrac{18}{x^4}$, $f''(x) = 0 \Rightarrow x = -3$

(d) $f(-3) = \dfrac{8}{3}$ and $f(-2) = \dfrac{11}{4}$

(e) Done in the above table!

(f) Done in the above table!

(g)



3. Evaluate the following limits

$$a)\ \lim_{x\to 0} \frac{1}{\sin x} \int_0^{\sin 2x} \cos 5t dt$$

Solution:

$\frac{\int_0^{\sin(2x)}}{\sin(x)} = \frac{0}{0}$ By applying L'Hospital's rule we get;
$\lim_{x\to 0} \frac{\cos(5(\sin(2x)))2\cos(2x)}{\cos(x)} = 2$

$$b)\ \lim_{n\to\infty} \left(\frac{1}{1+n} + \frac{1}{2+n} + \frac{1}{3+n} + \cdots + \frac{1}{n+n}\right)$$

Solution:

$$\frac{1}{1+n} + \frac{1}{2+n} + \frac{1}{3+n} + \cdots + \frac{1}{n+n} = \frac{\frac{1}{n}}{\frac{1}{n}+1} + \frac{\frac{1}{n}}{\frac{2}{n}+1} + \cdots + \frac{\frac{1}{n}}{\frac{n}{n}+1} = \sum_{k=1}^{n} \frac{1}{n}\frac{1}{\frac{k}{n}+1}$$

this is a Riemann sum corresponding to the function $\frac{1}{1+x}$ in [0,1]

$\lim_n \to \infty \sum_{k=1}^n \frac{1}{n}\frac{1}{\frac{k}{n}+1} = \int_0^1 \frac{dx}{1+x} = \ln|1+x| \,|_0^1 = \ln 2 - \ln 1 = \ln 2$

4. Let R be the region bounded by the curves $y = -x$ and $y^2 = 2 - x$ and the x-axis.

   (a) Sketch the region R and find its area.

   (b) Let S_1 be the solid generated by rotating the region R about the line $x = -2$. Set up an intergral to calculate the volume of S_1. (Do not evaluate the integral).

   (c) Let S_2 be the solid generated by rotating the region R about the x-axis. Set up an intergral to calculate the volume of S_2. (Do not evaluate the integral).

Solution:

(a)
$$\begin{aligned} 2 - x &= x^2 \\ 0 &= (x-1)(x+2) \end{aligned}$$
$x = -2 \qquad x = 1$

A$=\int_{-2}^{0} \sqrt{2-x} - (-x)dx + \int_0^2 \sqrt{2-x}dx$ or

A$=\int_0^2 2 - y^2 - (-y)dy = 2y - \frac{y^3}{3} + \frac{y^2}{2} \,|_0^2 = \frac{10}{3}$

$x = 2 - y^2 \quad x = -y$

(b) $V_1 = \pi \int_0^2 \left[((2-y^2)+2)^2 - (-y+2)^2\right] dy$

(c) $V_2 = \pi \left[\int_{-2}^{0}(2-x) - (x^2)dx + \int_0^2 (2-x)dx\right]$

B U Department of Mathematics

Math 101 Calculus I

Fall 1999 Second Midterm

1.) Find the area of the region bounded by the graph of $y = x \ln x$ and the x-axis between $x = 1$ and $x = e$.

Solution:

Since the function $y = x \ln x$ is positive on the interval $[1, e]$, the area of this region, A, is equal to the integral: $A = \int_1^e x \ln x dx$.

(Integration by parts) Put $u = \ln x, dv = x$. We get

$$A = \frac{x^2 \ln x}{2} - \int_1^e \frac{x}{2}\, dx = \frac{1}{4}(e^2 + 1)$$

2.) (a) Given $F(x) = \int_{x^2}^{x} \cos^3(t+1)dt$, find $\frac{d}{dx} F(x)$.

Solution:

$$\frac{d}{dx} F(x) = \cos^3(x+1) - 2x\cos^3(x^2+1)$$

(b) Find $\lim_{x \to +\infty} (1 + e^{-x})^{e^x}$.

Solution:

Substitute $u = e^x$: $\lim_{x \to +\infty} (1 + e^{-x})^{e^x} = \lim_{u \to +\infty} (1 + \frac{1}{u})^u = e.$

3.) Integrate $\int \sqrt{\frac{\sin x}{\cos^5 x}} dx.$

Solution:

$$\int \sqrt{\frac{\sin x}{\cos^5 x}} dx = \int \frac{1}{\cos^2 x} \sqrt{\frac{\sin x}{\cos x}} dx = \int \sec^2 x \sqrt{\tan x} dx$$

Substitute $u = \tan x$, then we get

$$= \int \sqrt{u} du = \frac{2}{3} u^{\frac{3}{2}} + c = \frac{2}{3} \tan^{\frac{3}{2}} x + c$$

4.) (a) Show that $\int_0^a f(x)dx = \int_0^a f(a-x)dx.$

Solution:

Substitute $u = a - x$: $\int_0^a f(x)dx = \int_a^0 -f(a-u)du = \int_0^a f(a-u)du.$

(b) Deduce that $\int_0^\pi \frac{x \sin^{100} x}{1 + \cos^2 x} dx = \frac{\pi}{2} \int_0^\pi \frac{\sin^{100} x}{1 + \cos^2 x} dx.$

Solution:

By part (a), $\displaystyle\int_0^{\pi} \frac{x\sin^{100} x}{1+\cos^2 x}\,dx = \int_0^{\pi} \frac{(\pi - x)\sin^{100}(\pi - x)}{1+\cos^2(\pi - x)}\,dx$

$\displaystyle = \pi \int_0^{\pi} \frac{\sin^{100} x}{1+\cos^2 x}\,dx - \int_0^{\pi} \frac{x\sin^{100} x}{1+\cos^2 x}\,dx.$

Hence $\displaystyle 2\int_0^{\pi} \frac{x\sin^{100} x}{1+\cos^2 x}\,dx = \pi \int_0^{\pi} \frac{\sin^{100} x}{1+\cos^2 x}\,dx$

5.) Determine whether the series $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{n(\ln n)^3}$ is convergent.

Solution:

By Integral Test, consider the integral $\displaystyle\int_1^{+\infty} \frac{1}{x(\ln x)^3}\,dx.$

By substitution $u = \ln x$, the improper integral $\displaystyle\int_1^{+\infty} \frac{1}{u^3}\,du$ is finite, hence the series is convergent.

B U Department of Mathematics

Math 101 Calculus I

Spring 2000 Second Midterm

1. Find $\displaystyle\int x(\ln x)^2 dx$.

Solution:

Using integration by parts: $(\ln x)^2 = u$ and $xdx = dv$

Then $2\frac{1}{x}\ln xdx = du$ and $\frac{x^2}{2} = v$

$\displaystyle\int x(\ln x)^2 dx = \frac{x^2}{2}(\ln x)^2 - \int \frac{x^2}{2}2\frac{1}{x}\ln xdx = \frac{x^2}{2}(\ln x)^2 - \int x\ln xdx.$

Using integration by parts once again: $\ln x = u$ and $xdx = dv$ we get:

$\displaystyle\int x(\ln x)^2 dx = \frac{x^2}{2}(\ln x)^2 - \left(\frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{1}{x}dx\right).$

Now combining everything, we obtain the final result to be:

$$\int x(\ln x)^2 dx = \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \frac{x^2}{4} + C.$$

2. Verify that the average value of a linear function $f(x) = ax+b$ over the interval $[x_1, x_2]$ is equal to the value of the function at the midpoint of the interval.

Solution:

$$\begin{aligned} \text{Average value} &= \frac{1}{x_2 - x_1}\int_{x_1}^{x_2}(ax+b)dx = \frac{1}{x_2 - x_1}\left(\frac{ax^2}{2} + bx\right)\Bigg|_{x_1}^{x_2} \\ &= \frac{1}{x_2 - x_1}\left(\frac{ax_2^2}{2} + bx_2 - \frac{ax_1^2}{2} - bx_1\right) \\ &= \frac{1}{x_2 - x_1}\left(\frac{a}{2}(x_2^2 - x_1^2) + b(x_2 - x_1)\right) \\ &= \frac{a}{2}(x_2 + x_1) + b = f\left(\frac{x_1 + x_2}{2}\right) \end{aligned}$$

where $\frac{x_1 + x_2}{2}$ is indeed the midpoint of interval $[x_1, x_2]$.

3. Find arc-length of $y = \dfrac{\sqrt{x}(3-x)}{3}$ for $0 \le x \le 3$.

Solution:

Recall that arc-length of $y = f(x)$ from $x = a$ to $x = b$ is given by the integral:

$$\int_a^b \sqrt{1 + (y')^2}dx.$$

Let us now compute the necessary quantities for the given curve:

$$y' = \frac{1}{3}\frac{1}{2\sqrt{x}}(3-x) + \frac{1}{3}\sqrt{x}(-1) = \frac{1}{2\sqrt{x}} - \frac{\sqrt{x}}{6} - \frac{\sqrt{x}}{3} = \frac{1}{2\sqrt{x}} - \frac{\sqrt{x}}{2},$$

$$(y')^2 = \frac{1}{4x} - 2\frac{1}{2\sqrt{x}}\frac{\sqrt{x}}{2} + \frac{x}{4} = \frac{1}{4x} - \frac{1}{2} + \frac{x}{4},$$

$$1 + (y')^2 = \frac{1}{4x} + \frac{1}{2} + \frac{x}{4} = \left(\frac{1}{2\sqrt{x}} + \frac{\sqrt{x}}{2}\right)^2 \text{ then } \sqrt{1+(y')^2} = \frac{1}{2\sqrt{x}} + \frac{\sqrt{x}}{2}.$$

Now we are ready to compute the required arc-length L:

$$L = \int_0^3 \sqrt{1+(y')^2}dx = \int_0^3 \frac{1}{2\sqrt{x}} + \frac{\sqrt{x}}{2}dx = \frac{x^{\frac{1}{2}}}{2}2 + \frac{x^{\frac{3}{2}}}{2}\frac{2}{3}\Bigg|_0^3 = 2\sqrt{3}.$$

4. **(a)** Show that $(\arctan x)' = \dfrac{1}{1+x^2}$.

Solution:

$(\tan(\arctan x))' = x'$ (taking the derivative of both sides)

$[1 + (\tan(\arctan x))^2](\arctan x)' = 1,$

Thus $(\arctan x)' = \dfrac{1}{1+x^2}$.

(b) Find $((\tan x)^{\arctan x})'$.

Solution:

We will use logarithmic differentiation:

$y = (\tan x)^{\arctan x} \Rightarrow \ln y = \arctan x \ln(\tan x)$.

Now take derivatives of both sides:

$$\frac{y'}{y} = \frac{1}{1+x^2}\ln(\tan x) + \arctan x \frac{1}{\tan x}(1 + (\tan x)^2),$$

$$y' = (\tan x)^{\arctan x}\left(\frac{1}{1+x^2}\ln(\tan x) + \arctan x\frac{1}{\tan x}(1+(\tan x)^2)\right).$$

(c) Find $\lim\limits_{x\to\infty}\left(\cos\dfrac{1}{x}\right)^x$.

Solution:

Instead of the function itself, it advantageous to consider its logarithm, we then convert back via exponentiation.

$$y = \left(\cos\frac{1}{x}\right)^x \Rightarrow \ln y = x \ln\cos\frac{1}{x}$$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty} x\ln\cos\frac{1}{x} = \lim_{h\to 0+}\frac{\ln\cos h}{h} \text{ (where } h = 1/x)$$

$$\Rightarrow \lim_{h\to 0+}\frac{-\frac{\sin h}{\cos h}}{1} = 0 \Rightarrow \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^0 = 1.$$

B U Department of Mathematics

Math 101 Calculus I

Spring 2001 Second Midterm

1. Find the integral $\int_1^3 \sqrt{x}\tan^{-1}\sqrt{x}dx$.

Solution:

We use integration by parts:
Let $u = \tan^{-1}\sqrt{x}$ and $dv = \sqrt{x}$.
Then $\dfrac{du}{dx} = \dfrac{1}{1+x}\dfrac{1}{2\sqrt{x}}$ and $v = \dfrac{x^{3/2}}{3/2}$.

$$\begin{aligned}
\int_1^3 \sqrt{x}\tan^{-1}\sqrt{x}dx &= \tan^{-1}\sqrt{x}\frac{x^{3/2}}{3/2}\bigg|_1^3 - \frac{2}{3}\int_1^3 x^{3/2}\frac{1}{1+x}\frac{1}{2\sqrt{x}}dx \\
&= \tan^{-1}\sqrt{x}\frac{2}{3}x^{3/2}\bigg|_1^3 - \frac{1}{3}\int_1^3\left(1-\frac{1}{1+x}\right)dx \\
&= \left[\tan^{-1}\sqrt{x}\frac{2}{3}x^{3/2} - \frac{1}{3}(x-\ln|1+x|)\right]_1^3 \\
&= 2\sqrt{3}\tan^{-1}\sqrt{3} - \frac{2}{3}\tan^{-1}1 - \frac{1}{3}(3-\ln|4|-1+\ln|2|) \\
&= 2\sqrt{3}\frac{\pi}{3} - \frac{\pi}{6} - \frac{2}{3} + \frac{\ln 2}{3}.
\end{aligned}$$

2. Find the integral $\int \dfrac{dx}{(9x^2-1)^{3/2}}$.

Solution:

Using trigonometric substitution: $x = \dfrac{1}{3}\sec\theta \Rightarrow \dfrac{dx}{d\theta} = \dfrac{1}{3}\sec\theta\tan\theta$. Hence:

$$\begin{aligned}
I = \int \frac{dx}{(9x^2-1)^{3/2}} &= \int \frac{dx}{27(x^2-\frac{1}{9})^{3/2}} = \int \frac{\frac{1}{3}\sec\theta\tan\theta d\theta}{\tan^3\theta} \\
&= \frac{1}{3}\int \frac{\sec\theta}{\tan^2\theta}d\theta = \frac{1}{3}\int\frac{\cos\theta}{\sin^2\theta}d\theta.
\end{aligned}$$

Now let $u = \sin\theta$, then $\dfrac{du}{d\theta} = \cos\theta$. Hence, the integral becomes:

$$I = \frac{1}{3}\int\frac{du}{u^2} = -\frac{1}{3u} + C = -\frac{1}{3\sin\Theta} + C = -\frac{x}{3\sqrt{x^2-\frac{1}{9}}} + C.$$

3. Find $f(x)$, given that $f(1) = 0$ and that $y = f(x)$ satisfies the differential equation $\frac{dy}{dx} = xe^{x-y}$.

Solution:

We can separate the variables:

$$\frac{dy}{dx} = x\frac{e^x}{e^y} \Rightarrow \int e^y\,dy = \int xe^x\,dx = xe^x - \int e^x\,dx$$

integrating the right hand side by parts letting $u = x$ and $dv = e^x dx$. Finishing the integration we get:

$$e^y = xe^x - e^x + C \Rightarrow y = \ln(xe^x - e^x + C)$$

where C is an integration constant and to be determined by the given condition: $f(1) = \ln(e - e + C) = 0 \Rightarrow C = 1$ so that the unique solution becomes:

$$y = \ln(xe^x - e^x + 1).$$

4. Evaluate $\frac{d}{dx}\int_{3x}^{x^2} \frac{t-1}{t+1}dt$.

Solution:

$$\frac{d}{dx}\int_{3x}^{x^2} \frac{t-1}{t+1}dt = \frac{d}{dx}\left[\int_{3x}^{a} \frac{t-1}{t+1}dt + \int_{a}^{x^2} \frac{t-1}{t+1}dt\right]$$

$$= \frac{d}{dx}\left[-\int_{a}^{3x} \frac{t-1}{t+1}dt + \int_{a}^{x^2} \frac{t-1}{t+1}dt\right] = -3\left(\frac{3x-1}{3x+1}\right) + \left(\frac{x^2-1}{x^2+1}\right)2x,$$

by using the fundamental theorem of calculus.

5. Evaluate the following limits, if they exist:

(a) $\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos 2x}$.

(b) $\lim_{x\to\infty} \left(\cos\frac{1}{x}\right)^x$.

Solution:

(a) This is indeterminate of the form $\frac{0}{0}$. Applying L'Hopital's Rule we get:

$$\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos 2x} = \lim_{x\to 0} \frac{e^x - e^{-x}}{2\sin 2x} = \lim_{x\to 0} \frac{e^x + e^{-x}}{4\cos 2x} = \frac{1}{2}.$$

(b) This is of the form 1^∞, so it is good to consider the logarithm of the given function:

$$L = \lim_{x\to\infty} \ln\left(\cos\frac{1}{x}\right)^x = \lim_{x\to\infty} x\ln\left(\cos\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln(\cos 1/x)}{1/x}.$$

Apply L'Hopital's rule since the limit has the indeterminate form $\frac{0}{0}$:

$$L = \lim_{x\to\infty} \frac{\frac{1}{\cos 1/x}(-\sin 1/x)(-1/x^2)}{-1/x^2} = \lim_{x\to\infty} \frac{-\sin 1/x}{\cos 1/x} = 0.$$

Hence the desired limit is:

$$\lim_{x \to \infty} \left(\cos \frac{1}{x} \right)^x = e^0 = 1.$$

B U Department of Mathematics

Math 101 Calculus I

Spring 2002 Second Midterm

1. Given $f(1) = 3,\ f(4) = 7,\ f(4),\ f(14) = 23$, evaluate $\displaystyle\int_1^2 (x^2+1)f'(x^3+3x)dx$.

Solution:

Let $u = x^3 + 3x$ then $du = 3(x^2+1)dx$. So the integral becomes:

$$\int_1^2 (x^2+1)f'(x^3+3x)dx = \frac{1}{3}\int_4^{14} f'(u)du = \frac{1}{3}[f(14) - f(4)] = \frac{1}{3}[23-7] = \frac{16}{3}.$$

2. Evaluate the following limits.

(a) $\displaystyle\lim_{x\to 0}(e^x + x)^{\frac{1}{x}}$.

Solution:

$\displaystyle\lim_{x\to 0}(e^x+x)^{\frac{1}{x}} = [1^\infty]$ exponential type indeterminacy. Instead of $y = (e^x+x)^{\frac{1}{x}}$ let us consider its logarithm $\ln y = \dfrac{\ln(e^x+x)}{x}$:

$$\lim_{x\to 0}(\ln y) = \lim_{x\to 0}\frac{\ln(e^x+x)}{x} = \left[\frac{0}{0}\right] = \lim_{x\to 0}\frac{\frac{e^x+1}{e^x+x}}{1} = \frac{2}{1} = 2.$$

Now exponentiate this to find the desired limit:

$$\lim_{x\to 0} y = \lim_{x\to 0}(e^x+x)^{\frac{1}{x}} = e^2.$$

(b) $\displaystyle\lim_{x\to 0+}\frac{1}{x}\int_0^x e^{t^2}dt$.

Solution:

We use the fundamental theorem of calculus when applying L'Hopital's rule:

$$\lim_{x\to 0+}\frac{1}{x}\int_0^x e^{t^2}dt = \lim_{x\to 0+}\frac{\displaystyle\int_0^x e^{t^2}dt}{x} = \left[\frac{0}{0}\right] = \lim_{x\to 0+}\frac{e^{x^2}}{1} = 1.$$

(c) $\lim_{n\to\infty}\sum_{k=1}^{n}\ln\sqrt[n]{2+\frac{k}{n}}$ (Hint:Recognize it as a Riemann sum).

Solution:

Observe $\lim_{n\to\infty}\sum_{k=1}^{n}\ln\sqrt[n]{2+\frac{k}{n}} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\ln\left(2+\frac{k}{n}\right)$. This is the Riemann sum for $f(x)=\ln x$ over $[2,3]$. Then

$$\lim_{n\to\infty}\sum_{k=1}^{n}\ln\sqrt[n]{2+\frac{k}{n}} = \int_2^3 \ln x dx.$$

We use integration by parts: let $\ln x = u$ and $dx = dv$ then $du = \frac{1}{x}dx$ and $v = x$. Then:

$$\int_2^3 \ln x dx = x\ln x|_2^3 - \int_2^3 dx = 3\ln 3 - 2\ln 2 - (3-2) = \ln\frac{27}{4} - 1.$$

3. Find the integrals:

(a) $\int \frac{e^{\frac{x}{2}}}{1+e^{\frac{x}{3}}}dx$.

Solution:

Let $u = e^{\frac{x}{6}} \Rightarrow \begin{cases} u^6 = e^x \\ u^3 = e^{\frac{x}{2}} \\ u^2 = e^{\frac{x}{3}} \end{cases}$.

Computing the differential: $6u^5 = e^x\frac{du}{dx} \Longrightarrow 6u^5du = u^6dx$ which means $\frac{6}{u}du = dx$. Plugging into the integral:

$$\begin{aligned}\int \frac{e^{\frac{x}{2}}}{1+e^{\frac{x}{3}}}dx &= 6\int \frac{u^2}{u^2+1}du = 6\int\left(1-\frac{1}{u^2+1}\right) \\ &= 6[u - \arctan u] + c = 6[e^{\frac{x}{6}} - \arctan e^{\frac{x}{6}}] + c.\end{aligned}$$

(b) $\int x\arcsin x dx$.

Solution:

By integration by parts: $\arcsin x = u$ and $xdx = dv \Longrightarrow du = \frac{dx}{\sqrt{1-x^2}}$ and $v = \frac{x^2}{2}$. Rewriting the integral:

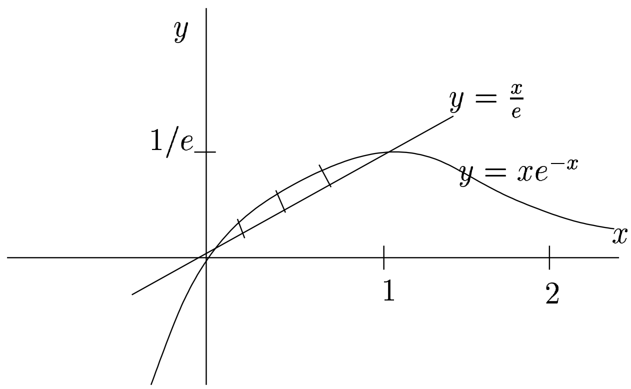
$$I = \int x\arcsin x dx = \frac{1}{2}x^2\arcsin x - \int \frac{x^2}{\sqrt{1-x^2}}dx.$$

Setting $x = \sin\theta$ we get:

$$\begin{aligned} I &= \frac{1}{2}x^2 \arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\cos\theta}\cos\theta d\theta \\ &= \frac{1}{2}x^2 \arcsin x - \frac{1}{4}\int (1-\cos 2\theta) d\theta \\ &= \frac{1}{2}x^2 \arcsin x - \frac{1}{4}[\theta - \frac{1}{2}\sin 2\theta] + c \\ &= \frac{1}{2}x^2 \arcsin x - \frac{1}{4}[\arcsin x - x\sqrt{1-x^2}] + c. \end{aligned}$$

4. Find the area bounded by the graphs of $y = xe^{-x}$ and $y = \dfrac{x}{e}$.

Solution:



We make a picture in order to see the situation. The second one is line through the origin. For $y = xe^{-x}$, we look at derivatives and asymptotes: $y' = e^{-x} - xe^{-x} = e^{-x}(1-x) \Longrightarrow y'' = e^{-x}(x-2)$. Moreover $\lim_{x\to-\infty} y = -\infty$, and $\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \dfrac{x}{e^x} = 0$.

We now check the intersection points of these two curves:

$$\frac{x}{e} = xe^{-x} \Longrightarrow x(1-e^{1-x}) = 0 \Longrightarrow x = 0,\ x = 1.$$

Then the area A bounded by both curves reads:

$$\begin{aligned} A &= \int_0^1 \left(xe^{-x} - \frac{x}{e}\right) dx = \int_0^1 xe^{-x} dx - \left.\frac{x^2}{2e}\right|_0^1 \\ &= -xe^{-x}|_0^1 + \int_0^1 e^{-x} dx - \left.\frac{x^2}{2e}\right|_0^1 \\ &= -e^{-x}(x+1)|_0^1 - \left.\frac{x^2}{2e}\right|_0^1 = -\frac{2}{e} + 1 - \frac{1}{2e} = \frac{2e-5}{2e}. \end{aligned}$$

5. The region between the graphs of $y = x^2$ and $y = \frac{1}{2} - x^2$ is revolved about the $y-$axis to form a lens. Calculate the volume of the lens.

Solution:

We first make a picture to see this area and find the intersection points that we need:

$$x^2 = \frac{1}{2} - x^2 \Longrightarrow x^2 = \frac{1}{4} \Longrightarrow x = \pm\frac{1}{2}.$$

We now apply the shell method:

$$\begin{aligned} V &= 2\pi \int_0^{\frac{1}{2}} x \left(\frac{1}{2} - x^2 - x^2\right) dx = 2\pi \int_0^{\frac{1}{2}} \left(\frac{x}{2} - 2x^3\right) dx \\ &= 2\pi \left[\frac{x^2}{4} - 2\frac{x^4}{4}\right]\Bigg|_0^{1/2} = 2\pi \left[\frac{1}{16} - \frac{1}{32}\right] = \frac{\pi}{16}. \end{aligned}$$

B U Department of Mathematics

Math 101 Calculus I

Spring 2004 Second Midterm

1. Evaluate the following limits, if they exist (justify your answer).

a) $\lim_{x\to\infty} x\tan^{-1}(\frac{2}{x})$.

Solution:

$$\lim_{x\to\infty} x\tan^{-1}(\frac{2}{x}) = [\infty\cdot 0]$$

$$= \lim_{x\to\infty}\frac{\tan^{-1}(\frac{2}{x})}{\frac{1}{x}} = \left[\frac{0}{0}\right]$$

By L'Hospital's $$= \lim_{x\to\infty}\frac{1}{1+(\frac{4}{x^2})}\cdot(-\frac{2}{x^2})\cdot\frac{1}{(-\frac{1}{x^2})} = \lim_{x\to\infty}\frac{2}{1+\frac{4}{x^2}} = 2$$

b) $\lim_{x\to\infty}(x+e^x)^{\frac{2}{x}}$

Solution:

$$\lim_{x\to\infty}(x+e^x)^{\frac{2}{x}} = [\infty^0]$$

For $y = (x+e^x)^{\frac{2}{x}} \Rightarrow lny = \frac{2ln(x+e^x)}{x} \to \left[\frac{\infty}{\infty}\right] \quad (x\to\infty)$

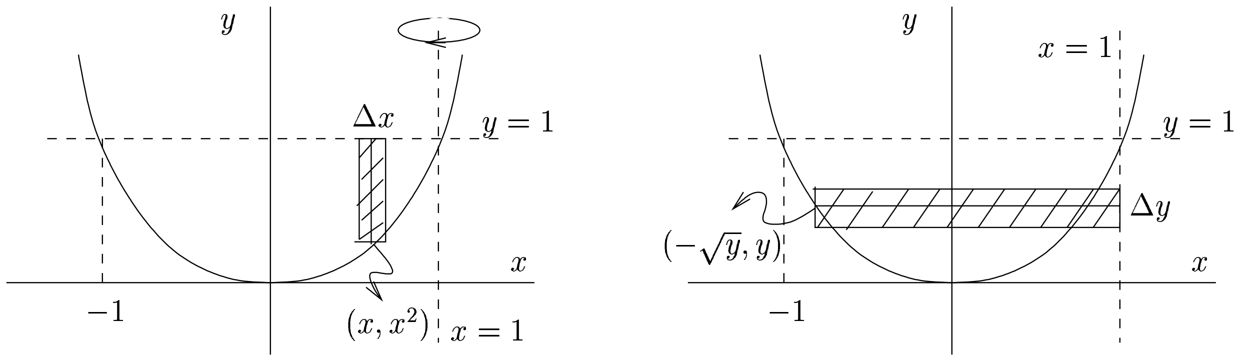
By L'H $$\lim_{x\to\infty}(lny) = \lim_{x\to\infty}\frac{2\left(\frac{1+e^x}{x+e^x}\right)}{1} = \left[\frac{\infty}{\infty}\right]$$

By L'H $$\lim_{x\to\infty} 2\cdot\frac{e^x}{1+e^x} = 2$$

So $$\lim_{x\to\infty}(x+e^x)^{\frac{2}{x}} = e^2$$

2. Find the volume of the solid generated by revolving the region bounded by the parabola $y = x^2$ and the line $y = 1$ about the line $x = 1$.

Solution:



By Shell Method: $V = 2\pi \int_{-1}^{1} (1-x)(1-x^2)dx = \dfrac{8\pi}{3}$

By Disc Method: $V = \pi \int_{0}^{1} \left[(1+\sqrt{y})^2 - (1-\sqrt{y})^2\right] dy = \dfrac{8\pi}{3}$

3. Find the length of the curve $y = \ln(\cos x)$ $0 \le x \le \dfrac{\pi}{3}$.

Solution:

$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$$
$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2 x = \sec^2 x$$
$$l = \int_0^{\frac{\pi}{3}} \sqrt{\sec^2 x} dx = \ln|\sec x + \tan x|\Big|_0^{\frac{\pi}{3}} = \ln(2+\sqrt{3}).$$

4. Evaluate the integrals;

a) $\int \cos(\sqrt{x}\,)dx$

Solution:

Put $t = \sqrt{x}$. We get $t^2 = x$, $2tdt = dx$ and

$I = \int \cos(\sqrt{x}\,)dx = 2\int t \cdot \cos t \cdot dt.$

Now let $t = u$ and $\cos t \cdot dt = dv$ so that $du = dt$ and $v = \sin t$. Then,

$I = 2\left[t \sin t - \int \sin t dt\right]$

$= 2\,(t \sin t + \cos t) + C$

$= 2\,(\sqrt{x} \sin(\sqrt{x}\,) + \cos(\sqrt{x}\,)) + C.$

b) $\int_0^2 \dfrac{dx}{|1-x^2|}$

Solution:

$$\int_0^2 \frac{dx}{|1-x^2|} = \int_0^1 \frac{dx}{1-x^2} - \int_1^2 \frac{dx}{1-x^2}$$

$$f(x) = \frac{1}{1-x^2} = \frac{1/2}{1-x} + \frac{1/2}{1+x}$$

and $\displaystyle \int \frac{dx}{1-x^2} = -\frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+x| + C = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|.$

$$\int_0^1 \frac{dx}{1-x^2} = \lim_{a\to 1^-} \int_0^a \frac{dx}{1-x^2} \qquad \text{(improper integral)}$$

$$= \lim_{a\to 1^-} \frac{1}{2}\ln\left|\frac{1+a}{1-a}\right| = -\infty.$$

$$-\int_1^2 \frac{dx}{1-x^2} = \lim_{b\to 1^+} \int_b^2 \frac{-dx}{1-x^2} = \lim_{b\to 1^+} \left[\frac{1}{2}\ln\left|\frac{1+b}{1-b}\right| - \frac{1}{2}\ln 3\right] = -\infty.$$

So the improper integrals both diverge. Hence the given improper integral diverges.

B U Department of Mathematics

Math 101 Calculus I

Spring 2005 Second Midterm

1. Evaluate the integrals below:

(a)[10] $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

Solution:

$$\text{Let } I = \int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$$

$$\ln x = u, \frac{dx}{x} = du$$

$$x = e \Rightarrow u = 1, x = e^4 \Rightarrow u = 4$$

$$\Rightarrow I = \int_1^4 \frac{du}{\sqrt{u}} = 2\sqrt{u}\ \Big]_1^4 = 2.$$

(b)[10] $\displaystyle\int \frac{\sin 2\theta}{\sin^2\theta + 1}\ d\theta$

Solution:

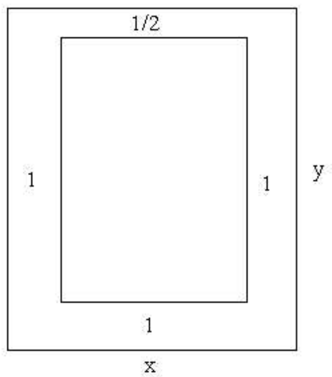
$$sin^2\theta + 1 = u \Rightarrow 2\sin\theta\cos\theta d\theta = \sin 2\theta d\theta = du$$

$$\Rightarrow \int \frac{\sin 2\theta}{\sin^2\theta + 1}\ d\theta = \int \frac{du}{u} = \ln|u| + C = \ln|sin^2\theta + 1| + C = \ln(sin^2\theta + 1) + C$$

since $sin^2\theta + 1 > 0$

2.[25] A page has an area of $90\,\mathrm{cm}^2$. A text is to be printed onto this page with $1\,\mathrm{cm}$ margins at the bottom and both sides and a $\frac{1}{2}\,\mathrm{cm}$ margin at the top. Find the dimensions of the page that will allow the largest printed area.

Solution:



$90 = xy$ fixed. Area to be fixed maximized is

$A = (x-2)(y-3/2)$, writing $y = 90/x, \ \ x \neq 0$

$\Rightarrow A = (x-2)(\frac{90}{x} - \frac{3}{2})$

$A = 90 - \frac{3}{2}x - \frac{180}{x} + 3 = 93 - \frac{3}{2}x - \frac{180}{x}$

$y \geq \frac{3}{2}, x \geq 2,$

If $y = 3/2 \Rightarrow x = 90 \cdot \frac{2}{3}$. Hence, $2 \leq x \leq 60$

$A' = -\frac{3}{2} + \frac{180}{x^2} = 0 \Rightarrow x^2 = 120.$

$x = \sqrt{120} = 2\sqrt{30}$ since $x > 0$

There are three values to check:

Critical point:
$x = 2\sqrt{30} \Rightarrow y = \dfrac{90}{2\sqrt{30}} = \dfrac{45}{\sqrt{30}}.$

Boundary points:

$x = 2 \Rightarrow y = 45$ and $A = 0$ $[\lim_{x\to 2^+} A = 0]$

$x = 60 \Rightarrow y = 3/2$ and $A = 0$ $[\lim_{x\to 60^-} A = 0]$

Therefore, dimensions $x = 2\sqrt{30}, y = \dfrac{45}{\sqrt{30}}$ give maximum printed area.

$A = 93 - \frac{3}{2} \cdot 2\sqrt{30} - \dfrac{180}{2\sqrt{30}} = 93 - 6\sqrt{30}$

3. Evaluate the following limits:

(a)[13] $\lim_{x\to 0^+}(\sin x)^x$

Solution:

Let $y = (sinx)^x$

$\lim_{x\to 0^+} y$ gives $[0^0]$. Take log of y:

$\ln y = x \ln \sin(x) \to 0 \cdot \infty \;\; [x \ln \sin(x) \geq 0 \text{ because } \cos x \to 0^+]$

$$\lim_{x\to 0^+} \frac{\ln \sin x}{1/x} \quad [\frac{\infty}{\infty}] = \lim_{x\to 0^+} \frac{\frac{\cos x}{\sin x}}{\frac{-1}{x^2}} = -\lim_{x\to 0^+} \frac{x^2 \cos x}{\sin x} = -\lim_{x\to 0^+} \frac{x}{\sin x} \cdot x \cdot \cos x$$

$$\Rightarrow \lim_{x\to 0^+} lny = -\lim_{x\to 0^+} \frac{x}{\sin x} \cdot \lim_{x\to 0^+} x \cdot \lim_{x\to 0^+} \cos x = 1 \cdot 0 \cdot 1 = 0.$$

$$\Rightarrow \lim_{x\to 0^+} y = \lim_{x\to 0^+} (\sin x)^x = e^0 = 1.$$

(b)[12] $\lim_{x\to 3} \frac{x^3 - 27}{\int_9^{3x} \cos \pi t^2 dt}$

Solution:

If we put $x = 3$, we get $\left[\frac{0}{0}\right]$

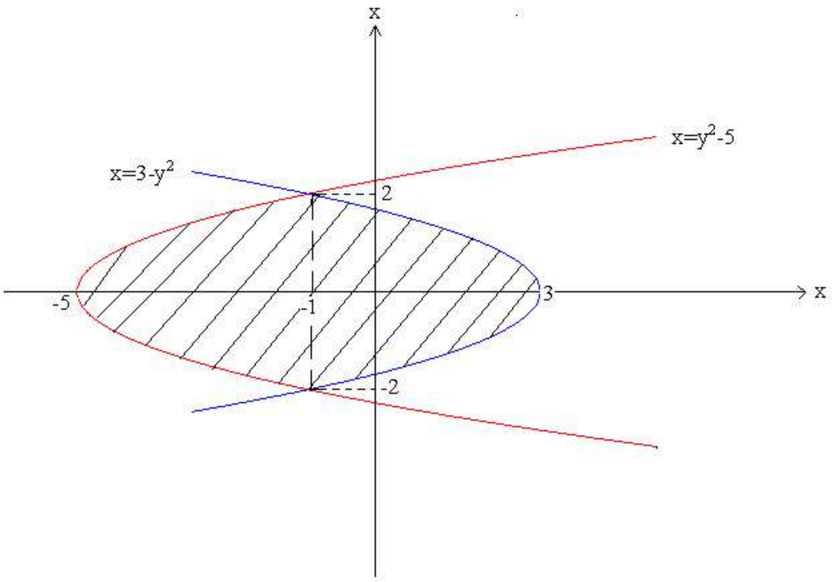
F.T.C must be used.

$$\lim_{x\to 3} \frac{x^3 - 27}{\int_9^{3x} \cos \pi t^2 dt} = \lim_{x\to 3} \frac{3x^2}{3\cos(\pi 9x^2)} = \frac{9}{\cos(\pi \cdot 81)} = -9.$$

4. Sketch the region enclosed by the graphs of $x = y^2 - 5$ and $x = 3 - y^2$.

(a)[10] Express the area of this region by setting up an integral with respect to x. (Do not evaluate.)

Solution:



Intersection points: $y^2 - 5 = 3 - y^2 \Rightarrow 2y^2 = 8 \Rightarrow y = \pm 2 \Rightarrow x = -1$.

$$A = \int_{-5}^{-1} \left(\sqrt{x+5} - \left(-\sqrt{x+5}\right)\right) dx + \int_{-1}^{3} \left(\sqrt{3-x} - (-\sqrt{3-x})\right) dx$$
$$= \int_{-5}^{-1} \left(2\sqrt{x+5}\right) dx + \int_{-1}^{3} \left(2\sqrt{3+x}\right) dx$$

(b)[10] Express the area of this region by setting up an integral with respect to y. (Do not evaluate.)

Solution:

$$A = \int_{-2}^{2} \left(3 - y^2\right) - \left(y^2 - 5\right) dy = \int_{-2}^{2} \left(8 - 2y^2\right) dy$$

(c)[10] Find the area of this region by evaluating one of the definite integrals found above. (Which one would you prefer? Of course, the simpler.)

Solution:

$$A = \int_{-2}^{2} \left(8 - y^2\right) dy = 8y - \frac{2}{3}y^3]_{-2}^{2}$$
$$= (16 - \frac{2}{3}8) - (-16 + \frac{2}{3}8) = \frac{64}{3}$$

B U Department of Mathematics

Math 101 Calculus I

Spring 2006 Second Midterm

1. Evaluate

(a) $\lim\limits_{x\to 0^+}\left[\sin(x^2)\right]^{1/\ln x}$

Solution:

$$y=\left(\sin x^2\right)^{1/\ln x}$$

$$\ln y=\frac{1}{\ln x}\ln\sin x^2 \Rightarrow \lim_{x\to 0^+}\ln y=\left[\frac{\infty}{\infty}\right]=\lim_{x\to 0^+}\frac{\dfrac{2x\cos x^2}{\sin x^2}}{1/x}$$

$$\ln y=\lim_{x\to 0^+}\frac{2x^2\cos x^2}{\sin x^2}=\lim_{x\to 0^+}\frac{x^2}{\sin x^2}\cdot\lim_{x\to 0^+}2\cos x^2=2$$

$$\lim_{x\to 0^+}\ln y=2\Rightarrow \ln\lim_{x\to 0^+}y=2\Rightarrow \lim_{x\to 0^+}y=e^2$$

$$\Rightarrow \lim_{x\to 0^+}\left(\sin x^2\right)^{1/\ln x}=e^2$$

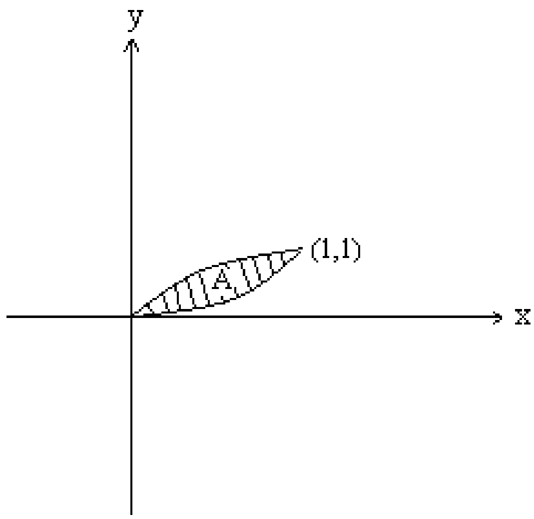
(b) $\lim\limits_{n\to\infty}\sum\limits_{j=1}^{n}\dfrac{1}{n+jc}$ for $c>1$

Solution:

$$\frac{1}{nj+c}=\frac{1/n}{1+\dfrac{jc}{n}}\Rightarrow \lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{n}\cdot\frac{1}{1+\dfrac{jc}{n}}=\int_0^1\frac{dx}{1+xc}$$

$$\int_0^1\frac{dx}{1+xc}=\frac{1}{c}\int_0^1\frac{c\,dx}{1+xc}=\frac{1}{c}\ln|1+xc|_0^1=\frac{1}{c}\ln(1+c)$$

2. (a) Find the area of the region R bounded by the curves of $y = x^2$ and $x = y^2$. Solution:



$$y = x^2 = y^4 \Rightarrow y - y^4 = 0 \Rightarrow y(1 - y^3) = 0, \quad y = 0, \; y = 1$$

$$A = \int_0^1 (\sqrt{x} - x^2)dx = \frac{2}{3}x^{3/2} - \frac{x^3}{3}|_0^1 = 2/3 - 1/3 = 1/3$$

(b) Find the value of $a > 0$ such that when the region bounded by the curve $y = 1 + \sqrt{x}e^{x^2}$, the line $y = 1$ and the line $x = a$ is rotated about the line $y = 1$, a volume of 2π is generated.

Solution:

$$V = \pi \int_0^a (\sqrt{x}e^{x^2})^2 dx = 2\pi$$

$$\pi \int_0^a xe^{2x^2} dx = 2\pi$$

$$2x^2 = u \Rightarrow 4xdx = du$$

$$\pi \int_0^{2a^2} \frac{e^u du}{4} = 2\pi$$

$$\frac{\pi}{4}(e^{2a^2} - 1) = 2\pi$$

$$e^{2a^2} - 1 = 8$$

$$e^{2a^2} = 9 \Rightarrow 2a^2 = \ln 9 = \ln 3^2 = 2\ln 3$$

$$a = \sqrt{\ln 3}$$

3. Integrate

(a) $\displaystyle\int \frac{5x}{(x+2)(x^2+1)}dx$

Solution:

$$I = \int \frac{5x}{(x+2)(x^2+1)}dx$$

$$\frac{5x}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+c}{x^2+1}$$

$$A(x^2+1) + (Bx+C)(x+2) = 5x$$

$$x = -2 \Rightarrow 5A = -10, A = -2$$

$$A + B = 0, \quad B = 2$$

$$x = 0 \Rightarrow A + 2C = 0, C = 1$$

$$I = \int \left(\frac{-2}{x+2} + \frac{2x}{x^2+1} + \frac{1}{x^2+1}\right) dx = -2\ln|x+2| + \ln(x^2+1) + \arctan x + C$$

$$I = \ln\frac{x^2+1}{(x+2)^2} + \arctan x + C$$

(b) $\displaystyle\int x \ln\sqrt{x}dx$

Solution:

$$I = \int x \ln\sqrt{x}dx$$

$$\sqrt{x} = u \Rightarrow \frac{1}{2\sqrt{x}}dx = du$$

$$x = u^2$$

$$I = \int u^2 \ln u\, 2udu = 2\int u^3 \ln udu = \left[\ln u \cdot \frac{u^4}{u} - \int \frac{u^4du}{4u}\right]$$

$$\ln u = U, \ \frac{1}{u}du = dU$$

$$u^3\, du = dV, \ \frac{u^4}{4} = V$$

$$\Rightarrow I = \frac{u^4 \ln u}{2} - \frac{u^4}{8} + C = \frac{x^2\ln\sqrt{x}}{2} - \frac{x^2}{8} + C$$

4. (a) If f and g are continuous functions on $[a,b]$ satisfying $\int_a^b f(x)dx = \int_a^b g(x)dx$, then show that there exists $c \in (a,b)$ such that $f(c) = g(c)$.

Solution:

Let $h(x) = f(x) - g(x)$, h is continuous so by Mean Value Theorem

$$\exists c \in (a,b) \;\; h(c) = \frac{1}{b-a}\int_a^b (f(t) - g(t))dt = \frac{1}{b-a}\left[\int_a^b f(t)dt - \int_a^b g(t)dt\right]$$

$$h(c) = \frac{1}{b-a} \cdot 0 = 0$$

So, $\exists c \in (a,b) \;\; h(c) = f(c) - g(c) = 0 \Rightarrow \exists c \in (a,b) \;\; f(c) = g(c)$

(b) Evaluate $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{\cos x}{\sin^2 x} dx$

Solution:

$$I = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{\sin^2 x} dx$$

This is an improper integral.

$$I = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{\sin^2 x} dx = \lim_{c \to 0^-} \int_{-\pi/2}^{c} \frac{\cos x}{\sin^2 x} dx + \lim_{b \to 0^+} \int_{b}^{\pi/2} \frac{\cos x}{\sin^2 x} dx$$

$$u = \sin x \Rightarrow du = \cos x dx, \;\; \int \frac{\cos x}{\sin^2 x} dx = \int \frac{du}{u^2} = -\frac{1}{u} + C$$

$$I = \lim_{c \to 0^-} \left(-\frac{1}{\sin x}\Big|_{-\frac{\pi}{2}}^{c}\right) + \lim_{b \to 0^+} \left(-\frac{1}{\sin x}\Big|_{b}^{\frac{\pi}{2}}\right)$$

$$\Rightarrow I = \lim_{c \to 0^-} -\frac{1}{\sin c} + (-1) + \lim_{b \to 0^+} -1 + \frac{1}{\sin b} = \infty, \text{ divergent.}$$

B U Department of Mathematics

Math 101 Calculus I

Spring 1999 Second Midterm

1. Evaluate $\lim_{h\to 0^+}\left(\frac{1}{h}\int_2^{2+h}\sqrt{t^2+2}\ dt\right)$.

Solution:

$$\lim_{h\to 0^+}\left(\frac{1}{h}\int_2^{2+h}\sqrt{t^2+2}\ dt\right)=\lim_{h\to 0^+}\frac{f(h)}{g(h)}$$

where $f(h)=\frac{1}{h}\int_2^{2+h}\sqrt{t^2+2}\ dt$ and $g(h)=h$.

$\lim_{h\to 0^+}\frac{f(h)}{g(h)}$ has $\frac{0}{0}$ indeterminacy. So L'Hôpital's rule can be applied:

$$\lim_{h\to 0^+}\left(\frac{1}{h}\int_2^{2+h}\sqrt{t^2+2}\right)\ dt=\lim_{h\to 0^+}\frac{\sqrt{(2+h)^2+2}.1-\sqrt{2^2+2}.0}{1}=\sqrt{6}.$$

2. Evaluate

a) $\lim_{n\to\infty}\frac{\pi}{n}\left[\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\sin\frac{3\pi}{n}+...+\sin\frac{n\pi}{n}\right]$

b) $\lim_{x\to\infty}\frac{x^2}{(\ln x)^3}$

Solution:

a) $\lim_{n\to\infty}\frac{\pi}{n}\left[\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\sin\frac{3\pi}{n}+...+\sin\frac{n\pi}{n}\right]=\lim_{n\to\infty}R_n$

where $R_n=\frac{\pi}{n}\left[\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\sin\frac{3\pi}{n}+...+\sin\frac{n\pi}{n}\right]$ is a Riemann sum for $f(x)=\sin x\quad x\in[0,\pi]$ with $\Delta x=\frac{\pi}{n}$. So

$$\lim_{n\to\infty}R_n=\int_0^{\pi}\sin x\ dx=-\cos x|_0^{\pi}=\cos 0-\cos\pi=2.$$

b) $\lim_{x\to\infty}\frac{x^2}{(\ln x)^3}$ has $\left[\frac{\infty}{\infty}\right]$ indeterminacy. Hence we can apply L'Hôpital's rule:

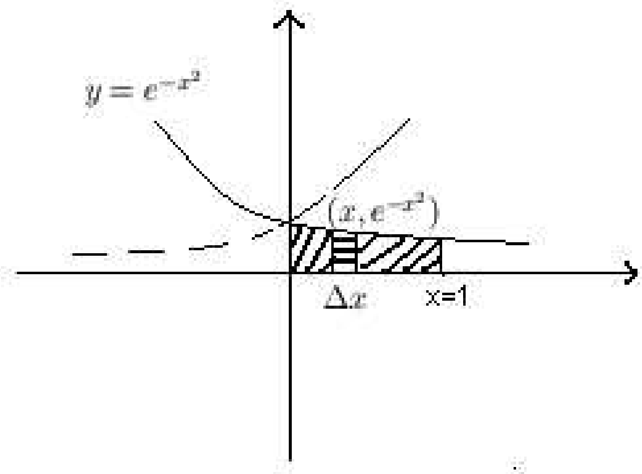
$\lim_{x\to\infty}\frac{x^2}{(\ln x)^3}=\lim_{x\to\infty}\frac{2x}{\frac{3(\ln x)^2)}{x}}=\lim_{x\to\infty}\frac{2x^2}{3(\ln x)^2}$ which has $\left[\frac{\infty}{\infty}\right]$ indeterminacy. Reapplying L'Hôpital's rule gives:

$\lim_{x\to\infty}\frac{2x^2}{3(\ln x)^2}=\lim_{x\to\infty}\frac{4x}{\frac{6\ln x}{x}}=\lim_{x\to\infty}\frac{2x^2}{3\ln x}$ has $\left[\frac{\infty}{\infty}\right]$ indeterminacy again. We apply l'Hôpital's rule for the last time to get:

$$\lim_{x\to\infty}\frac{2x^2}{3\ln x}=\lim_{x\to\infty}\frac{4x}{\frac{3}{x}}=\lim_{x\to\infty}\frac{4x^2}{3}=\infty.$$

3. Find the volume of the solid generated if the region bounded by the graphs of $y = e^{-x^2}$, $x = 0$, $x = 1$, $y = 0$ is revolved about y-axis.

Solution:



By Shell Method:

$V = 2\pi \int_0^1 xe^{-x^2}\,dx$. Put $u = -x^2$ so that $du = -2xdx$.

$\Rightarrow V = -\pi \int_0^{-1} e^u\,du = \pi(e^u)|_{-1}^{0} = \pi(1 - \frac{1}{e})$.

4. Evaluate

a) $\int \frac{dx}{e^x + e^{x/2}}$ b) $\int_0^1 x \ln x dx$

Solution:

a) $\int \frac{dx}{e^x + e^{x/2}}$ $u = e^{x/2}$ $2 \ln u = x$

$\int \frac{2du}{u(u^2 + u)} = \int \frac{2du}{u^2(u + 1)}$

$\frac{2}{u^2(u+1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+1}$

$2 = Au(u + 1) + B(u + 1) + Cu^2$

$u = 0 \Rightarrow B = 2$

$u = -1 \Rightarrow C = 2$

and $A + C = 0 \Rightarrow A = -2$.

$$\begin{aligned}\int \frac{2du}{u(u^2+u)} &= -\int \frac{2}{u}du + \int \frac{2}{u^2}du + \int \frac{2}{u+1}du \\ &= -2\ln|u| - \frac{2}{u} + 2\ln|u+1| + C = 2\ln\left|\frac{e^{x/2}+1}{e^{x/2}}\right| - \frac{2}{e^{x/2}} + C\end{aligned}$$

b) $\int_0^1 x\ln x dx$ is an improper integral.

Hence, $\int_0^1 x\ln x dx = \lim_{a\to 0^+} \int_a^1 x\ln x dx, \qquad (1 > a > 0).$

Using integration by parts: $u = \ln x,\ \ xdx = dv \Rightarrow du = \frac{dx}{x},\ \ v = \frac{x^2}{2}$

$$\begin{aligned}\int_0^1 x\ln x dx &= \lim_{a\to 0^+} \left[\frac{x^2\ln x}{2} - \int \frac{1}{2}xdx\right]_a^1 \\ &= \lim_{a\to 0^+} \left[-\frac{1}{4} - \frac{a^2\ln a}{2} + \frac{a^2}{1}\right] \\ &= -\frac{1}{4}.\end{aligned}$$

Note that

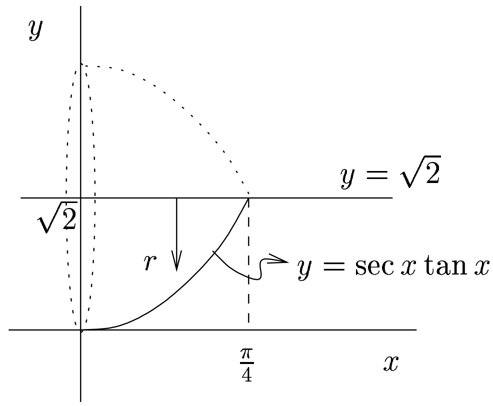
$$\begin{aligned}\lim_{a\to 0^+} a^2\ln a &= [0.\infty] = \lim_{a\to 0^+}\left[\frac{\ln a}{a^{-2}}\right] = \left[\frac{\infty}{\infty}\right] \\ &= \lim_{a\to 0^+} \frac{1/a}{-2a^{-3}} = -\frac{1}{2}a^2 = 0.\end{aligned}$$

B U Department of Mathematics
Math 101 Calculus I Summer 2004 Second Midterm

1. Find the volume of solid generated by revolving the region in the first quadrant bounded above by the line $y = \sqrt{2}$, below by the curve $y = \sec x \tan x$, and on the left by the y-axis, about the line $y = \sqrt{2}$.

Solution:

$$f(x) = \sec x \tan x, \qquad f(0) = 0,\ f(\tfrac{\pi}{4}) = \sqrt{2}.$$



$$\begin{aligned} V &= \pi \int_0^{\frac{\pi}{4}} r^2\,dx = \pi \int_0^{\frac{\pi}{4}} (\sqrt{2} - \sec x \tan x)^2\,dx \\ &= \pi \int_0^{\frac{\pi}{4}} (2 - 2\sqrt{2}\sec x \tan x + \sec^2 x \tan^2 x)\,dx \\ &= \pi \left[2x - 2\sqrt{2}\sec x \tan x + \frac{\tan^3 x}{3}\right]_0^{\frac{\pi}{4}} \\ &= \pi \left(\frac{\pi}{2} - 2\sqrt{2}.\sqrt{2} + \frac{1}{3} - (0 - 2\sqrt{2} + 0)\right) \\ &= \frac{\pi^2}{2} + \frac{\pi}{3}(6\sqrt{2} - 11) \end{aligned}$$

2. Evaluate:

(a) $\displaystyle\int_0^4 xe^{-x}\,dx$ **(b)** $\displaystyle\int_4^8 \frac{y}{y^2 - 2y - 3}\,dy$ **(c)** $\displaystyle\int_0^{\frac{\pi}{2}} \cot x\,dx$ **(d)** $\displaystyle\int \frac{x^3}{\sqrt{9 - x^2}}\,dx$

Solution:

(a) Take u=x and $e^{-x}\,dx = dv$. Then:

$$\begin{aligned} \int xe^{-x}\,dx &= -xe^{-x} - \int -e^{-x}\,dx \\ \int_0^4 xe^{-x}\,dx &= -xe^{-x} - e^{-x}\Big]_0^4 \\ &= (-4e^{-4} - e^{-4}) - (0 - 1) \\ &= 1 - 5e^{-4} \end{aligned}$$

(b)

$$\begin{aligned}
\frac{y}{y^2-2y-3} &= \frac{A}{y+1}+\frac{B}{y-3} \\
y &= \frac{A}{y+1}+\frac{B}{y-3} \\
3 &= 4B \implies B=\frac{3}{4} \\
-1 &= -4A \implies A=\frac{1}{4}
\end{aligned}$$

So,

$$\begin{aligned}
\int_4^8 \frac{y}{y^2-2y-3}\,dy &= \int_4^8 \left(\frac{1}{4}\,\frac{1}{y+1}+\frac{3}{4}\,\frac{1}{y-3}\right)dy \\
&= \left[\frac{1}{4}\ \ln|y+1|+\frac{3}{4}\ \ln|y-3|\ \right]_4^8 \\
&= \frac{1}{4}\ \ln 9+\frac{3}{4}\ \ln 5-\frac{1}{4}\ \ln 5-\frac{3}{4}\ \ln 1 \\
&= \frac{\ln 9}{4}+\frac{\ln 5}{2}
\end{aligned}$$

(c)

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \cot x\,dx &= \lim_{a\to 0^+}\int_a^{\frac{\pi}{2}} \frac{\cos x}{\sin x}\,dx \\
&= \lim_{a\to 0^+} \ln|\sin x|\ \Big]_a^{\frac{\pi}{2}} \\
&= \lim_{a\to 0^+} (\ln a) \\
&= +\infty,\quad \text{diverges.}
\end{aligned}$$

(d) Take $x=3\sin\theta$. Then,

$$\begin{aligned}
\int \frac{x^3}{\sqrt{9-x^2}}\,dx &= \int \frac{27\sin^3\theta}{\sqrt{9-9\sin^2\theta}}\,3\cos\theta\,d\theta \\
&= \int \frac{27\sin^3\theta}{3\cos\theta}\,3\cos\theta\,d\theta \\
&= \int 27\sin^3\theta\,d\theta \\
&= 27\int \sin\theta(1-\cos^2\theta)\,d\theta \\
&= 27\left(\int \sin\theta\,d\theta+\int \cos^2\theta(-\sin\theta)\,d\theta\ \right) \\
&= 27\left(-\cos\theta+\frac{\cos^3\theta}{3}\right) \\
&= 27\left(-\frac{(9-x^2)^{\frac{1}{2}}}{3}+\frac{(9-x^2)^{\frac{3}{2}}}{3\cdot 27}\right) \\
&= -9(9-x^2)^{\frac{1}{2}}+\frac{(9-x^2)^{\frac{3}{2}}}{3}+C
\end{aligned}$$

3. Determine whether the following series converge.

(a) $\sum_{n=1}^{\infty} \frac{1}{n(1+\ln^2 n)}$ (b) $\sum_{n=1}^{\infty} e^{-n}\, n^3$ (c) $\sum_{n=1}^{\infty} \frac{n^n}{(2^n)^2}$ (d) $\sum_{n=1}^{\infty} \frac{2n^2+3}{5n^3+4n+6}$

Solution:

(a) Integral test :

$$f(x) = \frac{1}{x(1+\ln^2 x)}, \quad x > 0. \qquad \text{clearly decreasing}$$

Take $u = \ln x$. Then:

$$\int_1^{\infty} \frac{1}{x(1+\ln^2 x)}\, dx = \int_0^{\infty} \frac{du}{1+u^2} = \tan^{-1} u \Big]_0^{\infty} = \frac{\pi}{2} \implies \text{series converges}$$

(b) Ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{e^{-(n+1)}(n+1)^3}{e^{-n}n^3} = \frac{(n+1)^3}{n^3}\frac{1}{e}$$

$$\implies \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{e} < 1 \implies \text{series converges}$$

(c) Root test:

$$\left(\frac{n^n}{(2^n)^2}\right)^{\frac{1}{n}} = \frac{n}{2^2} = \frac{n}{4} \longrightarrow +\infty \implies \text{series diverges}$$

(d) Limit comparison test:

Compare to $\sum_{n=1}^{\infty} \frac{1}{n}$. Observe that $\frac{(2n^2+3)}{5n^3+4n+6} n \longrightarrow \frac{2}{5}$.

$0 < \frac{2}{5} < 1 \implies$ series diverges since $\sum \frac{1}{n}$ diverges.