B U Department of Mathematics

Math 101 Calculus I

Fall 2000 First Midterm

1. **(a)** Find the value of a that makes the function

$$f(x) = \begin{cases} \frac{(1+x)^n - 1}{x} & \text{if} \quad x \neq 0 \\ a & \text{if} \quad x = 0 \end{cases}.$$

continous at x=0, where n is a positive integer.

Solution:

METHOD I:

$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k$, where $C(n,k) = \dfrac{n!}{k!(n-k)!}$ are the binomial coefficients.

$\Rightarrow (1+x)^n = 1 + nx + \dfrac{n(n-1)}{2}x^2 + ... + x^n$

$\Rightarrow \dfrac{(1+x)^n - 1}{x} = n+$(terms containing x)

Thus, $\lim\limits_{x \to \circ} \dfrac{(1+x)^n - 1}{x} = n + 0 = n \Rightarrow a = n$

METHOD II:

Using L'Hospital's Rule,

$\lim\limits_{x \to \circ} \dfrac{(1+x)^n - 1}{x} = \lim\limits_{x \to \circ} \dfrac{n(1+x)^{n-1} - 1}{1}$

Then, $\Rightarrow a = n$

METHOD III:

Consider the function $f(x) = (1+x)^n$ at the point x=0.f is differentiable at this point, and by the power rule

$f'(x) = n(1+x)^{n-1} \Rightarrow f'(0) = n$

On the other hand, using the definition of the derivative,

$f'(0) = \lim\limits_{h \to \circ} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to \circ} \dfrac{(1+h) - 1}{h}$, which is the same as the question.

$\Rightarrow \lim\limits_{x \to \circ} \dfrac{(1+x) - 1}{x} = f'(0) = n.$

Then, $\Rightarrow a = n$

(b) Find lim $\lim\limits_{x \to \circ} \dfrac{6x - \sin 2x}{2x + 3\sin 4x}$, if it exists.

Solution:

METHOD I:

$\lim\limits_{x \to \circ} \dfrac{6x - \sin 2x}{2x + 3\sin 4x} = \lim\limits_{x \to \circ} \dfrac{x(6 - \frac{2\sin 2x}{2x})}{x(2 + \frac{4.3\sin 4x}{4x})} = \dfrac{6-2}{2+14} = \dfrac{2}{7}$, using $\lim\limits_{x \to \circ} \dfrac{\sin x}{x} = 1$

METHOD I:

$\lim\limits_{x \to \circ} \dfrac{6x - \sin 2x}{2x + 3\sin 4x} = \lim\limits_{x \to \circ} \dfrac{6 - 2\cos 2x}{2 + 12\cos 4x} = \dfrac{6-2}{2+14} = \dfrac{2}{7}$

2. Consider the function $f(x) = \dfrac{2x^2+5x+1}{x-1}$

(a) What is the domain of f ?

Solution:

$$Df = \{x|x \neq 0\}.$$

(b) Find all horizontal, vertical and oblique (slant) asymptotes.

Solution:

$\lim_{x\to\infty} f(x) = \infty$ $\lim_{x\to-\infty} f(x) = -\infty$. Therefore , no horizontal asymptotes.
$x = 1$ is the vertical asymptote, because $\lim_{x\to1^+} f(x) = +\infty$ $\lim_{x\to1^-} f(x) = -\infty$.
Since $f(x) = 2x + 7 + \dfrac{8}{x-1}$, then $y = 2x + 7$ is the slant asymptote.

(c) Find the interval(s) on which f is increasing or decreasing.

Solution:

$f'(x) = 2 - \dfrac{8}{(x-1)^2} = \dfrac{2(x-1)^2 - 8}{(x-1)^2}$
$f'(x) = 0 \Rightarrow (x-1)^2 = 4 \Rightarrow$ Then $x = 3$ and $x = -1$ are critical points.
Increasing on $(-\infty, -1)$ and $(3, +\infty)$
Decreasing on (-1,1) and (1,3)

(d) Find all local extrema.

Solution:

$f''(x) = (-8)(-2)(x-1)^{-3} = \dfrac{16}{(x-1)^{-3}}$
$f''(3) > 0 \Rightarrow$ local minimum at $x = 3$.
$f''(-1) < 0 \Rightarrow$ local maximum at $x = -1$.

(e) Find the interval (c) where the graph of f is concave up and down.

Solution:

Concave up on $(1, \infty)$.
Concave down on $(-\infty, 1)$.

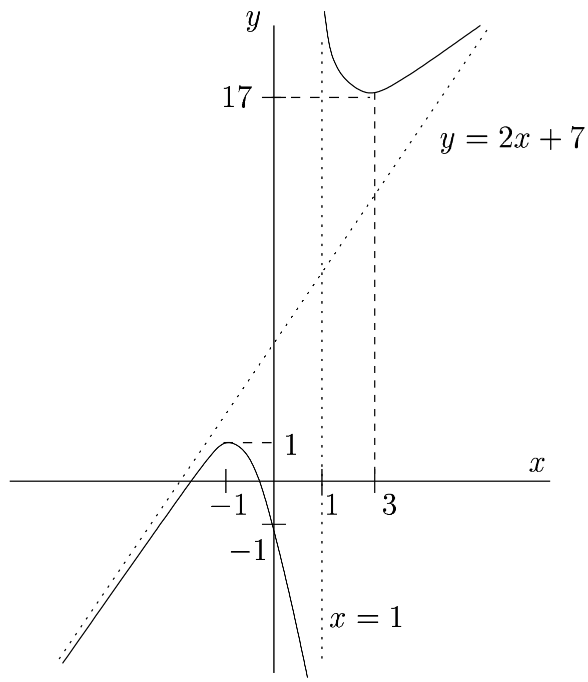
(f) Find the point(s) of inflection, if any.

Solution:

There is no inflection point.

(g) Sketch the graph of f.

Solution:



3. **(a)** Let

$$f(x) = \begin{cases} x^2+2 & \text{if} \quad x \neq 0 \\ 4x-2 & \text{if} \quad x = 0 \end{cases}.$$

Determine whether f has a tangent line at the point P(2,6); if so, find an equation of the tangent line.

Solution:

METHOD I:

For $x_0 > 2,\quad f'(x_0) = \left.\dfrac{d(x^2+2)}{dx}\right|_{x=x_0} = 2x_0.$

For $x_0 < 2,\quad f'(x_0) = \left.\dfrac{d(4x-2)}{dx}\right|_{x=x_0} = 4.$

Since $\lim_{x_0 \to 2} 2x_0 = 4$, the two one-sided limits agree at $x_0 = 2$, and $f'(2)$ exists.

Thus, the equation of the tangent line is $y - 6 = 4(x-2)$ or $y = 4x - 2$.

METHOD II:

Alternatively, one can show that $f'(2) = 2$ by evaluating the two one-sided limits.

$\lim_{\Delta x \to 0^+} \dfrac{\Delta f}{\Delta x}$ and $\lim_{\Delta x \to 0^-} \dfrac{\Delta f}{\Delta x}$ from the definition of the derivative.

(b) Find $\displaystyle\int \frac{x^2+1}{\sqrt{x}}dx$

Solution:

$$\int \frac{x^2+1}{\sqrt{x}}dx = \int x^{\frac{3}{2}}dx + \int x^{-\frac{1}{2}}dx = \frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{1}{2}} + C$$

4. Two ships leave from the same point at the same time. The angle between their paths is 60^0.One ship travels at 20 miles per hour and the other at 30 miles per hour. Find the rate of change of distance between the ships 2hrs after they seperated.

Solution:

At time t, first ship would travel $20t$ miles, and the second would travel $30t$ miles at time t. Since we are looking for the destance between these two ships at time t=2, by taking the derivative of distance function wrt t=2, then the rate of change will be found.

Using the Cosine theorem,

$x = (30t)^2 + (20t)^2 - 2 \times 30t \times 20t \times \cos 60 = 700t^2.$

$\left.\frac{dx}{dt}\right|_{t=2} = 1400t = 2800\text{miles/h}.$

B U Department of Mathematics

Math 101 Calculus I

Fall 2002 First Midterm

1. Evaluate the following limits :

(a) $\lim\limits_{x\to 0} \dfrac{\tan x}{\sqrt{1-\cos x}}$.

Solution:

$\lim\limits_{x\to 0} \dfrac{\tan x}{\sqrt{1-\cos x}}$ has $\left[\frac{0}{0}\right]$ type indeterminacy. We multiply and divide by by $\sqrt{1+\cos x}$:

$$\lim_{x\to 0} \frac{\tan x\sqrt{1+\cos x}}{\sqrt{1-\cos x}\sqrt{1+\cos x}} = \lim_{x\to 0} \frac{\tan x\sqrt{1+\cos x}}{\sqrt{1-\cos^2 x}} = \sqrt{2}\lim_{x\to 0} \frac{\tan x}{\sqrt{\sin^2 x}} = \sqrt{2}\lim_{x\to 0} \frac{\tan x}{|\sin x|}.$$

Now, we must consider two sided limit because of the absolute value (note that $\sin x > 0$ if $x > 0$ and $\sin x < 0$ if $x < 0$ provided that x is close to 0):

left-hand limit: $-\sqrt{2}\lim\limits_{x\to 0^-} \dfrac{\tan x}{\sin x} = -\sqrt{2}\lim\limits_{x\to 0^-} \dfrac{1}{\cos x} = -\sqrt{2}$.

right-hand limit: $\sqrt{2}\lim\limits_{x\to 0^+} \dfrac{\tan x}{\sin x} = \sqrt{2}\lim\limits_{x\to 0^+} \dfrac{1}{\cos x} = \sqrt{2}$.

Left-hand and right-hand limits are not equal, therefore limit does not exist.

(b) $\lim\limits_{x\to 0} \dfrac{\sin 4x - 2\sin 2x}{x^3}$.

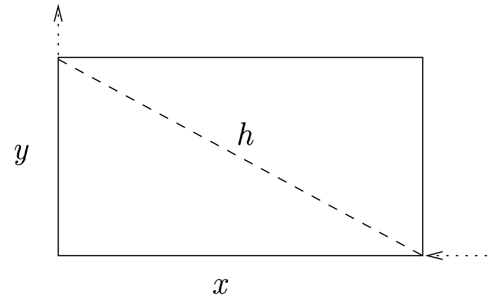
Solution:

$\lim\limits_{x\to 0} \dfrac{\sin 4x - 2\sin 2x}{x^3}$ has $\left[\frac{0}{0}\right]$ type indeterminacy. We use trigonometric relations and try to remove the indeterminacy. Let us denote the value of the limit by L. Then:

$$\begin{aligned} L &= \lim_{x\to 0} \frac{2\sin 2x\cos 2x - 2\sin 2x}{x^3} = \lim_{x\to 0} \frac{2\sin 2x(\cos 2x - 1)}{x^3} \\ &= \lim_{x\to 0} \frac{2\sin 2x(-2\sin^2 x)}{x^3} = -4\lim_{x\to 0} \frac{\sin 2x\sin^2 x}{xx^2} \\ &= -8\lim_{x\to 0} \frac{\sin 2x}{2x}\lim_{x\to 0} \frac{\sin^2 x}{x^2} = -8. \end{aligned}$$

2. One side of a right triangle decreases 1 cm/min and the other side increases 2 cm/min. At some time t_0 the first side is 8 cm and the second side is 4 cm long. How fast is the length of hypothenuse changing 2 minutes after t_0?

Solution:



We first note the hypothenuse relation: $h^2 = x^2 + y^2$. We are also given the rate of change of each side: $\frac{dx}{dt} = -1$ cm/min ($\searrow$) and $\frac{dy}{dt} = 2$ cm/min ($\nearrow$). At t_0 we know that $x = 8$ cm and $y = 4$ cm.

Related rates relation is obtained by differentiating the hypothenuse relation and given by:

$$h\frac{dh}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}.$$

Let t_1 denote the time 2 minutes after t_0. We need to find the values of x, y and h at t_1. Knowing the rates of change, this is easy: At t_1: $x = 8 - 2.1 = 6$ cm and $y = 4 + 2.2 = 8$ cm. So $h = 10$ cm by direct calculation. Then we use the related rates relation at t_1:

$$10\frac{dh}{dt} = 6.(-1) + 8.(2) = 10 \Longrightarrow \frac{dh}{dt} = 1 \text{ cm/min.}$$

This means h is increasing.

3. Graph the curve $y = f(x) = \frac{1-x^2}{x^3}$ by computing y' and y'', determining their signs, finding the critical and the inflection points, indicating the interval on which the function is increasing, decreasing, concave up, concave down, and finding the asymptotes.

Solution:

The function $y = \frac{1-x^2}{x^3}$ has $x-$intercepts: $y = 0 \Rightarrow x = \pm 1$ and no $y-$intercept.
At $x = 0$, y is undefined and hence $x = 0$ is a vertical asymptote. Checking the limits about the asymptote:

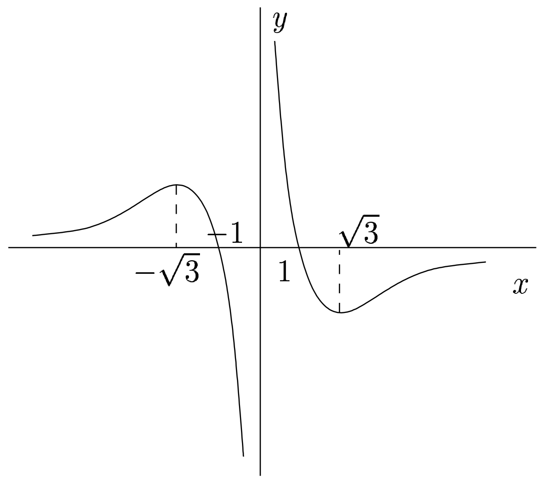
$$f(x) \to -\infty \text{ as } x \to 0^- \quad \text{and} \quad f(x) \to +\infty \text{ as } x \to 0^+.$$

Behaviour for large x: $\lim_{x\to\pm\infty} \frac{1-x^2}{x^3} = 0$ hence $y = 0$ is a horizontal asymptote.

First derivative: $y' = \frac{x^2-3}{x^4}$. Denominator does not affect the sign of y' ($x^4 > 0$).
$y' = 0 \Rightarrow x = \pm\sqrt{3}$ are critical points. Examining the sign of y' we get:
$y \nearrow$ on $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, +\infty)$ as $y' > 0$, and $y \searrow$ on $(-\sqrt{3}, +\sqrt{3})$ as $y' < 0$.
Hence, at $x = -\sqrt{3}$, y has a local maximum, and at $x = +\sqrt{3}$, y has a local minimum.

Second derivative: $y'' = -2\frac{x^2-6}{x^5}$.

$y'' = 0 \Rightarrow x = \pm\sqrt{6}$ are candidates for inflection points. Analyzing the sign of y'': $y'' > 0$ on $(-\infty, -\sqrt{6}) \cup (0, \sqrt{6})$ and hence y is concave-up. $y'' < 0$ on $(-\sqrt{6}, 0) \cup (\sqrt{6}, \infty)$ and hence y is concave-down. Then $x = \pm\sqrt{6}$ are inflection points indeed.



4. Suppose that the sum of the surface areas of a sphere and a cube is a constant c. Show that the sum of their volumes is smallest when the radius of the sphere is equal to the half of the length of an edge of the cube.

Solution:

$x \geq 0$: the length of an edge of the cube.
$r \geq 0$: the radius of the sphere.
Given that:

$$4\pi r^2 + 6x^2 = c = \text{constant}$$

Minimize the sum of the volumes:

$$V = \frac{4}{3}\pi r^3 + x^3$$

Let us choose r to be the independent variable. Critical points are found by:

$$\frac{dV}{dr} = V' = 4\pi r^2 + 3x^2\frac{dx}{dr} = 0.$$

Differentiating the first equation with respect to r we get :

$$8\pi r + 12x\frac{dx}{dr} = 0 \Longrightarrow \frac{dx}{dr} = -\frac{2\pi r}{3x}, \; x \neq 0.$$

Now substitute dx/dr in $V' = 0$ equation and solve:

$$V' = 2r^2 - rx = 0 \Rightarrow r = 0 \text{ or } r = \frac{x}{2}.$$

On the other hand we have boundaries for $0 \leq r \leq \frac{\sqrt{c}}{2\sqrt{\pi}}$, where the right boundary corresponds to $x = 0$ case. Then the points which possibly minimize V are:

$$P_1 \ : \ r = 0, \ x = \frac{\sqrt{c}}{\sqrt{6}} \quad \text{end point}$$

$$P_2 \ : \ r = \frac{\sqrt{c}}{2\sqrt{\pi}}, \ x = 0 \quad \text{end point}$$

$$P_3 \ : \ r = \frac{\sqrt{c}}{2\sqrt{\pi + 6}}, \ x = \frac{\sqrt{c}}{\sqrt{\pi + 6}} \quad \text{interior critical point.}$$

Evaluate V at P_1, P_2 and P_3 and compare to find:

$$V(P_3) < V(P_1) < V(P_2).$$

Hence V achieves its minimum at P_3 where $r = \dfrac{x}{2}$.

MATH 101 First Midterm Examination

November 3, 2003
17:00-18:00
Student number: 1
Name: Atilla Yılmaz
Signature: $\mathcal{AY}$

1	/15 points
2	/20 points
3	/20 points
4	/20 points
5	/25 points
Total	/100 points

In order to get full credit, you have to show your work and explain what you are doing. Calculators are not allowed, nor needed. The last page is blank. Good luck.

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(1) *Evaluate* $\lim_{t\to 0} \frac{\sin^2 t}{(1+\cos t)t}$. (5 points)

$$\lim_{t\to 0} \frac{\sin^2 t}{(1+\cos t)t} = \lim_{t\to 0} \sin t \frac{\sin t}{t} \frac{1}{1+\cos t}$$
$$= \lim_{t\to 0} \sin t \cdot \lim_{t\to 0} \frac{\sin t}{t} \cdot \lim_{t\to 0} \frac{1}{1+\cos t} = 0 \cdot 1 \cdot \frac{1}{1+1} = 0$$

or, alternatively

$$\lim_{t\to 0} \frac{\sin^2 t}{(1+\cos t)t} = \lim_{t\to 0} \frac{\sin^2 t}{(1+\cos t)t} \frac{1-\cos t}{1-\cos t}$$
$$= \lim_{t\to 0} \frac{1-\cos t}{t} \frac{\sin^2 t}{1-\cos^2 t} = \lim_{t\to 0} \frac{1-\cos t}{t} \cdot 1 = 0.$$

b) *Evaluate* $\lim_{x\to +\infty} \frac{\sin x}{x}$. (5 points)

For every $x > 0$, we have

$$\frac{-1}{x} \leqslant \frac{\sin x}{x} \leqslant \frac{1}{x}$$

and

$$\lim_{x\to +\infty} \frac{-1}{x} = 0, \qquad \lim_{x\to +\infty} \frac{1}{x} = 0,$$

so the squeezing theorem yields

$$\lim_{x\to +\infty} \frac{\sin x}{x} = 0.$$

c) *Find* $\frac{dh}{dx}$ *if* $h = \tan(\sqrt{x^2+\sin x})$ *wherever* $\frac{dh}{dx}$ *exists.* (5 points)

By the chain rule,

$$\frac{dh}{dx} = \sec^2(\sqrt{x^2+\sin x}) \cdot \frac{1}{2}(x^2+\sin x)^{-1/2} \cdot (2x+\cos x)$$
$$= \frac{1}{2} \frac{(2x+\cos x)\sec^2(\sqrt{x^2+\sin x})}{\sqrt{x^2+\sin x}}.$$

(2) *A string of length 18 m can be cut into two pieces to make a circle with one piece and a square with the other. How much string should be used for the circle if the total area enclosed by the figures is to be as large as possible?* (20 points)

Let x meters be used for the circle, $18 - x$ meters for the square. Then the radius of the circle is $\frac{x}{2\pi}$ m, the sides of the circle are $\frac{18-x}{4}$ m. The total area, in square meters, is

$$A(x) = \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{18-x}{4}\right)^2 = \frac{81}{4} - \frac{9}{4}x + (\frac{1}{4\pi} + \frac{1}{16})x^2$$

and $0 \leqslant x \leqslant 18$.

We find $A'(x) = -\frac{9}{4} + (\frac{1}{4\pi} + \frac{1}{16})2x = -\frac{9}{4} + (\frac{1}{2\pi} + \frac{1}{8})x$ and $A'(x) = 0$ when $x = x_0 := \frac{\frac{9}{4}}{\frac{1}{2\pi} + \frac{1}{8}} = \frac{18\pi}{4+\pi}$.

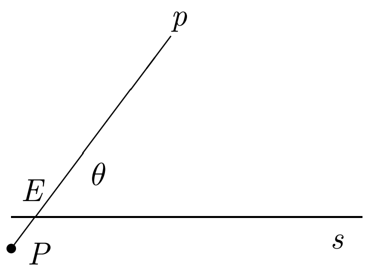
We compute and compare:

$$\begin{aligned}
A(0) &= \frac{81}{4} - \frac{9}{4}0 + (\frac{1}{4\pi} + \frac{1}{16})0^2 = \frac{81}{4},\\
A(18) &= \pi\left(\frac{18}{2\pi}\right)^2 + \left(\frac{18-18}{4}\right)^2 = \frac{81}{\pi} > \frac{81}{4} = A(0),\\
A(x_0) &= \pi\left(\frac{x_0}{2\pi}\right)^2 + \left(\frac{18-x_0}{4}\right)^2\\
&= \frac{1}{4\pi}\left(\frac{18\pi}{4+\pi}\right)^2 + \left(\frac{18 - \frac{18\pi}{4+\pi}}{4}\right)^2\\
&= \frac{1}{4}18^2\frac{\pi}{(4+\pi)^2} + \frac{18^2}{(4+\pi)^2}\\
&= \frac{18^2}{(4+\pi)^2}[\frac{\pi}{4} + 1] = \frac{81}{4+\pi} < \frac{81}{\pi} = A(18).
\end{aligned}$$

Thus 18 yields the absolute maximum of A on $[0, 18]$. We see that the whole piece of string must be used for the circle.

[Alternatively, at x_0, the second derivative of A is $A''(x_0) = (\frac{1}{2\pi} + \frac{1}{8}) > 0$, so x_0 is a relative minimum of A, it cannot be the absolute maximum of A on $[0, 18]$.]

(3) *A stick p has a pivot point P about which p rotates. The pivot P is one meter away from another stick s. Let E denote the intersection point of p and s, and let θ be the counterclockwise angle at E from s to p.*



What is the speed of the point E in terms of the rate of change of θ?

(10 points)

Let x be the distance between E and the foot of the perpendicular from P to s. Then

$$\frac{1}{x} = \tan\theta,$$
$$\frac{1}{x(t)} = \tan\theta(t)$$

and, differentiating with respect to t, we get

$$-\frac{1}{x^2}\frac{dx}{dt} = \sec^2\theta\frac{d\theta}{dt}.$$

The speed of E is

$$\frac{dx}{dt} = -x^2\sec^2\theta\frac{d\theta}{dt}.$$

If θ decreases at a rate of $1\,\text{rad/sec}$, what is the speed of the point E at the instant when E is 10^5 m away from P?

(10 points)

We are given $\frac{d\theta}{dt} = -1\,\text{rad/sec}$. At the instant t_0 when $\sqrt{x^2+1} = 10^5$ m, we have $\sqrt{x(t_0)^2+1} = 10^5$ and $x(t_0) = \sqrt{10^{10}-1}$ and $\sec\theta(t_0) = \frac{\sqrt{x(t_0)^2+1}}{x(t_0)} = \frac{10^5}{\sqrt{10^{10}-1}}$, so

$$\left.\frac{dx}{dt}\right|_{t_0} = -x^2\cdot\left(\frac{10^5}{\sqrt{10^{10}-1}}\right)^2\cdot(-1) = \sqrt{10^{10}-1}^{\,2}\left(\frac{10^5}{\sqrt{10^{10}-1}}\right)^2 = 10^{10}\ \text{m/sec}.$$

(4) *Define the* greatest integer function *by the formula*

$$\lfloor x \rfloor = \text{greatest integer less than or equal to } x$$

i.e.,

$$\lfloor x \rfloor = n \qquad \text{if} \quad n \leqslant x < n+1, \ n \in \mathbb{Z}.$$

At which points is the function $f(x) = (x - \lfloor x \rfloor)^2 + \lfloor x \rfloor$ *discontinuous? Why?*
(20 points)

If a is not an integer, then the greatest integer function is continuous at a, so $\lim_{x\to a} \lfloor x \rfloor = \lfloor a \rfloor$ and

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a} [(x - \lfloor x \rfloor)^2 + \lfloor x \rfloor] \\
&= (\lim_{x\to a} x - \lim_{x\to a} \lfloor x \rfloor)^2 + \lim_{x\to a} \lfloor x \rfloor = (a - \lfloor a \rfloor)^2 + \lfloor a \rfloor \\
&= f(a),
\end{aligned}$$

therefore f is continuous at a.

At an integer a, we have

$$\begin{aligned}
\lim_{x\to a^+} f(x) &= \lim_{x\to a^+} [(x - \lfloor x \rfloor)^2 + \lfloor x \rfloor] \\
&= (\lim_{x\to a^+} x - \lim_{x\to a^+} \lfloor x \rfloor)^2 + \lim_{x\to a^+} \lfloor x \rfloor = (a - a)^2 + a = a \\
&= f(a)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{x\to a^-} f(x) &= \lim_{x\to a^-} [(x - \lfloor x \rfloor)^2 + \lfloor x \rfloor] \\
&= (\lim_{x\to a^-} x - \lim_{x\to a^-} \lfloor x \rfloor)^2 + \lim_{x\to a^-} \lfloor x \rfloor = (a - (a-1))^2 + (a-1) = a \\
&= f(a),
\end{aligned}$$

so $\lim_{x\to a} f(x) = f(a)$. Thus f is continuous at a.

We conclude that f is continuous at every point a.

(5) *Consider the function* $f(x) = x \cdot |x|$.

a) *Is* f *differentiable everywhere? Justify your answer.*

We have $f(x) = \begin{cases} x^2 & \text{if } x > 0, \\ -x^2 & \text{if } x < 0, \\ 0 & \text{if } x = 0. \end{cases}$

If a is any point distinct from 0, then f coincides with a differentiable function in some open interval containing a, so f is differentiable at a.

We have

$$\lim_{x \to 0^\pm} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^\pm} \frac{\pm x^2}{x} = \pm \lim_{x \to 0^\pm} x = \pm 0 = 0,$$

so

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0,$$

so f is differentiable also at 0 and $f'(0) = 0$.

Hence f is differentiable everywhere.

b) *At what points is* f *twice differentiable? Justify your answer.*

From Part a), we have $f'(x) = \begin{cases} 2x & \text{if } x > 0, \\ -2x & \text{if } x < 0, \\ 0 & \text{if } x = 0. \end{cases}$

If a is any point distinct from 0, then f coincides with a twice differentiable function in some open interval containing a, so f is twice differentiable at a.

We have

$$\lim_{x \to 0+} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0+} \frac{2x - 0}{x} = \lim_{x \to 0+} 2 = 2,$$
$$\lim_{x \to 0-} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0-} \frac{-2x - 0}{x} = \lim_{x \to 0-} (-2) = -2 \neq 2,$$

so $\lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0}$ does not exist and f is not twice differentiable at 0.

Hence f is twice differentiable at every point except at 0.

c) *What are the inflection points of* f*? Justify your answer.*

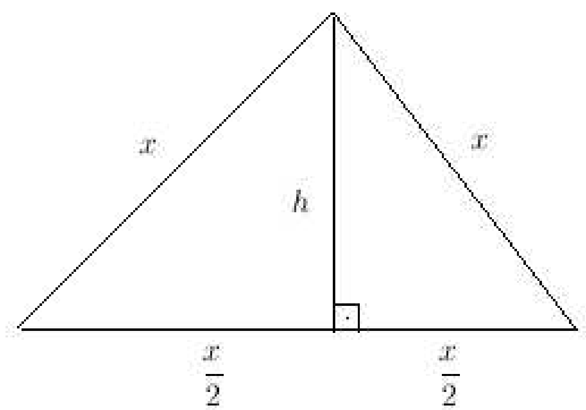
The second derivative of f is positive for $x > 0$, negative for $x < 0$, therefore f is concave up for $x > 0$, concave down for $x < 0$. The only point where the direction of concavity changes is $x = 0$. Thus 0 is the only inflection point of f.

B U Department of Mathematics

Math 101 Calculus I

Fall 2004 First Midterm

1.) The lengths of the sides of an equilateral triangle are increasing at a rate of 5 centimeters per hour. At what rate is the area of the triangle increasing when the side is 10 centimeters long ?



Solution:

$$x^2 - \frac{x^2}{4} = h^2 \Longrightarrow h = \frac{\sqrt{3}}{2}x$$

$$A = \frac{xh}{2} = \frac{\sqrt{3}}{4}x^2$$

Find $\frac{dA}{dt}$ when $x = 10$, $\frac{dx}{dt} = 5$ cm/h.

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4} 2x \frac{dx}{dt}$$

$$\Longrightarrow \left.\frac{dA}{dt}\right|_{x=10} = \frac{\sqrt{3}}{2} \cdot 10.5 = 25\sqrt{3} \text{ cm}^2/\text{h}.$$

2.) For both a) and b) express your results in their simplest form.

a) Find $\frac{d^2y}{dx^2}$ by implicit differentiation if $x^2 + 2xy - y^2 + 8 = 0$.

Solution:

$$2x + 2y + 2xy' - 2yy' = 0 \Longrightarrow y' = \frac{x+y}{y-x}$$

$$y'' = \frac{(1+y')(y-x) - (x+y)(y'-1)}{(y-x)^2} = \frac{y - x + y'y - y'x - (xy' - x + yy' - y)}{(y-x)^2}$$

$$\Longrightarrow y'' = \frac{2y - 2xy'}{(y-x)^2} = \frac{2y - 2x(\frac{x+y}{y-x})}{(y-x)^2} = \frac{2y(y-x) - 2x(x+y)}{(y-x)^3}$$

$$\implies y'' = \frac{2y^2 - 2xy - 2x^2 - 2xy}{(y-x)^3} = \frac{2(y^2 - 2xy - x^2)}{(y-x)^3} = \frac{16}{(y-x)^3}$$

b) Find $\dfrac{dy}{dx}$ if $y = \dfrac{\cos x}{1 - \sin x}$.

Solution:

$$\frac{dy}{dx} = \frac{(-\sin x)(1 - \sin x) - \cos x(-\cos x)}{(1 - \sin x)^2}$$

$$\implies \frac{dy}{dx} = \frac{-\sin x + \sin x^2 + \cos x^2}{(1 - \sin x)^2} = \frac{1}{1 - \sin x}$$

You are NOT allowed to use L'Hopital's rule !

3.) a)Evaluate $\lim_{x \to 0} \dfrac{|2x-1| - |2x+1|}{x}$.

Solution:

When $-\dfrac{1}{2} < x < \dfrac{1}{2}$, $\mid 2x - 1 \mid = -(2x-1)$ and $\mid 2x+1 \mid = 2x+1$.

$$\implies \lim_{x \to 0} \frac{\mid 2x-1 \mid - \mid 2x+1 \mid}{x} = \lim_{x \to 0} \frac{-(2x-1) - (2x+1)}{x} = \lim_{x \to 0} \frac{-4x}{x} = -4.$$

b)Is there a number a such that $\lim_{x \to -2} \dfrac{3x^2 + ax + a + 3}{x^2 + x + 2}$ exists? If so, find the value of a and the value of the limit.

Solution:

Since the denominator is 0 at $x = -2$, for the limit to exist we should have the numerator =0 at $x = -2$.

$3(-2)^2 - 2a + a + 3 = 0$ gives $a = 15$.

$$\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x + 2} = \lim_{x \to -2} \frac{3x^2 + 15x + 18}{x^2 + x + 2} = \lim_{x \to -2} \frac{(x+2)(3x+9)}{(x+2)(x-1)} = -1.$$

c) Evaluate $\lim_{x \to \frac{\pi}{4}} \dfrac{\sin(x + \frac{\pi}{4}) - 1}{x - \frac{\pi}{4}}$.

Solution:

Let $t = x - \frac{\pi}{4}$ then $x \to 0$ means $t \to 0$

$$\lim_{t\to 0} \frac{\sin(t + \frac{\pi}{2}) - 1}{t} = \lim_{t\to 0} \frac{\cos t - 1}{t} = 0.$$

4.) By using intermediate value theorem show that the graphs of the following functions $f(x) = x^4 - 5x^2$ and $g(x) = 2x^3 - 4x + 6$ intersect between $x = 3$ and $x = 4$.

Solution:

Let $h(x) = f(x) - g(x)$. Then h is continuous since f and g are continuous. (Polynomials are continuous everywhere.)

$h(x) = x^4 - 5x^2 - 2x^3 + 4x - 6$

$h(3) = 81 - 45 - 54 + 12 - 6 = -12 < 0$

$h(4) = 256 - 80 - 128 + 16 - 6 = 58 > 0 \Longrightarrow$ by intermediate value theorem there exists at least one $c \,\epsilon (3, 4)$ such that $h(c) = 0 = f(c) - g(c)$.

$\Longrightarrow f(c) = g(c)$.

5.) Suppose f is a differentiable function that satisfies the equation

$$f(x + y) = f(x) + f(y) + x^2y + xy^2$$

for all real numbers x and y. Suppose also that $\lim_{x\to 0} \frac{f(x)}{x} = 1$. Compute $f(0)$, $f'(0)$, $f'(x)$.

Solution:

$f(0 + 0) = f(0) + f(0) + 0^2.0 + 0.0^2$

$\Longrightarrow f(0) = 0.$

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x} = 1.$$

$$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x) + f(h) + x^2h + xh^2 - f(x)}{h}$$

$$\Longrightarrow \lim_{h\to 0} \frac{f(h)}{h} + \lim_{h\to 0}(x^2 + h) = 1 + x^2.$$

B U Department of Mathematics

Math 101 Calculus I

Fall 2005 First Midterm

1.) A stone dropped into a water pond sends out a circular ripple whose radius increases at a constant rate of 3ft/sec. How rapidly is the area enclosed by the ripple increasing at the end of 10 seconds?

Solution:

$\frac{dr}{dt} = 3ft/sec.$

$When t = 10\,, r = 30ft\ and\ A = \pi\, r^2.$

$\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt} = 2\,\pi r\,\frac{dr}{dt} = 2\pi 30\,3 = 180\pi ft^2/sec.$

2.) a) A function f defined for all positive real numbers satisfies the equation $f(x^2) = x^3$ for every $x > 0$. Determine $f'(4)$.

Solution:

$f(x^2) = x^3$
$f'(x^2)\,2x = 3\,x^2$
$f'(4)\,4 = 3\,2^2$
$f'(4) = 3.$

b) Using the definition of derivative, find the derivative of $g(x) = \sqrt{1+2x}$.

Solution:

$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{\sqrt{1+2x+2h}-\sqrt{1+2x}}{h}$

$= \lim_{h\to 0}\frac{1+2x+2h-(1+2x)}{h(\sqrt{1+2x+2h}+\sqrt{1+2x})}$

$= \lim_{h\to 0}\frac{2}{\sqrt{1+2x+2h}+\sqrt{1+2x}}$

$= \frac{2}{2\sqrt{1+2x}} = \frac{1}{\sqrt{1+2x}}.$

3.) a) Is there a number a such that $\lim\limits_{x\to -2}\dfrac{3x^2+ax+a+3}{x^2+x-2}$ exists? If so, find the value of a and the value of the limit.

Solution:

$3(-2)^2 + (-2)a + a + 3 = 0$ implies $15 - a = 0$, hence, $a = 15$.

$$\lim_{x\to -2}\frac{3(x^2+5x+6)}{x^2+x-2} = \lim_{x\to -2}\frac{3(x+2)(x+3)}{(x+2)(x-1)} = \lim_{x\to -2}\frac{3(x+3)}{x-1} = \frac{3}{-3} = -1.$$

b)Evaluate the following limits.

i) $\lim_{x\to 1} \frac{x^{\frac{1}{3}}-1}{\sqrt{x}-1} = \lim_{x\to 1} \frac{(x^{\frac{1}{3}}-1)(x^{\frac{2}{3}}+x^{\frac{1}{3}}+1)(\sqrt{x}+1)}{(x^{\frac{2}{3}}+x^{\frac{1}{3}}+1)(\sqrt{x}+1)(\sqrt{x}-1)} = \lim_{x\to 1} \frac{(x-1)(\sqrt{x}+1)}{(x^{\frac{2}{3}}+x^{\frac{1}{3}}+1)(x-1)} = \frac{2}{3}$

ii) $\lim_{x\to+\infty} \sin(\frac{\pi x}{2-3x}) = \sin \lim_{x\to+\infty} \left(\frac{\pi x}{2-3x}\right) = \sin\left(-\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}.$

4.) Find all points on the curve $x^2y^2+xy=2$ where the slope of the tangent line is -1.

Solution:

$2x^2yy'+2xy^2+y+xy'=0$

$y'(2x^2y+x)=-y(2xy+1)$

$y'=-\frac{y(2xy+1)}{x(2xy+1)}=-\frac{y}{x}=-1$ which implies y=x.

$x^2\,x^2+x^2=2$

$x^2(x^2+1)=2$ let $x^2=t$ $t(t+1)=2$

$t^2+t-2=0$ $t=1$ *and* $t=-2$ $x^2=t \Rightarrow x=\pm1$ $x^2=t=-2$ is not possible.
The points which gave slope -1 are (-1,-1) and (1,1).

B U Department of Mathematics

Math 101 Calculus I

Fall 1999 First Midterm

1.) State the Mean Value Theorem. If the mean value theorem is applied to the function $f(x) = x^3 + qx^2 + 5x - 6$ on the interval $[0, 2]$, consider the number c determined by the theorem. If $c = 2$, what is q?

Solution:

Let f be a continuous function on $[a, b]$, and f be differentiable on (a, b). Then there exist a number c between a and b sush that $f'(x) = \dfrac{f(b) - f(a)}{b - a}$.

Consider the function $f(x) = x^3 + qx^2 + 5x - 6$ on the interval $[0, 2]$, $f'(x) = 3x^2 + 2qx + 5$, if $c = 2$, we have $f'(2) = \dfrac{f(2) - f(0)}{2 - 0}$, then we get $q = -4$.

2.) (a) Find $\lim_{x \to 1} \dfrac{\sin(x^2 - 1)}{x - 1}$ (L'Hôpital's rule is not allowed).

Solution:

$$\lim_{x \to 1} \frac{\sin(x^2 - 1)}{x - 1} = \lim_{x \to 1} (x + 1) \frac{\sin(x^2 - 1)}{x^2 - 1} =$$
$$\lim_{x \to 1} \frac{\sin(x^2 - 1)}{x^2 - 1} \cdot \lim_{x \to 1} (x + 1) = \lim_{u \to 0} \frac{\sin(u)}{u} \cdot 2 = 2$$

(b) The function f is defined by

$$f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ mx + c & \text{if } x \geq 0 \end{cases}.$$

Find the numbers m and c such that f is differentiable on $\mathbb{R}$.

Solution:

f should be differentiable at $x = 0$, in particular, f should be continuous at $x = 0$. For continuity at $x = 0$, $\lim_{x \to 0^-} f(x) = f(0) = \lim_{x \to 0+} f(x)$. Hence $c = 1$.

For differentiability at $x = 0$, $\lim_{x \to 0^-} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \dfrac{f(x) - f(0)}{x - 0}$. Hence $m = 0$.

(c) Find the indefinite integral $\displaystyle\int \sqrt{\frac{x + 1}{x - 1}} \frac{1}{(x - 1)^2}\, dx$.

Solution:

Let $u = \dfrac{x + 1}{x - 1}$, then $du = -2\dfrac{1}{(x - 1)^2}\, dx$.

Hence $\displaystyle\int \sqrt{\frac{x + 1}{x - 1}} \frac{1}{(x - 1)^2}\, dx = \frac{-1}{2} \int \sqrt{u} du = \frac{-1}{3} u^{\frac{3}{2}} + c = \frac{-1}{3} (\frac{x + 1}{x - 1})^{\frac{3}{2}} + c.$

3.) The sum of two nonnegative number is 16. Find the maximum possible value and the minimum possible value of the sum of their cube roots.

Solution:

Let x, y be two nonnegative number s.t. $x, y > 0$ and $x + y = 16$, i.e. $y = 16 - x$. We need to find the extremal points of the function $f(x) = \sqrt[3]{x} + \sqrt[3]{16 - x}$ on the interval $[0, 16]$.

$f'(x) = \frac{1}{3}x^{\frac{-2}{3}} - \frac{1}{3}(16 - x)^{\frac{-2}{3}}$, and $f'(8) = 0$. Hence $x = 0, 8, 16$ are the possible extremal points.

Observe that $f(0) = 2\sqrt[3]{2}$, $f(8) = 4$, $f(16) = 2\sqrt[3]{2}$
The maximum possible value is 4 and the minimum possible value is $2\sqrt[3]{2}$.

4.) One side of a right triangle is increasing at a rate of 2 units per second, while the other side is decreasing in such a way that the hypotenuse remains constant at 10 units. When the length of the first side is 6 units, is the area of the triangle increasing or decreasing, and at what rate ?

Solution:

Let x, y be the sides of a right triangle, such that $x^2 + y^2 = 100$, i.e. $y^2 = 100 - x^2$. The area $A(x)$ of right triangle, depends on x, and x depends on t, where t denotes the time. Then $A(x) = \frac{1}{2}x\sqrt{100 - x^2}$, and $\frac{dx}{dt} = 2$. We need to find $\frac{dA}{dt}$, which can be obtained by chain rule $\frac{dA}{dt} = \frac{dA}{dx}\frac{dx}{dt}$. Hence we get $\frac{dA}{dt} = \sqrt{100 - x^2} - x^2\frac{1}{\sqrt{100 - x^2}}$

When $x = 6$, $\frac{dA}{dt} = 3.5$ which is positive. Therefore the area of the triangle is **increasing** with a rate of 3.5 per second.

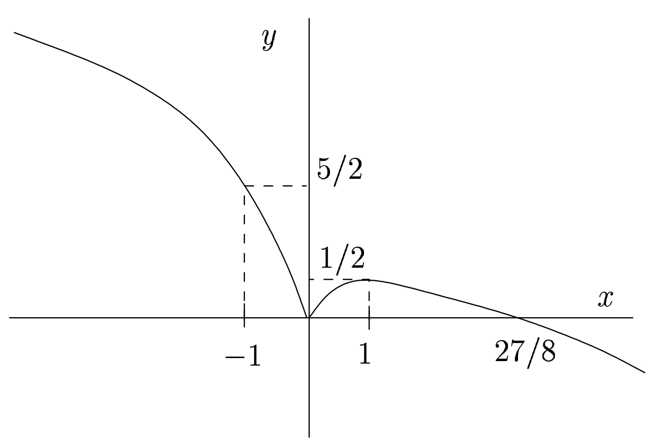
5.) Sketch the curve $y = \frac{3}{2}x^{\frac{2}{3}} - x$, indicating the asymptotes, maximum, minimum and inflection points, if any.

Solution:

Domain : $\mathbb{R}$; roots of f: $x = 0, \frac{27}{8}$.
f has no vertical or horizontal asymptotes.

$f'(x) = \frac{1 - x^{-\frac{1}{3}}}{x^{\frac{1}{3}}}$, $f'(x) = 0$ when $x = 1$ and f' is undefined at $x = 0$. Hence critical points are $x = 0, 1$.

$f''(x) = -\frac{1}{3}x^{-\frac{4}{3}}$, which is always negative except at $x = 0$ where it is not defined.



B U Department of Mathematics

Math 101 Calculus I

Spring 2001 First Midterm

1. Show that the sum of the $x-$ and $y-$intercepts of any tangent line to the graph of $x^{1/2}+y^{1/2}=c^{1/2}$ is constant.

Solution:

Differentiating both sides we get, $\dfrac{x^{-1/2}}{2}+\dfrac{y^{-1/2}}{2}\dfrac{dy}{dx}=0 \Rightarrow \dfrac{dy}{dx}=-\dfrac{\sqrt{y}}{\sqrt{x}}$. Hence, the equation of the tangent line will be:

$$y-y_0=-\frac{\sqrt{y_0}}{\sqrt{x_0}}(x-x_0).$$

Setting $x=0$ we get $y=y_0+\dfrac{\sqrt{y_0}}{\sqrt{x_0}}x_0$ as the $y-$intercept.

Setting $y=0$ we get $-y_0=\dfrac{-\sqrt{y_0}}{\sqrt{x_0}}x+\dfrac{\sqrt{y_0}}{\sqrt{x_0}}x_0 \Rightarrow x=\dfrac{\sqrt{x_0}}{\sqrt{y_0}}y_0+x_0$ as the $x-$intercept.

$\Rightarrow$ Sum $=y_0+\sqrt{x_0y_0}+\sqrt{x_0y_0}+y_0=(\sqrt{x_0}+\sqrt{y_0})^2=C$ as the point (x_0,y_0) belongs to the given curve.

2. Evaluate $\displaystyle\lim_{t\to 0}\frac{\sin(t+\frac{\pi}{2})-1}{t}$ if it exists. (Do not use L'Hopital's Rule!)

Solution:

Observe that $\displaystyle\lim_{t\to 0}\frac{f(t+\frac{\pi}{2})-f(\frac{\pi}{2})}{t}=f'(\frac{\pi}{2})$ if f differentiable at $x=\pi/2$. Now let $f(x)=\sin x$ and note that $\sin x$ is everywhere differentiable. Moreover $\sin\pi/2=1$. So the required limit is just value of the derivative of $\sin x$ at $x=\pi/2$. This is $f'(x)=\cos x \Rightarrow f'(\frac{\pi}{2})=\cos\frac{\pi}{2}=0$.

Hence, $\displaystyle\lim_{t\to 0}\frac{\sin(t+\frac{\pi}{2})-1}{t}=0$.

3. Find the critical points and classify all the extreme values of

$$f(x)=\frac{1}{4}\left(x^3-\frac{3}{2}x^2-6x+2\right) \text{ on } [-2,\infty).$$

Solution:

$f'(x)=\dfrac{1}{4}(3x^2-3x-6)=\dfrac{3}{4}(x+1)(x-2)=0 \Rightarrow x=-1, x=2$ are the critical points.

$f''(x)=\dfrac{1}{4}(6x-3)$ which will be used for classification purposes.

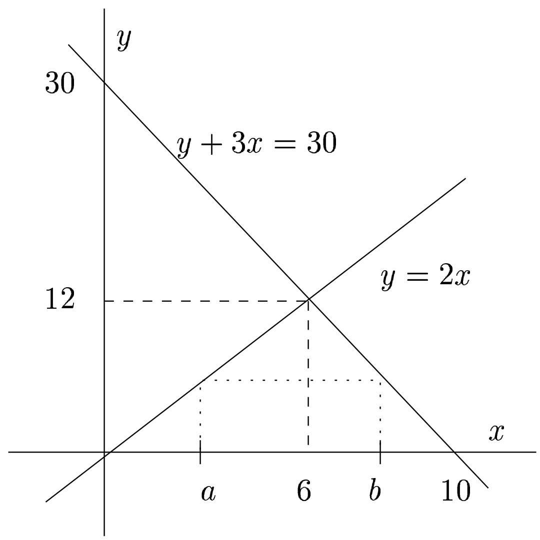
$f''(-1) = -\dfrac{9}{4} < 0 \Rightarrow$ Relative maximum at $(-1, 11/8)$.

$f''(2) = \dfrac{9}{4} > 0 \Rightarrow$ Relative minimum at $(2, -2)$.

Since $f(x) \to \infty$ as $x \to \infty$ there is no absolute maximum of f. $f(-2) = 0$ is an endpoint minimum because f' is positive in $[-2, -1)$ so that f is increasing. The point $(2, -2)$ is also an absolute minimum which can be seen by the comparison $f(2) < f(-2)$.

4. The base of a rectangle is on the $x-$axis. The upper left vertex is on the line whose equation is $y = 2x$ and the upper right vertex is on the line $y + 3x = 30$. For what value of y will the area of the rectangle be a maximum?

Solution:



$2x = 30 - 3x \Rightarrow x = 6 \Rightarrow y = 12$. So $(6, 12)$ is the intersection point.

The area is given by $A = (b - a)(30 - 3b) = (b - a)2a$.

By similar triangles, $\dfrac{b-a}{10} = \dfrac{12-2a}{12} \Rightarrow b - a = \dfrac{5}{6}(12 - 2a) = \dfrac{5}{3}(6 - a)$.

So the area has become:

$$A = \frac{5}{3}(6 - a)2a = \frac{5}{3}(12a - 2a^2).$$

We now look at the critical points of A:

$$\frac{dA}{da} = \frac{5}{3}(12 - 4a) = 0 \Rightarrow a = 3 \Rightarrow b - 3 = \frac{5}{3}(6 - 3) \Rightarrow b = 8$$

and $y = 2a = 6$.

Second derivative test $\dfrac{d^2A}{da^2} = -\dfrac{20}{3} < 0$ implies maximum at $y = 6$.

5. Use the intermediate value theorem to show that there is a square with a diagonal of length that is between r and $2r$ and an area that is half the area of a circle of radius r.

Solution:

Let the square have sides of length x and diagonal of length d. Then the area is given by:

$$A = x^2 = \frac{d^2}{2}$$

where $r < d < 2r$, r being the radius of the circle. We want to show that there is a d such that

$$A = \frac{d^2}{2} = \frac{\pi r^2}{2}.$$

We define the function: $A(x) = \frac{x^2}{2} - \frac{\pi r^2}{2}$ which is continuous for $x \in [r, 2r]$. We seek an $x_0 \in (r, 2r)$ so that $A(x_0) = 0$. This x_0 will work as d.
Let us check the end-point values of $A(x)$:
$A(r) = \frac{r^2}{2} - \frac{\pi r^2}{2} = \frac{r^2}{2}(1 - \pi) < 0$ and $A(2r) = \frac{4r^2}{2} - \frac{\pi r^2}{2} = \frac{r^2}{2}(4 - \pi) > 0$.
By the intermediate value theorem, there exists an $x_0 \in (r, 2r)$, such that

$$A(x_0) = 0 = \frac{x_0^2}{2} - \frac{\pi r^2}{2} \Rightarrow \frac{x_0^2}{2} = \frac{\pi r^2}{2}.$$

Choose $d = x_0$.

B U Department of Mathematics

Math 101 Calculus I

Spring 2002 First Midterm

1. **(a)** Find a linear approximation of $y = \sin x$ at $x_0 = \dfrac{\pi}{6}$.

Solution:

By Taylor approximation: $f(x) \cong f(x_0) + f'(x_0)(x - x_0)$. Letting $f(x) = \sin x$:

$$\sin x \cong \sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right).$$

(b) Using the approximation found in part **(a)** approximate $\sin 29°$.

Solution:

For $x = 29° = \dfrac{\pi}{6} - \dfrac{\pi}{180}$ we evaluate the value of the linear approximation:

$$\sin 29° \cong \frac{1}{2} + \frac{\sqrt{3}}{2}\left(-\frac{\pi}{180}\right).$$

2. Evaluate the following limits, if exist (L'Hopital's rule is not allowed).

(a) $\displaystyle\lim_{x\to 9} \frac{9 - x}{3 - \sqrt{x}}$

(b) $\displaystyle\lim_{x\to\infty} \frac{x + \sin x + 2\sqrt{x}}{x + \sin x}$

(c) $\displaystyle\lim_{x\to 0} \frac{\sin(x^2) - \tan(x^2)}{x^6}$

Solution:

(a) $\displaystyle\lim_{x\to 9} \frac{9 - x}{3 - \sqrt{x}} = \lim_{x\to 9} \frac{(3 - \sqrt{x})(3 + \sqrt{x})}{3 - \sqrt{x}} = 3 + \sqrt{9} = 6.$

(b) $\displaystyle\lim_{x\to\infty} \frac{x + \sin x + 2\sqrt{x}}{x + \sin x} = \lim_{x\to\infty} \frac{1 + \dfrac{\sin x}{x} + \dfrac{2}{\sqrt{x}}}{1 + \dfrac{\sin x}{x}} = 1,$

since as $x \to \infty$, $0 \le \left|\dfrac{\sin x}{x}\right| \le \dfrac{1}{|x|} \implies \displaystyle\lim_{x\to\infty} \frac{\sin x}{x} = 0$ and also $\displaystyle\lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$

(c) Letting $x^2 = t$ we get $\lim_{x\to 0} \frac{\sin(x^2) - \tan(x^2)}{x^6} = \lim_{t\to 0} \frac{\sin t - \tan t}{t^3}$ which is apparently easier:

$$\begin{aligned}
\lim_{t\to 0} \frac{\sin t - \tan t}{t^3} &= \lim_{t\to 0} \frac{\sin t \left(1 - \frac{1}{\cos t}\right)}{t^3} = \lim_{t\to 0} \frac{\sin t}{t} \frac{\cos t - 1}{t^2 \cos t} \\
&= \lim_{t\to 0} \frac{\sin t}{t} \frac{(\cos t - 1)(\cos t + 1)}{t^2 \cos t(\cos t + 1)} \\
&= \lim_{t\to 0} \frac{\sin t}{t} \frac{-\sin^2 t}{t^2} \frac{1}{\cos t} \frac{1}{\cos t + 1} = 1\,(-1)\,1\,\frac{1}{2} = -\frac{1}{2}.
\end{aligned}$$

3. Show that the equation $x^8 + x - 1 = 0$ has exactly two real roots. (Hint: It may be helpful to draw a graph.)

Solution:

Observe that $f(x) = x^8 + x - 1$ a continuous function. Computing some easy values of f:

$$f(0) = -1 < 0, \quad f(-2) = 256 - 3 = 253 > 0, \quad f(1) = 1 > 0.$$

So by Intermediate Value Theorem there are at least 2 roots of $x^8 + x - 1 = 0$; one is in $(-2, 0)$, the other is in $(0, 1)$. We shall now show there are no more roots.

Now, $f'(x) = 8x^7 + 1 = 0$ if $x_0 = \sqrt[7]{-\frac{1}{8}}$. This is, f has only one critical point. Being a continuous function which tends to $+\infty$ as $x \to \pm\infty$ this point is an absolute minimum. Furthermore the graph of f is concave up as $f'' = 56x^6 \geq 0$ for all x. So, graph of f decreases from $+\infty$, hits the $x-$axis at some point in $(-2, 0)$ then makes an absolute minimum at x_0, and finally starts increasing from this point and hits $x-$axis once more in $(0, 1)$ and goes to $+\infty$.

Thus, there exactly two roots.

4. Let $g(x) = 1 + \sqrt{x}$ and $(f \circ g)(x) = 3 + 2\sqrt{x} + x$. Find $f'(2)$.

Solution:

We first differentiate the composition:

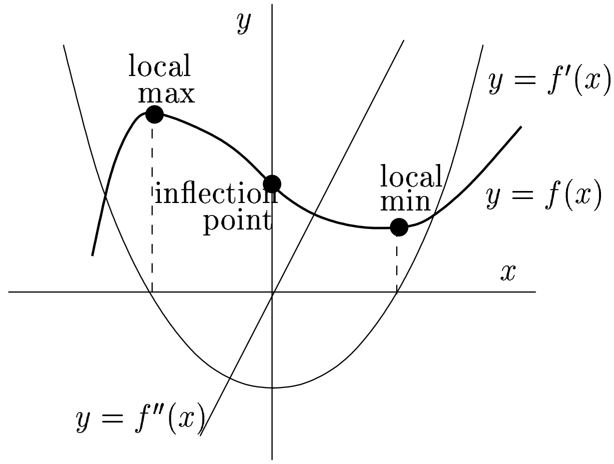
$$(f \circ g)' = f'(g(x))g'(x) \iff \frac{1}{\sqrt{x}} + 1 = f'(g(x)) \left(\frac{1}{2\sqrt{x}}\right).$$

We also have $g(x) = 2$ if $x = 1$. Then putting $x = 1$:

$$\frac{1}{\sqrt{1}} + 1 = f'(2)\frac{1}{2\sqrt{1}} \implies f'(2) = 4.$$

5. The graphs of the first and second derivatives of a function $y = f(x)$ are given below. Sketch an appropriate graph of f on the same coordinate plane so that the graph of f passes through the point P. Also label points of local extrema and inflection.

Solution:



6. Find the integrals:

(a) $\displaystyle\int \frac{2z dz}{\sqrt[3]{(z^2+1)}}$.

(b) $\displaystyle\int x^{\frac{1}{3}} \sin(x^{\frac{4}{3}} - 8) dx$.

Solution:

(a) Substitute $u = z^2 + 1 \Longrightarrow du = 2z dz$. Then:

$$\int \frac{2z dz}{\sqrt[3]{(z^2+1)}} = \int \frac{du}{\sqrt[3]{u}} = \frac{3}{2}u^{\frac{2}{3}} + c = \frac{3}{2}(z^2+1)^{\frac{2}{3}} + c.$$

(b) Substitute $u = x^{\frac{4}{3}} - 8 \Longrightarrow du = \frac{4}{3}x^{\frac{1}{3}} dx$. Then:

$$\int x^{\frac{1}{3}} \sin(x^{\frac{4}{3}} - 8) dx = \frac{3}{4}\int \sin u du = -\frac{3}{4}\cos u + c = -\frac{3}{4}\cos(x^{\frac{4}{3}} - 8) + c.$$

B U Department of Mathematics

Math 101 Calculus I

Spring 2003 First Midterm

1. Find the slope of the line tangent to the curve $\sin(xy) = x^2 \cos y$ at the point $(2, \pi/2)$.

Solution:

Using implicit differentiation: $\dfrac{d}{dx}\sin(xy) = \dfrac{d}{dx}(x^2\cos y) \implies \cos(xy)(xy' + y) = 2x\cos y - x^2(\sin y)y'$. Leaving y' alone we get:

$$y'(x\cos(xy) + x^2\sin y) = 2x\cos y - y\cos(xy) \Longrightarrow y' = \frac{2x\cos y - y\cos(xy)}{x\cos(xy) + x^2\sin y}.$$

We now evaluate y' at the point $P(2, \pi/2)$ to find the slope of the tangent:

$$y'(P) = \frac{2.2\cos(\pi/2) - (\pi/2)\cos\pi}{2\cos\pi + 4\sin(\pi/2)} = \frac{0 + (\pi/2)}{-2+4} = \pi/4.$$

2. Given the function $f(x) = \begin{cases} \dfrac{\sin x}{|x|} & \text{if } \ x \neq 0 \\ 1 & \text{if } \ x = 0 \end{cases}$

(a) Determine whether f is continuous at $x = 0$ or not.

(b) Determine whether f is differentiable at $x = 0$ or not.

Solution:

(a) Checking the right-hand and left-hand limit at $x = 0$:
$\lim\limits_{x\to0+} f(x) = \lim\limits_{x\to0+} \dfrac{\sin x}{x} = 1$ and $\lim\limits_{x\to0-} f(x) = \lim\limits_{x\to0-} \dfrac{\sin x}{-x} = -1$.
Since $1 \neq -1$ f is not continuous at $x = 0$.

(b) f is not continuous at $x = 0$ hence f is not differentiable at $x = 0$.

3. Evaluate the following limit: $\lim\limits_{y\to1^-} \dfrac{\sqrt{1-y^2}}{y-1}$.

Solution:

$$\lim_{y\to1^-} \frac{\sqrt{1-y^2}}{y-1} = \lim_{y\to1^-} \frac{\sqrt{(1-y)(1+y)}}{-(1-y)} = \lim_{y\to1^-} \frac{\sqrt{1+y}}{-\sqrt{1-y}} = -\sqrt{2}/0 = -\infty.$$

Alternative solution: Since $\lim\limits_{y\to1^-} \dfrac{\sqrt{1-y^2}}{y-1} = [0/0]$, using L'Hopital's Rule we have:

$$\lim_{y\to1^-} \frac{\sqrt{1-y^2}}{y-1} = \lim_{y\to1^-} \frac{\dfrac{-2y}{2\sqrt{1-y^2}}}{1} = \lim_{y\to1^-} \frac{-y}{\sqrt{1-y^2}} = -1/0 = -\infty.$$

4. Consider the function $f(x) = x^4 + 2x^3 - 1$.

(a) Determine the interval(s) on which f is increasing or decreasing.

(b) Find and classify all local extrema of f, if any.

(c) How many zeroes does the function f have?

(d) Determine the interval(s) on which f is concave up or concave down.

(e) Find all the inflection points of f, if any.

(f) Sketch the graph of f.

Solution:

We check first and second derivative of f:
$y = x^4 + 2x^3 - 1$ and $y' = 4x^3 + 6x^2 = 2x^2(2x+3)$ and $y'' = 12x^2 + 12x = 12x(x+1)$.
We now find the critical points: $y' = 0$ when $x = 0, -3/2$ are only critical points as f is everywhere continuous. $y'' = 0$ when $x = 0, -1$ which are inflection points because f'' changes sign at these points.
Asymptotic behaviour for large and small x: $\lim_{x \to \pm\infty} y = \infty$.

Some values of f:
$f(-3) = (-3)^4 + 2(-3)^3 - 1 = 26$.
$f(-3/2) = (-3/2)^4 + 2(-3/2)^3 - 1 = 81/16 - 54/8 - 1 = -27/16 - 1 = -43/16$.

x		-3		-2		$-3/2$		-1		0		1
y		26		-1		$-43/16$		-2		-1		2
y'	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$+$	$+$
y''	$+$	$+$	$+$	$+$	$+$	$+$	$+$	0	$-$	0	$+$	$+$

(a) From y', f is increasing on $(-3/2, \infty)$, decreasing on $(-\infty, -3/2)$.

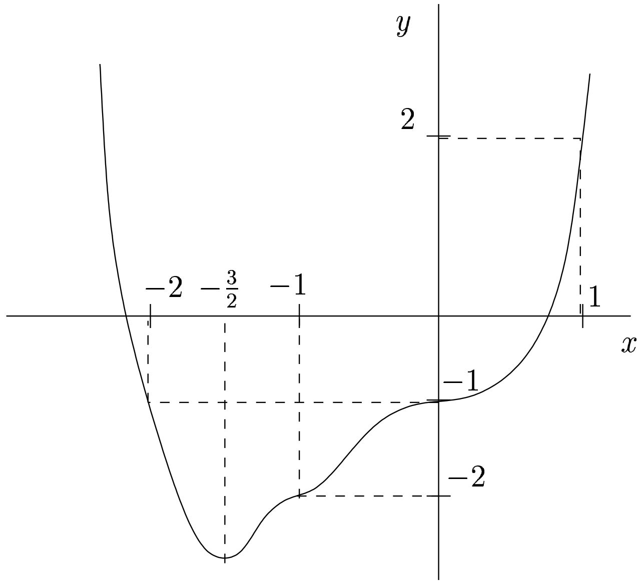
(b) $x = -3/2$ is a critical point and $y''(-3/2) > 0 \Longrightarrow x = -3/2$ is a local minimum point, by the 2nd derivative test. $x = 0$ is not a local maximum or minimum.

(c) y changes sign between 0 and 1, and between -3 and -2. By the Intermediate Value Theorem, there is at least one zero in each of these intervals. But f is monotone on these intervals, so there is exactly one root in each. Therefore, there are exactly 2 zeroes.

(d) From y'', f is concave up on $(-\infty, -1)$ and on $(0, \infty)$, concave down on $(-1, 0)$.

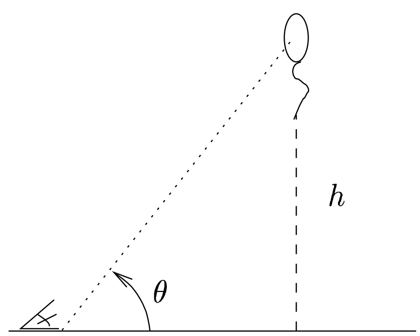
(e) $x = -1$ and $x = 0$ are inflection points.

(f)



5. A balloon is rising vertically from a point on the ground that is 200 meters from an observer at ground level. The observer determines that the angle of elevation θ between the observer and the balloon is increasing at a rate of $0.9°/\text{sec}$ when the angle of elevation is $45°$. How fast is the balloon rising at this time?

Solution:



From the figure $h = 200\tan\theta$ and the rate of change of h is $\dfrac{dh}{dt} = 200\sec^2\theta\dfrac{d\theta}{dt}$.

The angle θ changes with the rate: $\dfrac{d\theta}{dt} = 0.9°/\text{sec} = 0.9\dfrac{2\pi}{360} = \dfrac{\pi}{200}$ rad/sec as given in the problem.

When $\theta = 45°$, $\sec^2\theta = \sqrt{2}$ and $\dfrac{d\theta}{dt} = \dfrac{\pi}{200}$.

Then the height of the balloon has a rate of change at this moment:

$$\frac{dh}{dt} = 200(\sqrt{2})^2\frac{\pi}{200} = 2\pi \text{ m/sec}.$$

B U Department of Mathematics

Math 101 Calculus I

Spring 2004 First Midterm

1. Evaluate the following limits, if they exist (justify your answer).

a) $\lim_{\theta \to 0} \cos(\frac{\pi\theta}{\sin\theta})$

Solution:

$$\lim_{\theta\to 0} \cos(\frac{\pi\theta}{\sin\theta}) = \lim_{\theta\to 0} \cos\left(\frac{\pi}{\frac{\sin\theta}{\theta}}\right) = \cos\pi = -1 \text{ since } \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$

b) $\lim_{h\to 0} \dfrac{\sec^2(\frac{\pi}{4}+h) - \sec^2(\frac{\pi}{4})}{h}$

Solution:

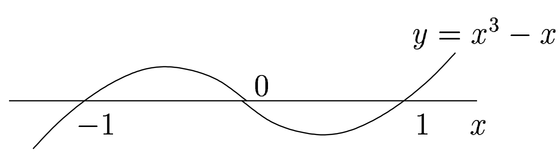
$$\lim_{h\to 0} \frac{\sec^2(\frac{\pi}{4}+h) - \sec^2(\frac{\pi}{4})}{h} = f'(\frac{\pi}{4}) \text{ if this limit exists for } f(x) = \sec^2 x.$$

For $f(x) = \sec^2 x$, $f'(x) = 2\cdot\sec^2 x \cdot \tan x$.

It follows that $f'(\frac{\pi}{4}) = 2(\sec\frac{\pi}{4})^2(\tan\frac{\pi}{4}) = 2\cdot(2)\cdot(1) = 4$

c) $\lim_{x\to 0^+} f(x^3 - x)$ if $\lim_{x\to 0^+} f(x) = 3$ and $\lim_{x\to 0^-} f(x) = 2$.

Solution:



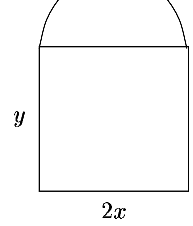
As $x \to 0^+$,

$x^3 - x = x(x-1)(x+1) \to 0^-$

So $\lim_{x\to 0^+} f(x^3 - x) = \lim_{x\to 0^-} f(x) = 2$

2. A window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 8 meters, find the dimensions that will allow the maximum amount of light to enter.

Solution:



$P = 4x + 2y + \pi x = 8\text{m} \Rightarrow y = 4 - (\frac{\pi + 4}{2})x$

$A = \frac{\pi}{2}x^2 + 2xy$

$A(x) = \frac{\pi}{2}x^2 + x(8 - \pi x - 4x)$

$A'(x) = \pi x + (8 - \pi x - 4x) + x(-\pi - 4) = 8 - \pi x - 8x = 8 - (\pi + 8)x = 0$

if $x = \frac{8}{\pi + 8}$ then $A''(x) = -\pi - 8 \leq 0$

So by Second Derivative Test, $A(x)$ is maximum when $x = \frac{8}{\pi + 8}$.

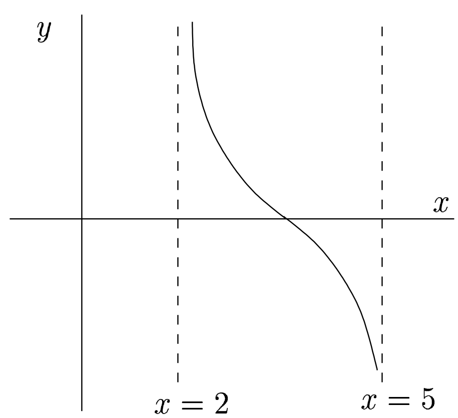
Hence dimensions: width$= 2x = \frac{16}{\pi + 8}$m

length$= y = \frac{16}{\pi + 8}$m so width=length=$\frac{16}{\pi + 8}$

3. $f(x) = \frac{2}{(x-2)^4} + \frac{3}{(x-5)^3}, \quad 2 < x < 5.$

Show that $f(c) = 0$ for some $c \in (2, 5)$. (Hint: Think of Intermediate Value Theorem).

Solution:



$f(x) = 2(x-2)^{-4} + 3(x-5)^{-3}$

$f'(x) = \frac{-8}{(x-2)^5} - \frac{9}{(x-5)^4} < 0$ for $2 < x < 5$. So f is continuous on (2,5) and $\lim_{x \to 2^+} f(x) = \infty$, $\lim_{x \to 5^-} f(x) = -\infty$. Hence there is some c between 2 and 5 such that $f(c) = 0$ by Intermediate Value Theorem.

4.a) Evaluate: $\int_{\frac{\pi^2}{36}}^{\frac{\pi^2}{4}} \frac{\cos\sqrt{t}}{\sqrt{t}\sin\sqrt{t}}dt$, (Hint:Let $u = \sin\sqrt{t}$)

Solution:

$$\int_{\frac{\pi^2}{36}}^{\frac{\pi^2}{4}} \frac{\cos\sqrt{t}}{\sqrt{t}\sqrt{\sin\sqrt{t}}}dt = 2\int_{\frac{1}{2}}^{1} \frac{du}{\sqrt{u}} \text{ where } u = \sin\sqrt{t},\ du = \frac{\cos\sqrt{t}}{2\sqrt{t}}dt$$

$$= 4\sqrt{u}\Big|_{\frac{1}{2}}^{1} = 4(1 - \frac{\sqrt{2}}{2})$$

b) Find: $I = \int (1 + \frac{1}{x})^{-3}(\frac{1}{x^2})dx$

Solution:

$$I = -\int u^{-3}du \qquad \text{(where } u = 1 + \frac{1}{x}, \quad du = -\frac{1}{x^2}dx)$$

$$= \frac{1}{2}u^{-2} + C = \frac{1}{2}(1 + \frac{1}{x})^{-2} + C$$

c) Find a function f and a constant a which satisfy:

$$2\int_a^x f(t)dt = 2\sin x - 1.$$

Solution:

$$\left(2\int_a^x f(t)dt\right)' = (2\sin x - 1)' \Rightarrow 2f(x) = 2\cos x \Rightarrow f(x) = \cos x$$

So $2\int_a^x \cos t dt = 2\sin x - 1$ i.e. $2(\sin t)\big|_a^x = 2\sin x - 1$

$$2\sin x - 2\sin a = 2\sin x - 1$$

$$\sin a = \frac{1}{2} \Rightarrow a = \frac{\pi}{6}$$

B U Department of Mathematics

Math 101 Calculus I

Date: April 6, 2005	Full Name	:
Time: 17:00-18:00	Math 101 Number	:
	Student ID	:

Spring 2005 First Midterm - Solution Key

IMPORTANT

1. Write your name, surname on top of each page. **2.** The exam consists of 4 questions some of which have more than one part. **3.** Read the questions carefully and write your answers neatly under the corresponding questions. **4.** Show all your work. Correct answers without sufficient explanation might not get full credit. **5.** Calculators are not allowed.

Q1	Q2	Q3	Q4	TOTAL
25 pts	20 pts	25 pts	30 pts	100 pts

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1. Evaluate the limits below (do not use l'Hôpital's rule):

(a)[7] $\lim_{x\to 0} \dfrac{1-\cos \pi x}{\sin^2 x}$

Solution:

This is a $[\frac{0}{0}]$ type indeterminacy. We multiply and divide by $1+\cos\pi x$ to obtain:

$$\lim_{x\to 0} \frac{1-\cos \pi x}{\sin^2 x} = \lim_{x\to 0} \frac{1-\cos^2 \pi x}{(1+\cos\pi x)\sin^2 x} = \lim_{x\to 0} \frac{\sin^2 \pi x}{(1+\cos\pi x)\sin^2 x}.$$

Now noting that $1+\cos\pi x \to 2$ as $x \to 0$, the limit above is just:

$$\frac{1}{2}\lim_{x\to 0} \frac{\sin^2 \pi x}{\sin^2 x} = \frac{\pi^2}{2}$$

by means of the fact $\lim_{x\to 0} \dfrac{\sin ax}{\sin bx} = \dfrac{a}{b}$, if $b \neq 0$.

(b)[8] $\lim_{t\to\infty} \dfrac{t^{2/3}+t^{-1}}{t^{2/3}+\cos^2 t}$

Solution:

First we remove t^{-1} term by multiplying and dividing by t, hence the required limit becomes:

$$\lim_{t\to\infty} \frac{t^{5/3}+1}{t^{5/3}+t\cos^2 t}.$$

Inspired by the fact that $\cos^2 t$ is a bounded function; by 1 from above and by 0 from below, we write:

$$f(t) := \frac{t^{5/3}+1}{t^{5/3}+t} \leq \frac{t^{5/3}+1}{t^{5/3}+t\cos^2 t} \leq \frac{t^{5/3}+1}{t^{5/3}} := g(t).$$

We easily see that $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = 1$, since large t behaviour is determined by the highest degree terms in denominator and numerator, which are $t^{5/3}$ in this case (This can be proved quite easily). But then the Sandwich Theorem entails that the middle function must have the same limit, i.e.

$$\lim_{t\to\infty} \frac{t^{2/3}+t^{-1}}{t^{2/3}+\cos^2 t} = 1.$$

(c)[10] Let $f(x) = \begin{cases} \dfrac{\sqrt{ax+b}-2}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

Determine the values of the real numbers a and b so that $f(x)$ is continuous at the point $x = 0$.

Solution:

Two conditions are to be met for continuity at $x = 0$: $\lim_{x\to 0} f(x)$ must exist and $f(0)$ must equal this limit value.

We observe that $f(x)$ is defined by the same formula $\dfrac{\sqrt{ax+b}-2}{x}$ on both sides of $x = 0$ and the denominator x approaches 0 as x tends to 0. Recalling that for the $\lim_{x\to x_0} \dfrac{p(x)}{q(x)}$ to exist we must have $p(x_0) = 0$ if $q(x_0) = 0$, although this does not guarantee the existence of the limit. We need this condition because otherwise the limit would be $+\infty$ or $-\infty$. Having exactly this situation, we force the numerator to vanish at $x = 0$, namely: $\sqrt{ax+b} - 2 = 0$ when $x = 0$. This is: $\sqrt{b} - 2 = 0$, giving us $b = 4$. With this b we have at least a $[\frac{0}{0}]$ indeterminacy.

Next task is to evaluate the limit and require it equals $f(0) = 1$. To this end, we multiply and divide by $\sqrt{ax+4} + 2$:

$$\lim_{x\to 0} \frac{\sqrt{ax+4}-2}{x} = \lim_{x\to 0} \frac{ax}{x(\sqrt{ax+4}+2)} = \frac{a}{4}.$$

This should be 1, thus we find that $a = 4$.

2. Consider the implicit relation defined by $x\sin(xy - y^2) = x^2 - 1$.

(a)[10] Find $\dfrac{dy}{dx}$.

Solution:

We use straightforward implicit differentiation together with the chain rule:

$$\sin(xy - y^2) + x\cos(xy - y^2)\,(y + xy' - 2yy') = 2x.$$

Collecting terms with y' together:

$$y' = \frac{2x - \sin(xy - y^2) - xy\cos(xy - y^2)}{x(x - 2y)\cos(xy - y^2)}.$$

(b)[10] Write down the equation(s) of the tangent line(s) to the graph of $x\sin(xy - y^2) = x^2 - 1$ at the intersection point(s) of $x\sin(xy - y^2) = x^2 - 1$ with the line $y = x$.

Solution:

Intersection points are found by setting $y = x$ in the relation:

$$x\sin(xx - x^2) = x^2 - 1 \Rightarrow x = \pm 1.$$

Hence two curves intersect at two points $P(1, 1)$ and $Q(-1, -1)$.

Now compute y' at P and Q: $y'(P) = -1$ and $y'(Q) = 3$ easily. Then:

the tangent line at $P(1, 1)$: $y - 1 = -1(x - 1) \Leftrightarrow y = -x + 2$.

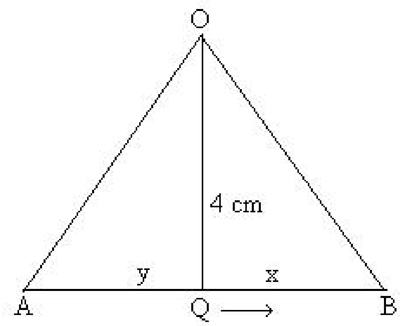
the tangent line at $Q(-1, -1)$: $y + 1 = 3(x + 1) \Leftrightarrow y = 3x + 2$.

3. Let AOB be any triangle where the point O is the top corner. Let OQ be the height from O to the side AB. Assume that the length of this height $|OQ| = 4\,\text{cm}$ (fixed), and $|OA| + |OB| = 15\,\text{cm}$ (fixed). Now, set $|AQ| = y$ and $|QB| = x$.

Suppose that the point B is moving away from the point Q at the rate of $\frac{1}{2}$ cm/sec. What is the rate of change of the distance y when $x = 3$? Is y decreasing or increasing?

Solution:

We first give a figure describing the situation:



By right triangle properties, we have the following relation between the distances x and y:

$$\sqrt{4^2 + y^2} + \sqrt{4^2 + x^2} = 15.$$

Differentiating with respect to t we receive:

$$\frac{yy'}{\sqrt{16 + y^2}} + \frac{xx'}{\sqrt{16 + x^2}} = 0.$$

We are asked to find y' when $x = 3$, and x' is given to be $1/2\,\text{cm/sec}$. Since B is moving away from Q, x is increasing and this rate of change is to be taken with positive signature. On the other, we need to know what y is, when $x = 3$. This is but easily calculated from the initial relation:

$$\sqrt{16 + y^2} + \sqrt{16 + 9} = 15 \Rightarrow 16 + y^2 = 100 \Rightarrow y = \sqrt{84} = 2\sqrt{21},$$

noting that $y > 0$ representing a distance. Then we substitute everything in the related rates equation:

$$\frac{2\sqrt{21}y'}{10} + \frac{3/2}{5} = 0 \Rightarrow y' = -\frac{3}{2\sqrt{21}}\,\text{cm/sec}$$

when $x = 3$. Thus, y is decreasing.

4. Sketch the graph of $y = f(x) = \dfrac{2 + x - x^2}{(x-1)^2}$ by examining (a) the domain, (b) intercepts, (c) asymptotes, (d) critical points and local extrema, (e) inflection points, (f) the intervals of increase and decrease, (g) concavity, and by making a table of all the data.

Solution:

Domain of $f(x)$ is clearly $\mathbb{R} - \{1\}$. $x = 0$ gives $y = 2$, and $y = 0$ gives $(x-2)(x+1) = 0$ which is to say $x = 2$ and $x = -1$. Hence the graph hits the $x-$axis at $(-1, 0)$ and $(2, 0)$, and the $y-$axis at $(0, 2)$.

The graph has a vertical asymptote at $x = 1$ as it undefined at that point. Checking the behaviour about $x = 1$: $\lim_{x\to 1^-} f(x) = +\infty$ and $\lim_{x \to 1^+} f(x) = +\infty$.

The behaviour as $|x|$ tends to ∞: $\lim_{x\to\infty} f(x) = \lim_{x\to -\infty} f(x) = -1$, hence the graph has a horizontal asymptote $y = -1$.

For the critical points we find $f'(x)$:

$$f'(x) = \frac{(1-2x)(x-1)^2 - (2+x-x^2)2(x-1)}{(x-1)^4} = \frac{x-5}{(x-1)^3}.$$

f' is defined everywhere f is defined. The only critical point then comes from $f'(x) = 0$, which is $x = 5$. Although it is not a critical point, we must note that f' changes sign before and after $x = 1$. $f' = 0$ holds at only one point, and it changes sign at that point. Thus f has a local extremum at $x = 1$. More precisely, $f' > 0$ if $x > 5$ and $f' < 0$ if $1 < x < 5$ meaning that f has a local minimum at $x = 5$. Noting also that $f' > 0$ if $x < 1$, we have: f is increasing in $(-\infty, 1) \cup (5, \infty)$, and decreasing in $(1, 5)$.

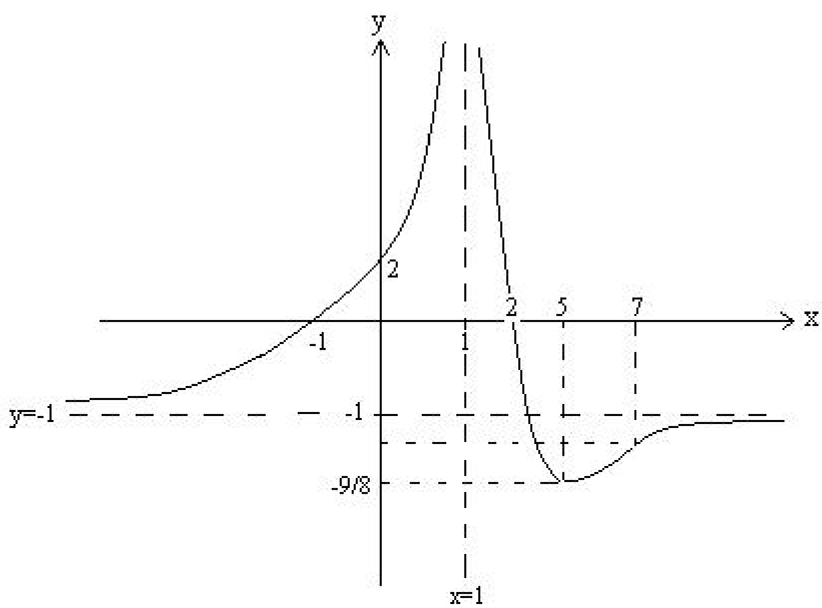
For the inflection point we calculate $f''(x)$:

$$f''(x) = \frac{(x-1)^3 - (x-5)3(x-1)^2}{(x-1)^6} = -2\frac{x-7}{(x-1)^4}.$$

f'' exists everywhere f is defined, and $f'' = 0$ gives $x = 7$. It also changes sign at this point, hence $x = 7$ is an inflection point. That is, $f'' > 0$ if $x < 7$, and $f'' < 0$ if $x > 7$. In terms of f we understand that the graph of f is concave up if $x < 7$, and concave down if $x > 7$.

We are now ready for the table of the data we have collected:

	1	5	7	
f''	+	+	+	−
f'	+	−	+	+
f	↗ concave up	↘ concave up	↗ concave up	↗ concave down



B U Department of Mathematics

Math 101 Calculus I

Spring 2006 First Midterm

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

   (a) f is differentiable $\forall x \in \mathbb{R}$

   (b) $f(a+b) = f(a) + f(b) + 2ab, \ \forall a, b \in \mathbb{R}$.

   Show that $f'(x) = f'(0) + 2x$.

   Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x) + f(h) + 2xh - f(x)}{h} = \lim_{h\to 0} \frac{f(h) + 2xh}{h}$$

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{f(h)}{h} + \lim_{h\to 0} 2x = \lim_{h\to 0} \frac{f(h)}{h} + 2x$$

$$f(0+0) = f(0) + f(0) + 2\cdot 0 \cdot 0 \ \ (a = b = 0)$$

$$\Rightarrow f(0) = 0$$

$$\lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = f'(0)$$

$$\Rightarrow f'(x) = f'(0) + 2x$$

2. (a) The point $T = (\frac{-4\sqrt{10}}{5}, \sqrt{10})$ lies on the ellipse with equation

$$9(x+y)^2 + (x-y)^2 = 36$$

Using implicit differentiation, determine the equation of the tangent line to the ellipse at the point T.

Solution:

$18(x+y)(1+y') + 2(x-y)(1-y') = 0$

At point T, $18\left(\frac{-4\sqrt{10}}{5} + \sqrt{10}\right)(1+y') + 2\left(\frac{-4\sqrt{10}}{5} - \sqrt{10}\right)(1-y') = 0$

$\Rightarrow y' = 0$

Equation of the tangent line:

$y - \sqrt{10} = 0\left(x + \frac{4\sqrt{10}}{5}\right) x$

$y = \sqrt{10}$

Tangent line is parallel to x-axis

(b) Find all points of discontinuities of the function $f(t) = \frac{\sin t \cos t}{t - t^2}$. In each case, decide whether the discontinuity is removable or not. In case of a removable discontinuity, determine how f should be defined in order to remove the discontinuity.

Solution:

$t(t-1) = 0 \Rightarrow t = 0,\ \ t = 1$ are points of discontinuity

$\lim_{t\to 1} f(t) = \lim_{t\to 1} \frac{\sin(2t)}{2t(1-t)}$

$\lim_{t\to 1^+} \frac{\sin(2t)}{2t(1-t)} = -\infty, \ \lim_{t\to 1^-} \frac{\sin(2t)}{2t(1-t)} = +\infty$

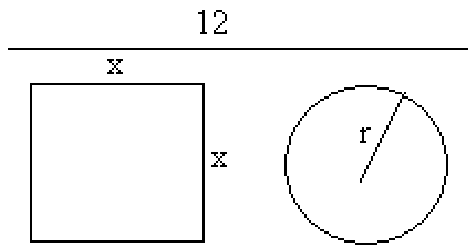
t=1 is essential discontinuity which is not removable.

$\lim_{t\to 0} f(t) = \lim_{t\to 0} \frac{\sin(2t)}{2t(1-t)} = 1$ which is a removable discontinuity.

Define $\tilde{f}(x) = \begin{cases} \frac{\sin(2t)}{2t(1-t)} & t \neq 0 \\ 1 & t = 0 \end{cases}$

3. A piece of wire 12cm. long is cut into 2 lengths, one of which is bent into a circle, the other into a square. Determine the ratio of the side length of the square to the radius of the circle if the sum of the areas of the circle and the square is minimum and find this minimum area.

Solution:



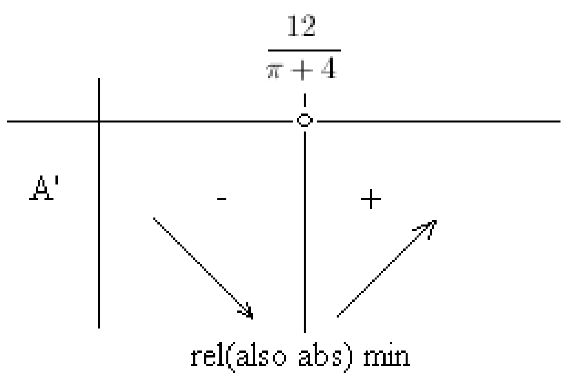
$$4x + 2\pi r = 12, \;\; 0 \le x \le 3, \;\; 0 \le r \le \frac{6}{\pi}$$

$$\Rightarrow r = \frac{12-4x}{2\pi} = \frac{6-2x}{\pi}$$

$$A = x^2 + \pi r^2$$

$$A(x) = x^2 + \pi\left(\frac{6-2x}{\pi}\right)^2 = x^2 + \frac{(6-2x)^2}{\pi} = \frac{\pi x^2 + 36 - 24x + 4x^2}{\pi}$$

$$A'(x) = \frac{2\pi x - 24 + 8x}{\pi} = 0 \Rightarrow x = \frac{24}{24\pi + 8} = \frac{12}{\pi + 4}$$



$$\frac{x}{r} = \frac{12}{\pi+4}\frac{\pi}{6 - 2\frac{12}{\pi+4}} = \frac{12\pi}{6\pi} = 2$$

$$x = \frac{12}{\pi+4} \text{ is relative min by the 1st derivative test}$$

$$A(0) = \frac{36}{\pi}, \;\; A(3) = 9$$

$$A(\frac{12}{\pi+4}) = \frac{12^2}{(\pi+4)^2} + \pi\left(\frac{6}{\pi+4}\right)^2 = \frac{36}{\pi+4} = A_{min}$$

4. Consider the function

$$f(x) = \frac{x^2 + x - 5}{x - 2}$$

(a) Determine the domain of f.

(b) Find all horizontal, vertical and oblique asymptotes of f.

(c) Find the intervals on which f is increasing or decreasing.

(d) Find all local extrema of f, if any.

(e) Find the intervals on which the graph of f is concave up or down.

(f) Sketch the graph of f.

Solution:

Domain: $\mathbb{R} \setminus \{2\}$

$x = 0 \Rightarrow f(0) = 5/2 \rightarrow$ y intercept

$y = 0 \Rightarrow x^2 + x - 5 = 0 \Rightarrow x_{1,2} = \dfrac{-1 + \sqrt{21}}{2} \rightarrow$ x intercepts

$\lim_{x\to\infty} f(x) = +\infty, \quad \lim_{x\to-\infty} f(x) = -\infty$

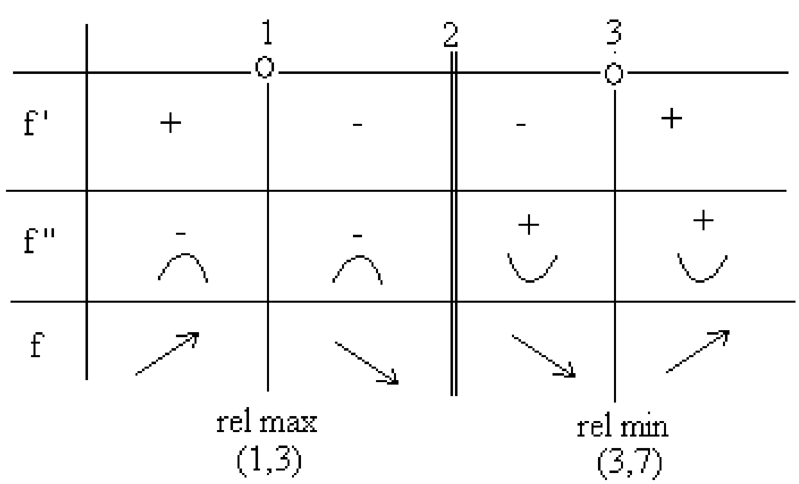
$x = 2$ vertical asym and $\lim_{x\to 2^+} = +\infty, \quad \lim_{x\to 2^-} = -\infty$

$f(x) = x + 3 + \dfrac{1}{x-2} \quad y = x + 3$ oblique asym

$\lim_{x\to+\infty} \dfrac{1}{x-2} = 0^+, \quad \lim_{x\to-\infty} \dfrac{1}{x-2} = 0^-$

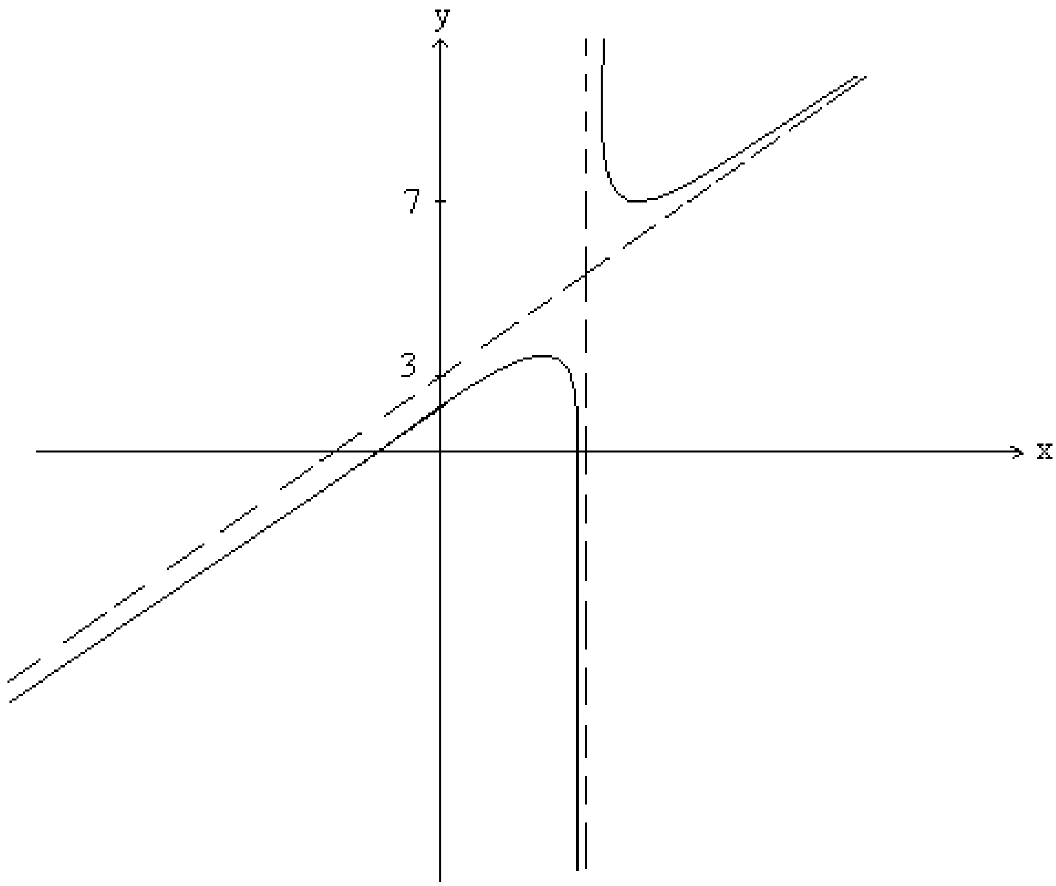
$f'(x) = \dfrac{x^2 - 4x + 3}{(x-2)^2}, \quad f'(x) = 0 \Rightarrow x = 1, x = 3$

$f''(x) = \dfrac{2}{(x-2)^3}, \quad f''(x) = 0 \Rightarrow x = 2$



$x = 1, x = 3$ are relative max and min, respectively by the 1st derivative test

The sketch of the curve is shown below.



B U Department of Mathematics

Math 101 Calculus I

Spring 1999 First Midterm

1. Find $y'(1) = 1$ if $y(1) = 1$ and $\sin^2\left(\frac{\pi}{4}xy\right) + y^2 + x = 3$.

Solution:

Take derivative of both sides of $\sin^2\left(\frac{\pi}{4}xy\right) + y^2 + x = 3$ with respect to x:
$2\sin(\frac{\pi}{4}xy).\cos(\frac{\pi}{4}xy).\frac{\pi}{4}[y + xy'] + 2yy' + 1 = 0.$
For $y(1) = 1:$
$2\sin(\frac{\pi}{4}).\cos(\frac{\pi}{4}).\frac{\pi}{4}[1 + y'] + 2y' + 1 = 0.$
Then,
$-\left(\frac{\pi}{4}\right) = y'\left(2 + \frac{\pi}{4}\right)$
and
$y'(1) = -\frac{\pi + 4}{\pi + 8}.$

2. Evaluate the following limits, if exist. (L'Hôpital's rule is not allowed.)

a) $\lim_{x\to\infty} \frac{x + \cos x}{x}$

b) $\lim_{x\to 1} \frac{x^2 + 5x + 4}{x^2 - 1}$

c) $\lim_{h\to 0} \frac{\sin\sqrt{3+h} - \sin\sqrt{3}}{h}$

Solution:

a) $\lim_{x\to\infty} \frac{x + \cos x}{x} = \lim_{x\to\infty}\left(1 + \frac{\cos x}{x}\right) = 1$

because

$0 \le \left|\frac{\cos x}{x}\right| \le \frac{1}{|x|} \Rightarrow \lim_{x\to\infty}\left(\frac{\cos x}{x}\right) = 0$ by Pinching Rule.

b) $\lim_{x\to 1} \frac{x^2 + 5x + 4}{x^2 - 1} = \lim_{x\to 1} \frac{(x+4)(x+1)}{(x+1)(x-1)} = \lim_{x\to 1} \frac{x+4}{x-1}$

$\lim_{x\to 1^+} \frac{x+4}{x-1} = \infty, \ \lim_{x\to 1^-} \frac{x+4}{x-1} = -\infty$

hence limit does not exist.

c) $\lim_{h\to 0} \frac{\sin\sqrt{3+h} - \sin\sqrt{3}}{h} = \left(\sin\sqrt{x}\right)'_{x=3} = \frac{1}{2\sqrt{3}}\cos\sqrt{3}$

3. Show that the equation $x + \cos x = 2$ has at least one solution in the interval $[0, \pi]$.

Solution:

Put $f(x) = x + \cos x - 2$. Then f is continuous on $[0, \pi]$ and $f(0) = -1 < 0$, $f(\pi) = \pi - 3 > 0$. So by Intermediate Value Theorem there exists $c \in (0, \pi)$ such that $f(c) = c + \cos c - 2 = 0$i.e. $c + \cos c = 2$.

4. **a)** Show that $|\sin x - \sin y| \leq |x - y|, \forall x, y \in \mathbb{R}$ (Hint: Use the Mean Value Theorem).

b) Using part(a) above to show that $|\sin x| \leq x, x > 0$.

Solution:

a) $f(x) = \sin x$ is continuous on $(-\infty, \infty)$ and $f'(x) = \cos x$ exist on $(-\infty, \infty)$. So for $x, y \in \mathbb{R}$ with $x \neq y$ and $y < x$there exists $c \in (y, x)$ such that $\sin x - \sin y = \cos c(x - y)$. It follows that,
$|\sin x - \sin y| \leq |x - y|$ since $|\cos c| \leq 1$.

b) Let $x > 0$. Then the function $y = \sin x$ satisfies conditions of Mean Value Theorem over [0, x] hence $\sin x - \sin 0 = \cos c(x - 0)$ for some $c \in (0, x)$. It follows that $|\sin x| \leq |x|$.

5. Integrate: **a)** $\displaystyle\int x\sqrt{2x+1}\ dx$ **b)** $\displaystyle\int \sin^2\left(\frac{x}{2}\right)\ dx$

Solution:

a) Let $u = 2x + 1 \quad du = 2dx$

$$\int x\sqrt{2x+1}\ dx = \frac{1}{2}\int\left(\frac{u-1}{2}\right)\sqrt{u}\ du$$
$$= \frac{1}{4}\int u^{3/2}\ du - \frac{1}{4}\int u^{1/2}\ du$$
$$= \frac{1}{10}u^{5/2} - \frac{1}{6}u^{3/2} + C = \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C$$

b) Let u=x/2:

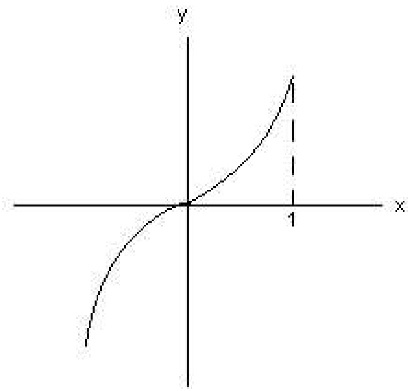
$$\int \sin^2\left(\frac{x}{2}\right)\ dx = 2\int \frac{1}{2}(1 - \cos 2u)du$$
$$= u - 1/2\sin 2u + C = \frac{x}{2} - \frac{1}{2}\sin x + C$$

6. Find the upper and the lower approximating sums ($\overline{S}_n$ and $\underline{S}_n$) for the area A of the region R bounded above by the graph of $f(x) = x^3$, below by the x-axis, on the left by y-axis, and on the right by $x = 1$ line.

Solution:

$$x_i = \frac{i}{n} \qquad i = 0, 1, 2, ..., n$$

$$\Delta x = \frac{i}{n}$$



$$\underline{S}_n = \sum_{i=0}^{n-1} f(x_i)\Delta x = \left[(\frac{1}{n})^3 + (\frac{2}{n})^3 + ... + (\frac{n-1}{n})^3\right]\frac{1}{n} = [1^3 + 2^3 + ... + (n-1)^3]\frac{1}{n^4}$$

$$\Rightarrow \underline{S}_n = \left[\frac{n(n-1)}{2}\right]^2 \frac{1}{n^4} = \frac{n^4 - 2n^3 + n^2}{4n^4} = \frac{n^2 - 2n + 1}{4n^2}$$

$$\overline{S}_n = \sum_{i=1}^{n-1} f(x_i)\Delta x = \left[(\frac{1}{n})^3 + (\frac{2}{n})^3 + ... + (\frac{n}{n})^3\right]\frac{1}{n} = [1^3 + 2^3 + ... + n^3]\frac{1}{n^4}$$

$$\Rightarrow \underline{S}_n = \left[\frac{n(n+1)}{2}\right]^2 \frac{1}{n^4} = \frac{n^4 + 2n^3 + n^2}{4n^4} = \frac{n^2 + 2n + 1}{4n^2}$$

Pay attention to the first and last indices of upper and lower approximating sums!!!

B U Department of Mathematics

Math 101 Calculus I

Summer 2001 First Midterm

1. Evaluate the limits

(a) $\lim_{x\to\infty}(2x-1-\sqrt{4x^2-4x-3})$

Solution:

$$\begin{aligned}
&\lim_{x\to\infty}(2x-1-\sqrt{4x^2-4x-3}) \ (\infty-\infty \text{ indeterminacy})\\
&=\lim_{x\to\infty}\frac{(2x-1-\sqrt{4x^2-4x-3})(2x-1+\sqrt{4x^2-4x-3})}{2x-1+\sqrt{4x^2-4x-3}}\\
&=\lim_{x\to\infty}\frac{4}{2x-1+\sqrt{4x^2-4x-3}}=0.
\end{aligned}$$

(b) $\lim_{x\to-\infty}(2x-1-\sqrt{4x^2-4x-3})$

Solution:

$$\begin{aligned}
&\lim_{x\to-\infty}(2x-1-\sqrt{4x^2-4x-3})\\
&=\lim_{x\to-\infty}\left(2x-1-|x|\sqrt{4-\frac{4}{x}-\frac{3}{x^2}}\right)=-\infty-\infty=-\infty
\end{aligned}$$

(c) $\lim_{x\to0}\dfrac{\cos x-\sqrt{\cos 2x}}{\sin 2x}$

Solution:

$$\begin{aligned}
&\lim_{x\to0}\frac{\cos x-\sqrt{\cos 2x}}{\sin^2 x}\ \left(\frac{0}{0}\text{indeterminacy}\right)\\
&=\lim_{x\to0}\frac{(\cos x-\sqrt{\cos 2x})(\cos x+\sqrt{\cos 2x})}{(\sin^2 x)(\cos x+\sqrt{\cos 2x})}\\
&=\lim_{x\to0}\frac{\cos^2 x-\cos 2x}{(\sin^2 x)(\cos x+\sqrt{\cos 2x})}\\
&=\lim_{x\to0}\frac{\cos^2 x-2\cos^2 x+1}{(\sin^2 x)(\cos x+\sqrt{\cos 2x})}\\
&=\lim_{x\to0}\frac{1-\cos^2 x}{(\sin^2 x)(\cos x+\sqrt{\cos 2x})}=\lim_{x\to0}\frac{1}{\cos x+\sqrt{\cos 2x}}=\frac{1}{2}
\end{aligned}$$

2. Consider the function $f(x)=\dfrac{x^2+3}{x^2-1}$.

(a) Determine the domain of f.

Solution:

$x^2 - 1 = 0 \Leftrightarrow x = \pm 1$. So, $Df = \mathbb{R}\backslash\{-1, 1\}$.

(b) Find all horizontal, vertical and oblique asymptotes of f.

Solution:

$\lim_{x \to -1^-} \frac{x^2+3}{x^2-1} = \infty$

$\lim_{x \to -1^+} \frac{x^2+3}{x^2-1} = -\infty$

$\lim_{x \to 1^-} \frac{x^2+3}{x^2-1} = -\infty$

$\lim_{x \to 1^+} \frac{x^2+3}{x^2-1} = \infty$. So, $x = 1$ and $x = -1$ are vertical asymptotes.

$\lim_{x \to \pm\infty} \frac{x^2+3}{x^2-1} = 1$. So, $y = 1$ is the horizontal asymptote.

There is no oblique asymptote.

(c) Find the intervals on which f is increasing or decreasing.

Solution:

$f'(x) = \frac{2x(x^2-1) - 2x(x^2+3)}{(x^2-1)^2} = \frac{-8x}{(x^2-1)^2} = 0 \Leftrightarrow x = 0$.

Hence, $x = 0, x = 1, x = -1$ are the critical points.

$f'(x) > 0 \Leftrightarrow x \in (-\infty, -1) \cup (-1, 0)$ and $f'(x) < 0 \Leftrightarrow x \in (0, 1) \cup (1, \infty)$.

So, f is increasing on $(-\infty, -1)$ and $(-1, 0)$ and decreasing on $(0, 1)$ and $(1, \infty)$.

(d) Find all local extrema of f.

Solution:

$f''(x) = \frac{24x^4 - 16x^2 - 8}{(x^2-1)^4} = \frac{(24x^2-8)(x^2+1)}{(x^2-1)^4}$

$f''(0) = -8 < 0$ so f has a local maximum at $x = 0$.

At $x = 1$ and $x = -1$, f has neither a local minimum nor a local maximum.

(e) Find the intervals on which the graph of f is concave up or down.

Solution:

$f''(x) = 0 \Leftrightarrow x = \pm\frac{1}{\sqrt{3}}$.

$x \in (-\infty, -1) \Rightarrow f''(x) > 0 \Rightarrow$ concave up on $(-\infty, -1)$.

$x \in (-1, -\frac{1}{\sqrt{3}}) \Rightarrow f''(x) > 0 \Rightarrow$ concave up on $(-1, -\frac{1}{\sqrt{3}})$.

$x \in (-\frac{1}{\sqrt{3}}, 0) \Rightarrow f''(x) < 0 \Rightarrow$ concave down on $(-\frac{1}{\sqrt{3}}, 0)$.

$x \in (0, \frac{1}{\sqrt{3}}) \Rightarrow f''(x) < 0 \Rightarrow$ concave down on $(0, \frac{1}{\sqrt{3}})$.

$x \in (\frac{1}{\sqrt{3}}, 1) \Rightarrow f''(x) > 0 \Rightarrow$ concave up on $(\frac{1}{\sqrt{3}}, 1)$.

$x \in (1, \infty) \Rightarrow f''(x) > 0 \Rightarrow$ concave up on $(1, \infty)$.

(f) Find the points of inflection.

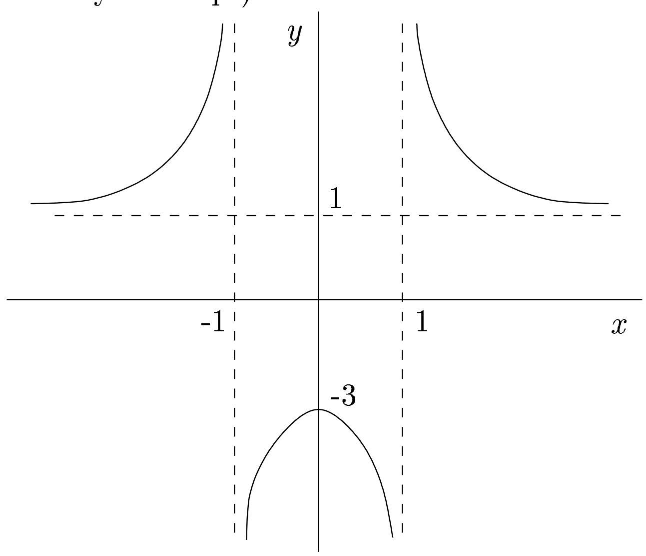
Solution:

$$f''(x) = 0 \Leftrightarrow x = \pm\frac{1}{\sqrt{3}}.\ \text{So, } x = \pm\frac{1}{\sqrt{3}} \text{ are the inflection points.}$$

(g) Sketch the graph of f.

Solution:

$x = 0 \Rightarrow y = -3$ (the only intercept)



3. Let $f(x) = \dfrac{1}{x}$.

(a) For $a \neq 0$ prove that an equation of the line tangent to the graph of f at $\left(a, \frac{1}{a}\right)$ is $y - \frac{1}{a} = -\frac{1}{a^2}(x - a)$.

Solution:

$$f'(x) = -\frac{1}{x^2} \Rightarrow m = f'(a) = -\frac{1}{a^2} \text{ is the slope of the line.}$$

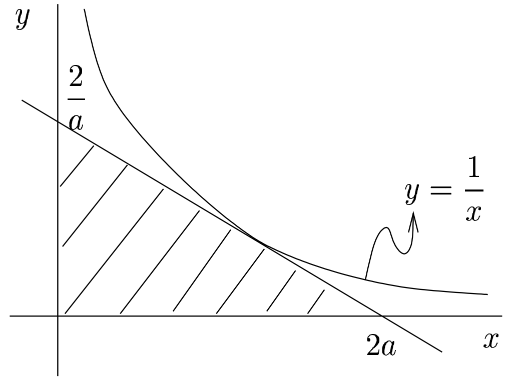
Hence the equation of the line is $y - \frac{1}{a} = m(x - a)$, that is, $y - \frac{1}{a} = -\frac{1}{a^2}(x - a)$.

(b) Show that the area of the triangle formed by the coordinate axes and the tangent line in part (a) is independent of the number a.

Solution:

$$x = 0 \Rightarrow y - \frac{1}{a} = -\frac{1}{a^2}(-a) \Rightarrow y = \frac{2}{a}$$

$$y = 0 \Rightarrow -\frac{1}{a} = -\frac{1}{a^2}(x - a) \Rightarrow a = x - a \Rightarrow x = 2a$$

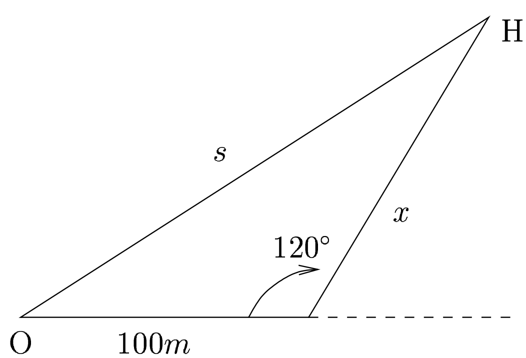


Hence, the area of the triangle is $A = \dfrac{1}{2} \cdot x \cdot y = \dfrac{1}{2} \cdot 2a \cdot \dfrac{2}{a} = 2$ which is independent of the number a.

4. A helicopter is taking off at an angle of 60°, flying at 20 m/s in a straight line away from an observer at ground level, 100 m away from take off point.

(a) Find the distance between the helicopter and the observer at 5th second after take off.

Solution:



O: the observer

H: the helicopter

$x = 20 \cdot 5 = 100$ m

$$\begin{aligned} s^2 &= x^2 + 100^2 - 2 \cdot x \cdot 100 \cdot \cos 120° \\ &= 100^2 + 100^2 - 2 \cdot 100 \cdot 100 \cdot (-\tfrac{1}{2}) \\ &= 100^2 \cdot 3 \end{aligned}$$

$\Rightarrow s = 100\sqrt{3}$ m.

(b) Find the rate of change of distance at this time.

Solution:

For any t, we have

$$\begin{aligned} s^2 &= 100^2 + (20t)^2 - 2 \cdot 100 \cdot 20t \cdot \cos 120° \\ &= 10000 + 400t^2 + 2000t. \end{aligned}$$

Take derivative of both sides with respect to t.

Then $2s\dfrac{ds}{dt} = 800t + 2000$

$$\Rightarrow 2 \cdot 100\sqrt{3} \cdot \left.\frac{ds}{dt}\right|_{t=5} = 800 \cdot 5 + 2000 \Rightarrow \left.\frac{ds}{dt}\right|_{t=5} = \frac{30}{\sqrt{3}} = 10\sqrt{3}.$$

B U Department of Mathematics

Math 101 Calculus I

Summer 2003 First Midterm

1. Find the x-coordinates of all the points on the curve $y = \dfrac{1}{100}x^2$ where the tangent line passes through the point $(100, 50)$.

Solution:

Tangent line : $y - 50 = y'(x_0)(x - 100)$ and $y'(x_0) = \dfrac{x_0}{50}$

Desired points (x_0, y_0)

$$\begin{aligned} y_0 &= \frac{x_0}{50}(x_0 - 100) + 50 \\ \frac{x_0^2}{100} &= \frac{x_0}{50}(x_0 - 100) + 50 \\ x_0^2 - 200x_0 + 5000 &= 0 \\ x_0 = \frac{200 \pm \sqrt{200^2 - 20000}}{2} &= 100 \pm 50\sqrt{2} \end{aligned}$$

2. Evaluate the following limits:

a. $\displaystyle\lim_{x\to 2} \frac{\sqrt{x+2} - \sqrt{2x}}{x^2 - 2x}$

Solution:

$$\lim_{x\to 2} \frac{\sqrt{x+2} - \sqrt{2x}}{x^2 - 2x} = \frac{0}{0}.$$

So by l'Hôpital Rule,

$$\lim_{x\to 2} \frac{\sqrt{x+2} - \sqrt{2x}}{x^2 - 2x} = \lim_{x\to 2} \frac{\frac{1}{2\sqrt{x+2}} - \frac{1}{\sqrt{2x}}}{2x - 2} = -\frac{1}{8}$$

b. $\displaystyle\lim_{x\to\infty} x \tan\frac{1}{x}$

Solution:

$$\lim_{x\to\infty} x \tan\frac{1}{x} = \lim_{x\to\infty} \frac{\sin 1/x}{1/x}\frac{1}{\cos 1/x} = 1$$

3. Given the function $f(x) = x^{\frac{-1}{3}} - x^{\frac{-2}{3}}$,

a. Determine the interval(s) on which f is increasing or decreasing.
b. Find and classify the local extrema of f, if any.
c. Determine the interval(s) on which f is concave up or concave down.
d. Find the inflection points of f, if any.
e. Find the horizontal, vertical and slant asymptotes of f, if any.
f. Sketch the graph of f.

Solution:

a.

$$\begin{aligned} f'(x) &= -\frac{1}{3}x^{-\frac{4}{3}} + \frac{2}{3}x^{-\frac{5}{3}} = 0 \\ \Rightarrow \quad x^{-\frac{4}{3}} &= 2x^{-\frac{5}{3}} \\ \Rightarrow \quad x &= 8 \end{aligned}$$

So f is decreasing on $(-\infty, 0)$ and on $(8, \infty)$ and increasing on $(0, 8)$

b. f has local max at $(8, \frac{1}{4})$

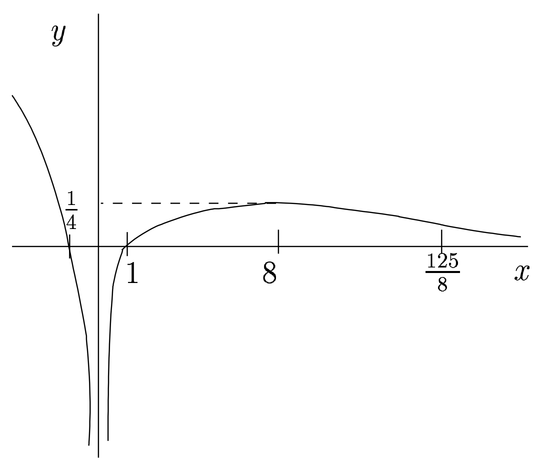
c.

$$\begin{aligned} f''(x) &= \frac{4}{9}x^{-\frac{7}{3}} - \frac{10}{9}x^{-\frac{8}{3}} = 0 \\ \Rightarrow \quad x &= \frac{125}{8} \end{aligned}$$

d. Inflection point occurs when $x = \dfrac{125}{8}$.

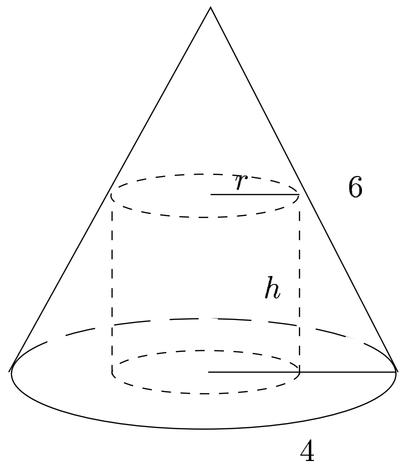
e. $x = 0$ is the vertical asymptote.
$\lim_{x\to\infty} x^{-\frac{1}{3}} - x^{-\frac{2}{3}} = 0 \Rightarrow y = 0$ is the horizontal asymptote.

f.



4. A right circular cylinder is inscribed in a cone with height 6 cm and base radius 4 cm. Find the largest possible volume of such a cylinder.

Solution:



$\dfrac{6-h}{6} = \dfrac{r}{4}$

$\Rightarrow h = 6 - \dfrac{3r}{2}$

$V = \pi r^2 h = \pi r^2 \left(6 - \dfrac{3r}{2}\right)$

$V(r) = 6\pi r^2 - \dfrac{3\pi}{2} r^3$

$\Rightarrow V'(r) = 12\pi r - \dfrac{9\pi}{2} r^2 = 0$

$\Rightarrow r = \dfrac{8}{3}$

$\Rightarrow V(\frac{8}{3}) = \dfrac{128}{9}\pi$

B U Department of Mathematics

Math 101 Calculus I

Summer 2004 First Midterm

1. Evaluate the following limits:

   (a) $\lim_{x\to\pi} \cos[x]$ **(b)** $\lim_{x\to-\infty} \dfrac{\sqrt{x^2+4}}{3x-2}$

   Justify your answers.

   Solution:

   (a) $\pi = 3.14\ldots$, therefore [x] is continuous at $x=\pi$ and $\lim_{x\to\pi}[x] = [x] = 3$. Since $\cos(x)$ is continuous everywhere,

   $$\lim_{x\to\pi}\cos[x] = \cos\lim_{x\to\pi}[x] = \cos 3\ .$$

   (b)

   $$\lim_{x\to-\infty}\frac{\sqrt{x^2+4}}{3x-2} = \lim_{x\to-\infty}\frac{\sqrt{x^2(1+\frac{4}{x^2})}}{3x-2} = \lim_{x\to-\infty}\frac{|x|\sqrt{1+\frac{4}{x^2}}}{x(3-\frac{2}{x})} = \lim_{x\to-\infty}\frac{-x\sqrt{1+\frac{4}{x^2}}}{x(3-\frac{2}{x})} = -\frac{1}{3}$$

2. Is it possible to make the function $f(x)=\dfrac{x-2}{|x|-2}$ continuous by defining $f(2)$ and $f(-2)$? Justify your answer.

   Solution:

   $\lim_{x\to 2}\dfrac{x-2}{|x|-2} = \lim_{x\to 2}\dfrac{x-2}{x-2} = 1$. So if we define $f(2)=1$ then we have
   $f(1) = \lim_{x\to 2} f(x)$ and f becomes continuous at $x=2$.
   $\lim_{x\to -2}\dfrac{x-2}{|x|-2} = \lim_{x\to -2}\dfrac{x-2}{-x-2}$ = does not exist. So however we define $f(-2)$, f cannot be made continuous at $x=-2$.

3. Show that among all rectangles with area A, the square has the minimum perimeter.

   Solution:

   Consider any rectangle with side lengths x and y, then $A = xy$ and $P = 2x+2y$.
   Hence $0 = y + x\dfrac{dy}{dx}$.
   Also $\dfrac{dP}{dx}=2+2\dfrac{dy}{dx}=0$, $0 < x < \infty$. $\Longrightarrow$ $2+2(\frac{-y}{x})=0$ $\Rightarrow$ $x=y$.
   As $x\to 0^+, x\to+\infty, P\to+\infty$, so $x=y$ is the absolute minimum.

4. Show that if f and g are functions for which $f'(x) = g(x),\ g'(x) = f(x)$, then $f^2(x) - g^2(x)$ is constant.

   Solution:

$$\frac{d}{dx}(f^2(x) - g^2(x)) = 2f(x)f'(x) - 2g(x)g'(x) = 2f(x)g(x) - 2g(x)f(x) = 0$$

for all x. Thus $f^2(x) - g^2(x)$ is constant by the Corollary of the Mean Value Theorem.

5. Sketch the graph of $f(x) = x^4 - 2x^2$.

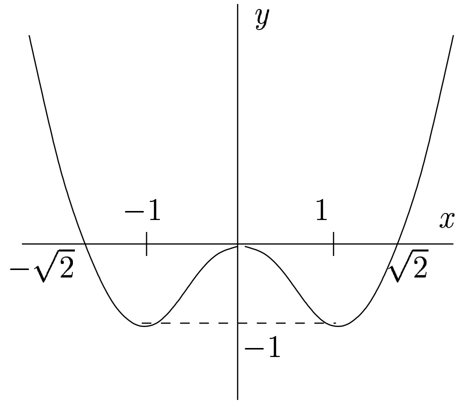
Solution:

$\lim_{x \to \pm\infty} x^4 - 2x^2 = +\infty$.

$f(x) = x^2(x^2 - 2) \;\Rightarrow\; f(0) = f(\pm\sqrt{2}) = 0$.

$f'(x) = 4x^3 - 4x = 0 \;\Rightarrow\; 4x(x^2 - 1) = 0 \;\Rightarrow\; x = 0, -1, +1$

Guess:



$f(x) \geq 0$ if $|x| \geq 2$, $\quad f(x) < 0$ if $|x| < \sqrt{2}$ because $f(x) = x^2(x^2 - 2)$.

$f' = 4x(x^2 - 1) > 0$ if $x \geq 1$ and $-1 < x < 0$ $\Big\} \Rightarrow$ increasing

$f' < 0$ if $x < -1$ and $0 < x < 1$ $\Big\} \Rightarrow$ decreasing

Therefore our guess is correct.