B U Department of Mathematics

Math 101 Calculus I

Fall 2001 Final Exam

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 2x & \text{if } x > 1 \end{cases}$.

(a) Is f continuous at $x = 1$?

Solution:

$\lim_{x\to 1^+} f(x) = 2 \cdot 1 = 2 \neq \lim_{x\to 1^-} f(x) = 1^2 = 1$. So $\lim_{x\to 1} f(x)$ does not exist.
$\therefore$ f is not continuous at $x = 1$.

(b) Is f differentiable at $x = 1$?

Solution:

Since differentiability implies continuity and f is not continuous at $x = 1$, f is not differentiable at $x = 1$.

2. Find $\dfrac{d}{dx}(f(x))$ if $\dfrac{d}{dx}(f(3x)) = 6x$.

Solution:

Let $u = 3x$. Then $6x = \dfrac{d}{dx}(f(u)) = \dfrac{d}{du}(f(u)) \cdot \dfrac{du}{dx} = f'(u) \cdot 3$
$\Rightarrow 2u = f'(u) \cdot 3 \Rightarrow f'(u) = \dfrac{2}{3}u$
or simply $f'(x) = \dfrac{2}{3}x$.

3. Determine whether the function $f(x) = 5x^5 + 4x^3$ has an inverse or not. If so, find $(f^{-1})'(9)$.

Solution:

$f'(x) = 25x^4 + 12x^2 > 0 \ \ \forall x \neq 0$
$\Rightarrow f$ is increasing. Hence f is invertible.
$f(x) = 9$ if $5x^4 + 4x^3 = 9$ or $x = 1$.
So $(f^{-1})'(9) = \dfrac{1}{f'(1)} = \dfrac{1}{25(1)^4 + 12(1)^2} = \dfrac{1}{37}$.

4. Find the arc length of the curve $y = \dfrac{x^2}{8} - \ln x$, for $4 \leq x \leq 8$.

Solution:

$$y' = \frac{2x}{8} - \frac{1}{x} = \frac{x^2-4}{4x} \Rightarrow 1 + (y')^2 = \left(\frac{x^2+4}{4x}\right)^2.$$

$$\ell = \int_4^8 \sqrt{\left(\frac{x^2+4}{4x}\right)^2}\,dx = \int_4^8 \left(\frac{x}{4} + \frac{1}{x}\right)dx$$

$$= \frac{1}{8}(64-16) + (\ln 8 - \ln 4) = 6 + \ln 2.$$

5. Evaluate the following integrals:

(a) $\displaystyle\int \frac{\cos x\,dx}{\sin^3 x - \sin x}$

Solution:

Put $u = \sin x$ so that $du = \cos x dx$.

$$I = \int \frac{du}{u^3-u} = \int \frac{du}{u(u-1)(u+1)}$$

$$\frac{1}{u(u-1)(u+1)} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{u+1} \quad \Rightarrow 1 = A(u-1)(u+1) + Bu(u+1) + Cu(u-1)$$

$u = 0 \Rightarrow A = -1$

$u = 1 \Rightarrow B = 1/2$

$u = -1 \Rightarrow C = 1/2$

$$I = \int -\frac{1}{u}du + \frac{1}{2}\int \frac{du}{u-1} + \frac{1}{2}\int \frac{du}{u+1}$$

$$= -\ln|u| + \frac{1}{2}\ln|u-1| + \frac{1}{2}\ln|u+1| + c = \ln\frac{\sqrt{\sin^2 x - 1}}{|\sin x|} + c.$$

(b) $\displaystyle\int \frac{x^2 dx}{\sqrt{9-x^2}}$

Solution:

Let $x = 3\sin\theta \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$. Then $dx = 3\cos\theta d\theta$.

$9 - x^2 = 9(1-\sin^2\theta) = 9\cos^2\theta$

$\sqrt{9-x^2} = 3\cos\theta.$

Then $\displaystyle I = \int \frac{9\sin^2\theta}{3\cos\theta} 3\cos\theta d\theta = \frac{9}{2}\int (1-\cos 2\theta)d\theta = \frac{9}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + c$

$$= \frac{9}{2}(\theta - \sin\theta\cos\theta) + c = \frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{9}{2} \times \left(\frac{x}{3}\right) \times \frac{\sqrt{9-x^2}}{3} + c$$

$$= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{x}{2}\sqrt{9-x^2} + c.$$

(c) $\displaystyle\int \ln(x + \sqrt{x^2+1})dx$

Solution:

Let $u = \ln(x + \sqrt{x^2+1})$, $dv = dx$.

Then $du = \dfrac{1 + \frac{x}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}}\, dx \Rightarrow du = \dfrac{1}{\sqrt{x^2+1}} dx$

and $v = x$.

Then by the method of integration by parts,

$I = x \cdot \ln(x + \sqrt{x^2+1}) - \displaystyle\int \frac{2x}{2\sqrt{x^2+1}} dx = x \cdot \ln(x + \sqrt{x^2+1}) - \sqrt{x^2+1} + c.$

6. Determine whether the series below converge or diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(1+\ln^2 n)}$

Solution:

Let $f(x) = \dfrac{1}{x(1+\ln^2 x)}$. It is decreasing as $x \geq 1$.

Let $F(n) = \displaystyle\int_1^n \frac{dx}{x(1+\ln^2 x)}$.

Let $u = \ln x$. Then $du = \dfrac{dx}{x}$.

Then $F(n) = \displaystyle\int_0^{\ln n} \frac{du}{1+u^2} = \arctan u\Big|_0^{\ln n} = \arctan(\ln n) - \arctan 0 = \arctan(\ln n).$

$\Rightarrow \displaystyle\lim_{n\to\infty} F(n) = \frac{\pi}{2} \Rightarrow$ The series converges by Integral Test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+2+\ldots+n}$

Solution:

Recall: $1+2+\ldots+n = \dfrac{n(n+1)}{2} = \dfrac{n^2+n}{2}$.

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+2+\ldots+n} = 2^{-1} \sum_{n=1}^{\infty} \frac{1}{n^2+n}.$

$0 \leq \dfrac{1}{n^2+n} \leq \dfrac{1}{n^2}$ and being p-series with $p = 2 > 1$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

So $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges by Comparison Test.

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{1+2+\ldots+n}$ is convergent.

7. Find the radius and interval of convergence of the power series

$$\sum_{n=2}^{\infty} \frac{(\ln n)(x-1)^n}{e^n}$$

Solution:

Let $a_n = \dfrac{(\ln n)(x-1)^n}{e^n}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\ln(n+1)(x-1)^{n+1}}{e^{n+1}} \cdot \frac{e^n}{\ln n (x-1)^n}\right| = \frac{1}{e}\frac{\ln(n+1)}{\ln n}|x-1| \to \frac{1}{e}|x-1|$$

as $n \to \infty$.

So the series converges absolutely when $|x-1| < e$ and diverges when $|x-1| > e$ by Ratio Test.

<u>End-Points</u>:

$|x-1| < e \Leftrightarrow -e < x-1 < e \Leftrightarrow 1-e < x < 1+e$.

<u>$x = 1+e$</u>: $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n (e)^n}{e^n} = \sum_{n=2}^{\infty} \ln n.$

Since $\ln n \to \infty$ as $n \to \infty$, the series diverges by n-th term test.

<u>$x = 1-e$</u>: $\displaystyle\sum_{n=2}^{\infty} (-1)^n \ln n.$

Again the series diverges by n-th term test.

So the radius of convergence $R = e$ and the interval of convergence is $(1-e, 1+e)$.

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Math 101 Calculus I Fall 2002 Final Exam

1. Show that $f(x) = \dfrac{x^2+1}{x-1} + \dfrac{x^4}{x-2}$ has at least one root in the open interval (1,2) (Hint: Check continuity and the behaviour at the end points).

   Solution:

   $\lim\limits_{x\to+\infty} f(x) = +\infty.$
   $\lim\limits_{x\to-\infty} f(x) = -\infty.$
   Therefore there exists an element c in $(1,2)$ such that $f(c) = 0$.

2. Compute $f'(0)$ by using the definition of derivative if

$$f(x) = \begin{cases} \frac{1}{x^2}\int_0^x \sin(t^2)dt & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

   Solution:

$$f'(0) = \lim_{x\to 0} \frac{1}{x^3}\int_0^x \sin(t^2)dt = \frac{0}{0}$$

   by l'Hôpital's rule

$$\lim_{x\to 0} \frac{1}{x^3}\int_0^x \sin(t^2)dt = \lim_{x\to 0} \frac{\sin(x^2)}{x^2} = 1.$$

3. Evaluate the following limits :

   (a) $\lim\limits_{x\to\infty} x(\frac{\pi}{2} - \arctan x)$

   Solution:

$$\lim_{x\to\infty} x(\frac{\pi}{2} - \arctan x) = \infty.0$$

   By l'Hôpital's rule $\lim\limits_{x\to\infty} x(\frac{\pi}{2} - \arctan x) = \lim\limits_{x\to\infty} \dfrac{(\frac{\pi}{2} - \arctan x)}{\frac{1}{x}} = \lim\limits_{x\to\infty} \dfrac{x^2}{1+x^2} = 1.$

   (b) $\lim\limits_{x\to 0^+} [\sin(x^2)]^{\frac{1}{\ln x}}$

   Solution:

   If $y = [\sin(x^2)]^{\frac{1}{\ln x}}$ then $\ln y = \dfrac{\ln\sin(x^2)}{\ln x}$

   $\lim\limits_{x\to 0^+} \ln y = \lim\limits_{x\to 0^+} \dfrac{\ln\sin(x^2)}{\ln x} = \dfrac{\infty}{\infty}$. By l'Hôpital's rule

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} 2\frac{\cos(x^2)x^2}{\sin(x^2)} = 2.$$

$$\lim_{x\to 0^+} y = e^2$$

4. Evaluate $\int_{\frac{3\pi}{2}}^{\frac{3\pi}{2}} f(x)dx$ if $f'(x) = \dfrac{\cos x}{x}$ and $f(\frac{\pi}{2}) = f(\frac{3\pi}{2}) = 1$. (Hint : Use integration by parts.)

Solution:

If $u = f(x)$ then $du = f'(x)dx$.

If $dv = dx$ then $v = x$.

By integration by parts

$$\int f(x)dx = xf(x) - \int xf'(x)dx = \frac{3\pi}{2} - \frac{\pi}{2} - \int_{\frac{3\pi}{2}}^{\frac{3\pi}{2}} \cos(x)dx = \pi - 2.$$

5. Evaluate $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(1+x)}$.

Solution:

This is an improper integral.

$$\int_0^\infty \frac{dx}{\sqrt{x}(1+x)} = \lim_{c\to 0^+}\int_c^1 \frac{dx}{\sqrt{x}(1+x)} + \lim_{t\to+\infty}\int_1^t \frac{dx}{\sqrt{x}(1+x)}$$
$$= \lim_{c\to 0^+} 2\arctan(\sqrt{x}) + \lim_{t\to+\infty} 2\arctan(\sqrt{x}) = 0 + \frac{\pi}{2} = \frac{\pi}{2}$$

6. Determine whether the following series are convergent or divergent.

(a) $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{k}{\sqrt{k^2+k+1}}$

Solution:

$\lim_{n\to\infty} a_n \neq 0$.Therefore this series is divergent.

(b) $\displaystyle\sum_{k=0}^{\infty} 2^{-k}\sin^2(e^{2k})$

Solution:

$|2^{-k}\sin^2(e^{2k})| \leq \dfrac{1}{2^k}$ and we know $\displaystyle\sum_{k=0}^{\infty}\frac{1}{2^k}$ is a convergent geometric series,by comparison test our series converges absolutely. Hence convergent.

(c) $\sum_{k=0}^{\infty}(1+(-1)^k)\frac{3^k}{(k-1)!}$

Solution:

$\sum_{k=0}^{\infty}(1+(-1)^k)\frac{3^k}{(k-1)!}$ *and* $\sum_{i=0}^{\infty}\frac{2.9^i}{(2i-1)!}$ have the same character.

By the ratio test $\lim_{i\to\infty}\frac{9}{(2i+1)(2i)}=0$. Hence the series converges.

7. **(a)** Given the power series $\sum_{k=0}^{\infty}(-1)^k\frac{(x-2)^k}{2^k(k+1)^{\frac{3}{4}}}$ find the radius and interval of convergence.

Solution:

$$r=\lim_{k\to+\infty}\left|\frac{(x-2)^{k+1}}{2^{k+1}(k+2)^{\frac{3}{4}}}\cdot\frac{2^k(k+1)^{\frac{3}{4}}}{(x-2)^k}\right|=\frac{|x-2|}{2}$$

thus the series converges absolutely if $|x-2|<2$.

To determine the convergence behavior at the end points $x=0, x=4$

If $x=0$ then the series becomes

$\sum_{k=0}^{\infty}\frac{1}{(k+1)^{\frac{3}{4}}}$ which is a convergent series.

If $x=4$ then the series becomes

$\sum_{k=0}^{\infty}(-1)^k\frac{1}{(k+1)^{\frac{3}{4}}}$ which is a convergent alternating series.

so the interval of convergence is [0,4].

(b) Write down the sixth Taylor polynomials about x=0 for the function $f(x)=\sin^2 x$.

Solution:

Since for $\cos x$ $p_6=\sum_{k=0}^{3}(-1)^k\frac{x^{2k}}{(2k)!}$ we have $\cos 2x=\sum_{k=0}^{3}(-1)^k\frac{(2x)^{2k}}{(2k)!}$

so for $\sin^2(x)$ the sixth Taylor polynomial about x=0 is equal to $\frac{1}{2}(1-\sum_{k=0}^{3}(-1)^k\frac{(2x)^{2k}}{(2k)!})$

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Math 101 Calculus I

Fall 2003 Final Exam

(1) *Find the derivative of* $x^{(e^x)}$ *with respect to* x $(x > 0)$.

Solution:

Put $y = x^{(e^x)}$. Then

$$\ln y = e^x \ln x,$$

$$\frac{y'}{y} = e^x \frac{1}{x} + e^x \ln x = e^x \left(\frac{1}{x} + \ln x\right),$$

$$y' = x^{(e^x)} e^x \left(\frac{1}{x} + \ln x\right).$$

(2) *Find the coordinates of all points on the graph of* $y = 1 - x^2$ *at which the tangent line passes through the point* $(2, 0)$.

Solution:

Let (a, b) be a point on the graph of the function. The tangent line to the graph at (a, b) has slope $y'(a) = -2a$. Since the tangent line is required to pass through the point $(2, 0)$, it follows that $-2a = \dfrac{b-0}{a-2} = \dfrac{1-a^2}{a-2}$ so that

$$-2a^2 + 4a = 1 - a^2,$$

$$a^2 - 4a + 1 = 0,$$

$$a = 2 + \sqrt{3} \text{ or } a = 2 - \sqrt{3}.$$

Therefore the points are $(2 - \sqrt{3}, -6 + 4\sqrt{3})$ and $(2 + \sqrt{3}, -6 - 4\sqrt{3})$.

(3) *(a) Find the smallest and largest values of the function $f(x) = x - \sin 2x$ on the interval $[0, \pi]$.*

Solution:

The extrema occur at the critical points on $[0, \pi]$ or at the end points 0 and π. The critical points, by definition, satisfy $f'(x) = 1 - 2\cos 2x = 0$. Hence $\cos 2x = 1/2$ so that $x = \pi/6$ or $x = \pi - \pi/6 = 5\pi/6$. Comparing the values:

$$f(0) = 0,$$
$$f(\pi/6) = \frac{\pi}{6} - \frac{\sqrt{3}}{2} < 0,$$
$$f(5\pi/6) = \frac{5\pi}{6} - \frac{-\sqrt{3}}{2} > \pi,$$
$$f(\pi) = \pi,$$

we get $f(\pi/6) < f(0) < f(\pi) < f(5\pi/6)$. Therefore $f(\pi/6)$ is absolute minimum and $f(5\pi/6)$ is absolute maximum on $[0, \pi]$.

(b) Show that $f(x) = x - \sin 2x$ has at least two roots on $[0, \pi]$.

Solution:

Since $f(\pi/6) < 0 < f(5\pi/6)$ and $f(x)$ is continuous, by Intermediate Value Theorem there is at least one point on $[\pi/6, 5\pi/6]$ at which f is zero. Furthermore $f(0) = 0$. Hence there are at least two roots of f on $[0, \pi]$.

(4) *Is the improper integral $\int_0^{+\infty} xe^{-x^2}\,dx$ convergent? Justify your answer.*

Solution:

Setting $u = x^2$, we find $\int xe^{-x^2}\,dx = \frac{1}{2}\int e^{-u}du = -\frac{1}{2}e^{-u} + c = -\frac{1}{2}e^{-x^2} + c$ and

$$\begin{aligned}
\int_0^{+\infty} xe^{-x^2}\,dx &= \lim_{l\to+\infty}\int_0^l xe^{-x^2}\,dx \\
&= \lim_{l\to+\infty}\frac{1}{2}\left[-e^{-x^2}\right]_{x=0}^{l} \\
&= -\frac{1}{2}\lim_{l\to+\infty}\left[e^{-l^2} - e^{0^2}\right] \\
&= -\frac{1}{2}[0-1] = \frac{1}{2},
\end{aligned}$$

so the given improper integral converges to $1/2$.

(5) *Find the indefinite integral* $\int \tan^{-1} x\, dx = \int \arctan x\, dx$.

Solution:

Writing $u = \tan^{-1} x$ and $v = x$, and using integration by parts, we find $du = \dfrac{1}{x^2+1}\, dx$ and $v = x$, say, and so

$$\begin{aligned}
\int \tan^{-1} x\, dx &= (\tan^{-1} x)x - \int x \frac{1}{1+x^2}\, dx \\
&= x \tan^{-1} x - \frac{1}{2} \int \frac{2x\, dx}{1+x^2} \\
&= x \tan^{-1} x - \frac{1}{2} \int \frac{dy}{y} \qquad \text{(on setting } y = x^2+1) \\
&= x \tan^{-1} x - \frac{1}{2} \ln|y| + c \\
&= x \tan^{-1} x - \frac{1}{2} \ln(x^2+1) + c
\end{aligned}$$

since $y = x^2 + 1 > 0$.

(6) *Let R be the region enclosed by the curve $y = \dfrac{1}{x^2+1}$ and the x-axis from 0 to 1. Find the volume of the solid generated when the region R is revolved about the x-axis.*

Solution:

$$\begin{aligned}
V &= \int_0^1 \pi\left(\frac{1}{x^2+1}\right)^2 dx \\
&= \pi \int_0^1 (x^2+1)^{-2} dx \qquad \left(\text{put } x = \tan u,\, dx = \sec^2 u\, du;\, -\pi/2 < u < \pi/2\right) \\
&= \pi \int_0^{\pi/4} (\tan^2 u + 1)^{-2} \sec^2 u\, du \\
&= \pi \int_0^{\pi/4} (\sec u)^{-4} \sec^2 u\, du \\
&= \pi \int_0^{\pi/4} \cos^2 u\, du \\
&= \frac{\pi}{2} \int_0^{\pi/4} (\cos 2u + 1)\, du \\
&= \frac{\pi}{2} \left[\frac{\sin 2u}{2} + u\right]_0^{\pi/4} \\
&= \frac{\pi}{2}(1/2 + \pi/4) \\
&= \frac{\pi}{8}(2+\pi).
\end{aligned}$$

(7) *Find the integral* $\int \frac{x-3}{x^3-1}\,dx$.

Solution:

Express the integrand in partial fractions:

$$\frac{x-3}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1},$$
$$x-3 = (A+B)x^2 + (A-B+C)x + (A-C),$$
$$A = -2/3, B = 2/3, C = 7/3.$$

It follows that

$$\begin{aligned}
\int \frac{x-3}{x^3-1}\,dx &= \int \Big(\frac{-2/3}{x-1} + \int \frac{2x/3+7/3}{x^2+x+1}\Big)dx \\
&= -\frac{2}{3}\ln|x-1| + \frac{1}{3}\int \frac{2x+1+6}{x^2+x+1}dx \\
&= -\frac{2}{3}\ln|x-1| + \frac{1}{3}\ln|x^2+x+1| + 2\int \frac{dx}{x^2+x+1} \\
&= \cdots + 2\int \frac{dx}{x^2+x+\frac{1}{4}+\frac{3}{4}} \\
&= \cdots + 2\int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} \qquad \Big(\text{put } u = \frac{2}{\sqrt{3}}(x+\frac{1}{2}), du = \frac{2}{\sqrt{3}}dx\Big) \\
&= \cdots + 2\int \frac{\frac{\sqrt{3}}{2}du}{\frac{3}{4}u^2+\frac{3}{4}} \\
&= \cdots + \frac{4}{\sqrt{3}}\int \frac{du}{u^2+1} \\
&= \cdots + \frac{4}{\sqrt{3}}\tan^{-1} u \\
&= -\frac{2}{3}\ln|x-1| + \frac{1}{3}\ln(x^2+x+1) + \frac{4}{\sqrt{3}}\tan^{-1}(\frac{2}{\sqrt{3}}(x+\frac{1}{2})).
\end{aligned}$$

(8) *Investigate the convergence behavior of the following series.*

(a) $\sum_{n=1}^{+\infty}(-1)^n\left[\left(1+\frac{1}{n}\right)^n - 1\right]$.

Solution:

Since $\left(1+\frac{1}{n}\right)^n - 1$ has the limit $e-1 \neq 0$ as $n \to +\infty$, the nth term does not tend to 0, therefore the given series diverges.

(b) $\sum_{n=1}^{+\infty} \frac{n^{3/2}+n}{n^{11/4}+\ln n}$.

Solution:

We have

$$\lim_{n\to+\infty}\frac{\dfrac{n^{3/2}+n}{n^{11/4}+\ln n}}{\dfrac{n^{3/2}}{n^{11/4}}}=\lim_{n\to+\infty}\frac{n^{3/2}+n}{n^{3/2}}\,\frac{n^{11/4}}{n^{11/4}+\ln n}=1\neq 0,+\infty$$

and by the limit comparison test, the given series behaves in the same way as $\sum_{n=1}^{+\infty}\frac{n^{3/2}}{n^{11/4}}$ does. The latter series $\sum_{n=1}^{+\infty}\frac{n^{3/2}}{n^{11/4}}=\sum_{n=1}^{+\infty}\frac{1}{n^{11/4-6/4}}=\sum_{n=1}^{+\infty}\frac{1}{n^{5/4}}$ is a p-seris with $p=5/4>1$, so it is convergent. We conclude that the given series $\sum_{n=1}^{+\infty}\frac{n^{3/2}+n}{n^{11/4}+\ln n}$ is convergent, too.

(9) *Find all real numbers x for which the power series*

$$\sum_{n=1}^{+\infty}\frac{x^n}{2n-1}=x+\frac{x^2}{3}+\frac{x^3}{5}+\frac{x^4}{7}+\cdots+\frac{x^n}{2n-1}+\cdots$$

is convergent. [Don't forget the endpoints of the interval of convergence.]

Solution:

By the Ratio Test for absolute convergence, the series is convergent whenever $1>\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right|$. Since

$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to+\infty}\left|\frac{\frac{x^{n+1}}{2(n+1)-1}}{\frac{x^n}{2n-1}}\right|=\lim_{n\to+\infty}\left|\frac{x(2n-1)}{2n+1}\right|=|x|,$$

the series is convergent if $|x|<1$ or equivalently $-1<x<1$.

As for the end points, for $x=1$, we get $\sum_{n=1}^{+\infty}\frac{1}{2n-1}$ which is divergent because $\sum_{n=1}^{+\infty}\frac{1}{n}$ is divergent (Limit Comparison Test). For $x=-1$, we get $\sum_{n=1}^{+\infty}\frac{(-1)^n}{2n-1}$. This series is absolutely divergent but is convergent (therefore conditionally convergent). To see this, observe: (1) $\frac{1}{2n-1}$ is always positive; (2) $\frac{1}{2n-1}$ is decreasing; (3) $\lim_{n\to+\infty}\frac{1}{2n-1}=0$ and apply Alternating Series Test.

We conclude that $\sum_{n=1}^{+\infty} \frac{x^n}{2n-1}$ is convergent on $[-1, 1)$ and divergent otherwise.

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Math 101 Calculus I

Fall 2004 Final

1.) Let

$$f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ 3 - x & \text{if } 0 \leq x < 3 \\ (x-3)^2 & \text{if } x > 3 \end{cases}$$

Determine the point(s) at which $f(x)$ is discontinuous. Explain in detail.

Solution:

Check the points $x = 0$ and $x = 3$ because these are candidate points of discontinuity.

$\underline{x = 0}$:

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \sqrt{-x} = 0$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (3 - x) = 3$

$\Rightarrow \lim_{x \to 0} f(x)$ doesn't exist and therefore f is discontinuous at $x = 0$.

$\underline{x = 3}$:

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (3 - x) = 0$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x - 3)^2 = 0$

$\Rightarrow \lim_{x \to 3} f(x) = 0$ but $f(3)$ is undefined and therefore f is discontinuous at $x = 3$, too because for continuity we must have $\lim_{x \to a} f(x) = f(a)$.

2.) For what values of r does the function $y = e^{rx}$ satisfy the equation $y'' + 5y' - 6y = 0$?

Solution:

$y = e^{rx} \Rightarrow y' = re^{rx}$ and $y'' = r^2 e^{rx}$

Now substitute in the equation above

$r^2 e^{rx} + 5re^{rx} - 6e^{rx} = 0$

$\Rightarrow e^{rx}(r^2 + 5r - 6) = 0$

e^{rx} cannot be zero and hence $r^2 + 5r - 6 = 0 \Rightarrow (r - 1)(r + 6) = 0$

$r = 1$ or $r = -6$

3.) Find a function f and a number a such that $6 + \int_a^x \frac{f(t)}{t^2} dt = 2\sqrt{x}$.

Solution:

Differentiate both sides with respect to x:

$$\frac{d}{dx}\left(6 + \int_a^x \frac{f(t)}{t^2} dt\right) = \frac{d}{dx}\left(2\sqrt{x}\right)$$

$$\frac{f(x)}{x^2} = \frac{1}{\sqrt{x}} \Rightarrow f(x) = x^{3/2}$$

Then, $6 + \int_a^x \frac{t^{3/2}}{t^2} dt = 2\sqrt{x}$

$$2\sqrt{t}\ ]_a^x = 2\sqrt{x} - 6$$

$$\Rightarrow 2\sqrt{x} - 2\sqrt{a} = 2\sqrt{x} - 6$$

$$\sqrt{a} = 3 \Rightarrow a = 9.$$

4)Evaluate **a)** $\int \frac{5x^3 - 3x^2 + 7x - 3}{(x^2+1)^2} dx$

Solution:

By partial fractions

$$\frac{5x^3 - 3x^2 + 7x - 3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

$$5x^3 - 3x^2 + 7x - 3 = (Ax+B)(x^2+1) + Cx + D = Ax^3 + Bx^2 + (A+C)x + B + D$$

Comparing the coefficients, we get:

$$A = 5 \quad B = -3 \qquad C + A = 7 \qquad B + D = -3$$

$$C = 2, \quad D = 0.$$

$$\Rightarrow I = \int \frac{5x}{x^2+1} dx - \int \frac{3}{x^2+1} dx + \int \frac{2x}{(x^2+1)^2} dx$$

$$\int \frac{5x}{x^2+1} dx \qquad u = x^2 + 1 \;\; du = 2xdx$$

$$\Rightarrow \int \frac{5x}{x^2+1} dx = \int \frac{5}{2u} = \frac{5}{2} ln|u| = \frac{5}{2} ln|x^2+1|$$

$$\int \frac{3}{x^2+1} dx = 3tan^{-1}x$$

$$\int \frac{2x}{(x^2+1)^2}dx \quad u = x^2+1 \;\; du = 2xdx$$

$$\Rightarrow \int \frac{2x}{(x^2+1)^2}dx = \int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{x^2+1}$$

Therefore, $I = \frac{5}{2}ln|x^2+1| - 3tan^{-1}x - \frac{1}{x^2+1} + C$

b) $\int \frac{(1-x^2)^{3/2}}{x^6}dx$

Solution:

Let $x = \sin\theta \Rightarrow dx = \cos\theta d\theta$

$1 - x^2 = \cos^2\theta$

$$\int \frac{(1-x^2)^{3/2}}{x^6}dx = \int \frac{(\cos^2\theta)^{3/2}}{\sin^6\theta}\cos\theta d\theta = \int \frac{\cos^4\theta}{\sin^4\theta}\frac{1}{\sin^2\theta}d\theta = \int \cot^4\theta\csc^2\theta d\theta$$

Now let $u = \cot\theta \Rightarrow du = -\csc^2\theta d\theta$, Then,

$$\int \cot^4\theta\csc^2\theta d\theta = -\int u^4 du = -\frac{u^5}{5} + C = -\frac{\cot^5\theta}{5} + C = -\frac{1}{5}\left(\frac{\sqrt{1-x^2}}{x}\right)^5 + C$$

5) Evaluate **a)** $\int e^{-x}\sin\pi x dx$

Solution:

By parts: let $u = \sin\pi x \quad du = \pi\cos\pi x dx$

$dv = e^{-x}dx \quad v = -e^{-x}$

$$\int udv = uv - \int vdu$$

$$I = \int e^{-x}\sin\pi x dx = -\sin\pi x e^{-x} + \int e^{-x}\pi\cos\pi x dx$$

by parts again: $u = \pi\cos\pi x \quad du = -\pi^2\sin\pi x dx$

$dv = e^{-x}dx \quad v = -e^{-x}$

Then, $I = -\sin\pi x e^{-x} + \left[-\pi\cos\pi x e^{-x} - \int(-e^{-x})(-\pi^2\sin\pi x)dx\right]$

$$\Rightarrow I = -\sin\pi x e^{-x} - \pi\cos\pi x e^{-x} - \int e^{-x}\pi^2\sin\pi x dx$$

$$\Rightarrow I = -\sin\pi x e^{-x} - \pi\cos\pi x e^{-x} - \pi^2 I$$

$$(\pi^2+1)I = -e^{-x}(\sin\pi x + \pi\cos\pi x) + C$$

$$\Rightarrow I = -\frac{e^{-x}}{1+\pi^2}(\sin \pi x + \pi \cos \pi x) + C$$

Alternatively, one can start by letting $u = e^{-x}$ and $dv = \sin \pi x$

b) Show that $\displaystyle\int_0^\infty x^2 e^{-x^2} dx = \frac{1}{2}\int_0^\infty e^{-x^2} dx.$

Solution:

$$\int_0^\infty x^2 e^{-x^2} dx = \lim_{l \to \infty} \int_0^l x^2 e^{-x^2} dx$$

$$u = x \Rightarrow du = dx.$$

$$dv = xe^{-x^2} dx \Rightarrow v = \frac{e^{-x^2}}{-2} \quad \text{since} \int xe^{-x^2} dx = \frac{e^{-x^2}}{-2} + C$$

$$\int_0^\infty x^2 e^{-x^2} dx = \lim_{l \to \infty} \frac{xe^{-x^2}}{-2}]_0^l + \int_0^\infty \frac{e^{-x^2}}{2} dx$$

$$\int_0^\infty x^2 e^{-x^2} dx = \lim_{l \to \infty} (\frac{le^{-l^2}}{-2} - 0) + \int_0^\infty \frac{e^{-x^2}}{2} dx = \lim_{l \to \infty} \frac{l}{-2e^{l^2}} \left(\frac{\infty}{\infty}\right) + \int_0^\infty \frac{e^{-x^2}}{2} dx$$

$$\lim_{l \to \infty} \frac{l}{-2e^{l^2}} \left(\frac{\infty}{\infty}\right) = \lim_{l \to \infty} \frac{1}{-2e^{l^2} 2l} = 0$$

$$\Rightarrow \int_0^\infty x^2 e^{-x^2} dx = \frac{1}{2}\int_0^\infty e^{-x^2} dx$$

6) Evaluate $\displaystyle\lim_{x \to +\infty} (2e^x + x^2)^{3/x}$.

Solution:

$$\ln y = \frac{3}{x} ln(2e^x + x^2)$$

$$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{3\ln(2e^x + x^2)}{x} \left(\frac{\infty}{\infty}\right) = \lim_{x \to \infty} \frac{\frac{3(2e^x + 2x)}{2e^x + x^2}}{1} = \lim_{x \to \infty} \frac{3(2e^x + 2x)}{2e^x + x^2} \left(\frac{\infty}{\infty}\right)$$

$$\lim_{x \to \infty} \frac{3(2e^x + 2x)}{2e^x + x^2} = \lim_{x \to \infty} \frac{6e^x + 6}{2e^x + 2x} \left(\frac{\infty}{\infty}\right) = \lim_{x \to \infty} \frac{6e^x}{2e^x + 2} \left(\frac{\infty}{\infty}\right) = \lim_{x \to \infty} \frac{6e^x}{2e^x} \left(\frac{\infty}{\infty}\right) = 3$$

$$\Rightarrow \lim_{x \to \infty} y = e^3$$

7) Determine the convergence or divergence of the following series:

a) $\displaystyle\sum_{n=1}^\infty \frac{3n^2 + 5n}{2^n(n^2+1)}$

Solution:

Limit comparison test: compare with $\sum \frac{1}{2^n}$ which converges (a geometric series with r=1/2).

$$\rho = \lim_{n\to\infty} \frac{\dfrac{3n^2+5n}{2^n(n^2+1)}}{\dfrac{1}{2^n}} = \lim_{n\to\infty} \frac{3n^2+5n}{n^2+1} = 3 > 0 \Rightarrow \text{series converges.}$$

b) $\sum_{n=1}^{\infty} \frac{(n+3)!}{3!n!3^n}$

Solution:

Ratio test:

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{(n+4)!}{3!(n+1)!3^{n+1}}}{\dfrac{(n+3)!}{3!n!3^n}} = \lim_{n\to\infty} \frac{n+4}{3(n+1)} = 1/3 < 1 \Rightarrow \text{the series converges.}$$

c) $\sum_{k=1}^{\infty} \frac{e^k}{k^2}$

Solution:

Ratio test:

$$\rho = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{e^{k+1}}{(k+1)^2} \frac{k^2}{e^k} = \lim_{k\to\infty} e\left(\frac{k}{k+1}\right)^2 = e > 1 \Rightarrow \text{diverges}$$

or root test: $\lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \frac{e}{\sqrt[k]{k}} = e > 1 \Rightarrow$ diverges

or divergence test: $\lim_{k\to\infty} \frac{e^k}{k^2} \left(\frac{\infty}{\infty}\right) = \lim_{k\to\infty} \frac{e^k}{2k} \left(\frac{\infty}{\infty}\right) = \lim_{k\to\infty} \frac{e^k}{2} = \infty \Rightarrow$ diverges.

8) Find the interval of convergence of the series $\sum_{k=1}^{\infty} (-1)^k \frac{(x+2)^k}{k^2 3^k}$.

Solution:

$$\rho = \lim_{k\to\infty} \left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty} \left|\frac{(-1)^{k+1}(x+2)^{k+1}}{(k+1)^2 3^{k+1}} \frac{k^2 3^k}{(-1)^k (x+2)^k}\right|$$

$$\Rightarrow \rho = \lim_{k\to\infty} \left|\frac{x+2}{3}\right| \left(\frac{k}{k+1}\right)^2 = \frac{|x+2|}{3}$$

So the series converges absolutely if $\dfrac{|x+2|}{3} < 1$ and diverges if $\dfrac{|x+2|}{3} > 1$

Now focus on the interval of convergence: $\dfrac{|x+2|}{3} < 1 \Leftrightarrow -3 < x+2 < 3 \Leftrightarrow -5 < x < 1$ for convergence.

Check the endpoints:

$x = -5 \Rightarrow \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k(-3)^k}{k^2 3^k} = \sum_{k=1}^{\infty} \frac{1}{k^2}$ p series with $p = 2 \Rightarrow$ convergent.

$x = 1 \Rightarrow \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ converges absolutely since $\Rightarrow \displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent.

$\Rightarrow \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ converges.

Conclusion: interval of convergence is $[-5, 1]$.

B U Department of Mathematics

Math 101 Calculus I

Fall 2005 Final Exam

1. Assume that $f(x)$ is defined for all x such that $|x| \leq 1$ and satisfies

$$x \leq f(x) \leq x + x^2 \text{ for all } x \text{ with } |x| \leq 1.$$

Prove that $f'(0)$ exists and has the value 1.

Solution:

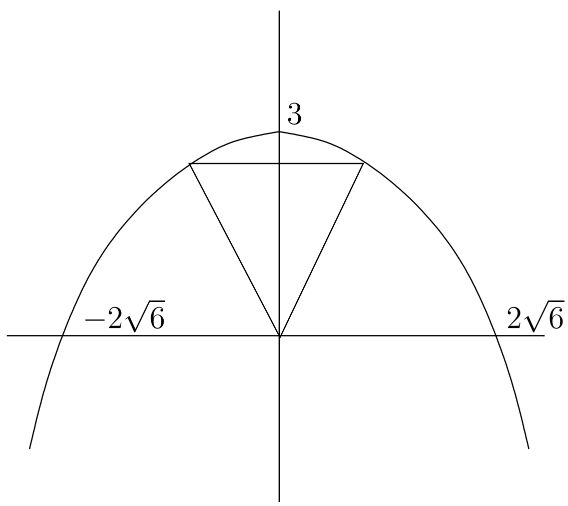
$$0 \leq f(0) \leq 0 \text{ implies } f(0) = 0. \; f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{f(x)}{x}$$

$$x \to 0^+,\ x > 0 \quad 1 \leq \frac{f(x)}{x} \leq 1 + x \quad \lim_{x\to 0^+} \frac{f(x)}{x} = 1$$

$$x \to 0^-,\ x < 0 \quad 1 \geq \frac{f(x)}{x} \geq 1 + x \quad \lim_{x\to 0^-} \frac{f(x)}{x} = 1 \text{ so } f'(0) = 0$$

2. An isosceles triangle is drawn with a vertex at the origin, its base parallel to and above the x-axis and the vertices of its base on the curve $12y = 36 - x^2$. Find the largest possible area of such a triangle.

Solution:



$A = \left(\frac{36 - x^2}{12}\right)\left(\frac{2x}{2}\right) = 3x - \frac{x^3}{12}$. $\frac{dA}{dx} = 3 - \frac{3x^2}{12}$ $\quad 1 - \frac{x^2}{12} = 0$, $\quad x^2 = 12$, $\quad x = \mp 2\sqrt{3}$. So the function A(x) decreases on $(-\infty, -2\sqrt{3})$ and $(2\sqrt{3}, +\infty)$ increases on $(-2\sqrt{3}, 2\sqrt{3})$. Hence $A(2\sqrt{3}) = 4\sqrt{3}$

3. Evaluate the following limits

   (a) $\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}$

   Solution:

$$y = (\cos x)^{\frac{1}{x^2}} \quad \ln y = \frac{1}{x^2} \ln \cos x$$

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln \cos x}{x^2} \text{ then}$$

$$= \lim_{x\to 0} \frac{\frac{-\sin x}{\cos x}}{2x} = \lim_{x\to 0} \frac{-\sin x}{2x\ \cos x} = \lim_{x\to 0} \frac{\sin x}{x} \lim_{x\to 0} \frac{-1}{2\cos x} = \frac{-1}{2}$$

$$\ln y \to \frac{-1}{2} \quad \lim_{x\to 0} y = \exp^{\frac{-1}{2}}$$

(b) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\pi}{4n} \tan \frac{k\pi}{4n}$

Solution:

$$\int_0^{\frac{\pi}{4}} \tan x\, dx = -\ln|\cos x||_0^{\frac{\pi}{4}} = -\ln\cos\tfrac{\pi}{4} = -\ln\tfrac{\sqrt{2}}{2} = \ln\sqrt{2}$$

4. Evaluate

(a) $\displaystyle\int_1^3 \frac{1}{(x-2)^4} dx$

Solution:

$$I = \int_1^3 \frac{1}{(x-2)^4} dx \text{ This is an improper integral}$$

$$I = \lim_{c\to 2^-} \int_1^c \frac{1}{(x-2)^4} dx + \lim_{b\to 2^+} \int_b^3 \frac{1}{(x-2)^4} dx = \lim_{c\to 2^-} \frac{(x-2)^{-5}}{-5} \ |_1^c + \lim_{b\to 2^+} \frac{(x-2)^{-5}}{-5} \ |_b^3$$

evaluating at the given points would give us $+\infty$ hence the integral diverges.

(b) $\displaystyle\int_1^4 e^{\sqrt{x}} dx$

Solution:

Let $\sqrt{x} = t$ then $\frac{1}{2\sqrt{x}} dx = dt$ the the integral becomes

$$\int_1^2 2te^t dt = 2\left[2te^t|_1^2 - \int_1^2 e^t dt\right] \text{ where } e^t dt = dv \ \ e^t = v \ \ t = u \ \ dt = du$$

$$2\left[te^t - e^t\right]|_1^2 = 2e^2$$

5. Evaluate

(a) $\displaystyle\int \frac{e^{4t}}{e^{2t} + 3e^t + 2} dt$

Solution:

Let $e^t = u$ and $e^t dt = du$ then the integral becomes $\displaystyle\int \frac{u^3}{u^2+3u+2} du$ and by polynomial division we have $\displaystyle\int \left(u - 3 + \frac{7u+6}{u^2+3u+2}\right)$ and by partial fractions method we will have the following $\displaystyle\left[\left(\frac{u^2}{2} - 3u\right) + \int \left(\frac{-1}{u+1} + \frac{8}{u+2}\right) du\right]$ which is equal to $\displaystyle\left[\left(\frac{u^2}{2} - 3u\right) - \ln|u+1| + 8\ln|u+2| + c\right] = \frac{e^{2t}}{2} - 3e^t + \ln\frac{(e^t+2)^8}{e^t+1} + c$

(b) $\int \frac{\sqrt{16-x^2}}{x^4}dx$

Solution:

Let $sin\theta = \frac{x}{4}$ then $dx = 4\cos\theta d\theta$ then we have the following integral after factoring out $\frac{1}{16}\int \frac{\cos^2\theta}{\sin^4\theta}d\theta$ letting $u = \cot\theta$ *and* $du = -\csc^2\theta$ we get $\frac{-1}{16}\int u^2 du = \frac{-\cot^3\theta}{48} + c = \frac{-(16-x^2)^{\frac{3}{2}}}{48x^3} + c$

6. Study the convergence (absolute and conditional) of

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$$

Solution:

Consider $\sum_{n=1}^{\infty}\left|\frac{(-1)^n \ln n}{n}\right| = \sum_{n=1}^{\infty}\frac{\ln n}{n}$ Apply integral test $\int_1^{\infty}\frac{\ln n}{n}dn = \lim_{b\to\infty}\int_1^b \frac{\ln n}{n}dn = \lim_{b\to\infty}\frac{(\ln x)^2}{2}|_1^b = \lim_{b\to\infty}\frac{\ln b^2}{2} = \infty$

The improper integral diverges so $\sum_{n=1}^{\infty}\frac{\ln n}{n}$ is divergent. Hence $\sum_{n=1}^{\infty}\frac{(-1)^n \ln n}{n}$ is not absolutely convergent.

Let $f(x) = \frac{\ln x}{x}$, $\lim_{n\to\infty}\frac{\ln n}{n} = 0$ then $f'(x) = \frac{1-\ln x}{x^2}$ for $x \geq 3$ $1 - \ln x < 0$ so $f'(x) < 0$ $forall$ $x \geq 3$ hence f is decreasing on $[3, \infty)$ so by alternating series test $\sum_{n=1}^{\infty}\frac{(-1)^n \ln n}{n}$ is convergent so the series is conditionally convergent.

7. Determine whether the following series converge and find the sum of the convergent ones.

a) $\sum_{n=1}^{\infty} \tan\frac{1}{n}$

b) $\sum_{n=1}^{\infty} \cos n\pi$

c) $\sum_{k=2}^{\infty} \frac{e^{-k}}{2^{k+1}}$

Solution:

a) Limit form of comparison $a_n = \tan\frac{1}{n}$ $b_n = \frac{1}{n}$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\tan\frac{1}{n}}{\frac{1}{n}} = 1$$ so since $\sum\frac{1}{n}$ is divergent $\sum\tan\frac{1}{n}$ is also divergent.

b) $\sum \cos n\pi = \sum (-1)^n$ is divergent by n'th term test $\lim\limits_{n\to\infty} (-1)^n \neq 0$

c) $\displaystyle\sum_{k=2}^{\infty} \frac{e^{-k}}{2^{k+1}} = \sum \frac{1}{2}\frac{e^{-k}}{2^k} = \frac{1}{2}\sum_{k=2}^{\infty}\left(\frac{1}{2e}\right)^k$ convergent geometric series since $\displaystyle\frac{1}{2e} < 1$

$$\sum_{k=0}^{\infty}\left(\frac{1}{2e}\right)^k = \frac{1}{1-\frac{1}{2e}} = \frac{2e}{2e-1} \quad \text{so}$$

$$\frac{1}{2}\sum_{k=2}^{\infty}\left(\frac{1}{2e}\right)^k = \frac{1}{2}\left[\frac{2e}{2e-1} - \frac{1}{2e} - 1\right] = \frac{1}{4e(2e-1)}$$

8. Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{3^n}{n}(2x-1)^n$$

Solution:

$$\lim_{n\to\infty}\left|\frac{3^{n+1}}{n+1}\frac{(2x-1)^{n+1}}{3^n}\frac{n}{(2x-1)^n}\right| = \lim_{n\to\infty}\left|\frac{3n(2x-1)}{n+1}\right| = |3(2x-1)|$$

$|3(2x-1)| < 1$ gives us $1/3 < x < 2/3$ looking at the end points

- For $x = \dfrac{2}{3}$ $\displaystyle\sum \frac{3^n}{n}(4/3-1)^n = \sum \frac{1}{n}$ so divergent!
- For $x = \dfrac{1}{3}$ $\displaystyle\sum \frac{3^n}{n}\frac{(-1)^n}{3^n} = \sum \frac{(-1)^n}{n}$ converges by alternating series test.

Hence the interval of convergence is $[\frac{1}{3}, \frac{2}{3})$.

B U Department of Mathematics

Math 101 Calculus I

Spring 2000 Final Exam

1. Find the volume of the solid obtained by revolving the region inside the circle $x^2+(y-b)^2=a^2$ $(0<a<b)$ about the $x-$axis.

Solution:

$V = 2\int_{b-a}^{b+a} 2\pi x y dy$ where $x=\sqrt{a^2-(y-b)^2}$.
Let $u = y-b$, so $du = dy$, and $b-a<y<b+a$ implies that $-a<u=y-b<a$. Then,

$$V = 4\pi\int_{-a}^{a}\sqrt{a^2-u^2}(u+b)\,du = 4\pi\left[\int_{-a}^{a}u\sqrt{a^2-u^2}\,du + b\int_{-a}^{a}\sqrt{a^2-u^2}\,du\right].$$

Looking at the summands separately:

$$\int_{-a}^{a}u\sqrt{a^2-u^2}\,du = \frac{-1}{2}\int_{-a}^{a}\sqrt{a^2-u^2}\,(-2u)du = \frac{-1}{2}\left[\frac{(a^2-u^2)^{3/2}}{3/2}\right]_{-a}^{a} = 0 \text{ and}$$

$$\int_{-a}^{a}\sqrt{a^2-u^2}du = a\int_{-a}^{a}\sqrt{1-\frac{u^2}{a^2}}du, \text{ letting } \cos\theta=\sqrt{1-\frac{u^2}{a^2}} \text{ we get:}$$

$\sin\theta = \frac{u}{a} \Rightarrow \cos\theta d\theta = \frac{du}{a}$, and consequently:

$$\int\sqrt{a^2-u^2}du = a^2\int\cos^2\theta d\theta = a^2\int\frac{\cos 2\theta+1}{2}d\theta = \frac{a^2}{2}\left[\frac{\sin 2\theta}{2}+\theta\right]$$

$$= a^2\left[\frac{\sin\theta\cos\theta}{2}+\frac{\theta}{2}\right] = a^2\left[\frac{\frac{u}{a}\sqrt{1-\frac{u^2}{a^2}}}{2}+\frac{\arcsin\frac{u}{a}}{2}\right]$$ after back substitution. Therefore:

$$b\int_{-a}^{a}\sqrt{a^2-u^2}du = ba^2\left[\frac{\frac{u}{a}\sqrt{1-\frac{u^2}{a^2}}}{2}+\frac{\arcsin\frac{u}{a}}{2}\right]_{-a}^{a} = \frac{ba^2}{2}[\arcsin 1-\arcsin(-1)]$$

$$= \frac{ba^2}{2}[\frac{\pi}{2}-\frac{-\pi}{2}] = \frac{ba^2\pi}{2}.$$

Hence the volume of the solid is $V = 4\pi\left[\frac{ba^2\pi}{2}\right] = 2ba^2\pi^2$.

2. Evaluate the following integrals.

   (a) $\int \ln x dx$

(b) $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$

Solution:

(a) $\begin{array}{lll} u = \ln x & \text{implies} & du = \dfrac{dx}{x} \\ dv = dx & \text{implies} & v = x \end{array}$

So by integration by parts: $\int u dv = uv - \int v du$ we get,

$$\int \ln x dx = x \ln x - \int x \frac{dx}{x} = x \ln x - \int dx = x \ln x - x + c.$$

(b) $u = \sqrt{x}$ implies $du = \frac{dx}{2\sqrt{x}}$ i.e. $\frac{dx}{\sqrt{x}} = 2du$. And $0 < x < 1$ implies $0 < u = \sqrt{x} < 1$.

Then, $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = \int_0^1 \frac{\ln \sqrt{x}^2}{\sqrt{x}}\,dx = \int_0^1 \frac{2\ln\sqrt{x}}{\sqrt{x}}\,dx$

$$= 2\int_0^1 \ln u \, 2du = 4\int_0^1 \ln u \, du = 4\lim_{t\to 0}\int_t^1 \ln u \, du = 4\lim_{t\to 0}[u \ln u - u]_t^1 \text{ by part (a).}$$

$$= 4\lim_{t\to 0}[1 \ \ln 1 - 1] - [t \ \ln t - t] = 4\lim_{t\to 0}(-1 + t - t \ \ln t) = -4 + 0 - 4\lim_{t\to 0} t \ \ln t$$

$$= -4 + 4\lim_{t\to 0}\frac{-\ln t}{\frac{1}{t}} = -4 + 4\lim_{t\to 0}\frac{\ln t^{-1}}{\frac{1}{t}} = -4 + 4\lim_{t\to 0}\frac{\ln \frac{1}{t}}{\frac{1}{t}}. \text{ By L'hospital rule:}$$

$$= -4 + 4\lim_{t\to 0}\frac{-\frac{1}{t}}{-\frac{1}{t^2}} = -4 + 4\lim_{t\to 0} t = -4.$$

3. Test the following for convergence.

(a) $\int_1^\infty \frac{2x-1}{\sqrt{x^5+2x-2}}\,dx$

(b) $\sum_{n=0}^{\infty} e^n(\sin^2 2^{-n})$

(c) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt{n^2-3}}$

Solution:

(a) For $1 \le x < \infty$, we have $x^5 + 2x - 2 = x^5 + 2(x-1) \ge x^5$. This entails:

$$\sqrt{x^5+2x-2} \geq x^{5/2} \Rightarrow \frac{2x-1}{\sqrt{x^5+2x-2}} \leq \frac{2x-1}{x^{5/2}} \leq \frac{2x}{x^{5/2}} = \frac{2}{x^{3/2}}.$$

$f(x) = \dfrac{2}{x^{3/2}}$ is a decreasing and continuous function in $[1, \infty)$. Furthermore we have:

$$\int_1^{\infty} \frac{2}{x^{3/2}}\, dx = 2 \lim_{t\to\infty} \int_1^t x^{-3/2} dx = 2 \lim_{t\to\infty} \left. \frac{x^{-1/2}}{-1/2} \right|_1^t = 2 \lim_{t\to\infty} \frac{-2}{\sqrt{t}} + 2 = 4 < \infty.$$

So by integral test $\displaystyle\int_1^{\infty} \frac{2x-1}{\sqrt{x^5+2x-2}}\, dx$ converges.

(b) Let $a_n = e^n \sin^2 2^{-n}$, then $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{e^{n+1}\sin^2 2^{-(n+1)}}{e^n \sin^2 2^{-n}}$

$$= \lim_{n\to\infty} e \frac{\sin^2 2^{-n-1}}{\sin^2 2^{-n}} \frac{2^{-2n}}{2^{-2n-2}} \frac{2^{-2n-2}}{2^{-2n}} = e \lim_{n\to\infty} \left(\frac{\sin^2 2^{-n-1}}{(2^{-n-1})^2} \right) \cdot \left(\frac{(2^{-n})^2}{\sin^2 2^{-n}} \right) \cdot \left(\frac{2^{-2n-2}}{2^{-2n}} \right)$$

$$= e \lim_{n\to\infty} \frac{2^{-2n-2}}{2^{-2n}} \quad \left(\text{since } \lim_{n\to 0} \frac{\sin n}{n} = 1; \text{ note that } 2^{-n-1} \to 0 \text{ as } n \to \infty\right)$$

$$= e \lim_{n\to\infty} \frac{2^{-2n}2^{-2}}{2^{-2n}} = e\, 2^{-2} = \frac{e}{4} < 1.$$

So by ratio test, $\displaystyle\sum_{n=0}^{\infty} e^n(\sin^2 2^{-n})$ converges.

(c) Consider the sequence (x_n) such that $x_n = \dfrac{1}{n\sqrt{n^2-3}}$.

$x_n = \dfrac{1}{n\sqrt{n^2-3}} > 0$ for each $n \geq 2$. Also, x_n is a decreasing sequence. Furthermore,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n\sqrt{n^2-3}} = 0.$$

Since x_n is a decreasing sequence of strictly positive numbers with limit 0, then by alternating series test,

$$\sum_{n=2}^{\infty} (-1)^n x_n = \sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt{n^2-3}} \text{ converges.}$$

4. Find $\displaystyle\sum_{n=2}^{\infty} \ln \frac{n(n+2)}{(n+1)^2}$.

Solution:

Let S_n denote the nth partial sum of the series. Then:

$$S_n = \sum_{i=2}^{n} \ln \frac{i(i+2)}{(i+1)^2} = \ln \frac{2.4}{3.3} + \ln \frac{3.5}{4.4} + \ln \frac{4.6}{5.5} + \cdots + \ln \frac{n(n+2)}{(n+1)(n+1)}$$

$$= \ln \prod_{i=2}^{n} \frac{i(i+2)}{(i+1)^2} = \ln \frac{2.4}{3.3} \frac{3.5}{4.4} \frac{4.6}{5.5} \cdots \frac{n(n+2)}{(n+1)(n+1)} = \ln \frac{2(n+2)}{3(n+1)}.$$

Therefore, $\displaystyle\sum_{n=2}^{\infty} \ln \frac{n(n+2)}{(n+1)^2} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln \frac{2(n+2)}{3(n+1)} = \ln \frac{2}{3}.$

5. **(a)** Find Maclaurin series expansion for $f(x) = \dfrac{x}{2x+1}$ about the origin. What is the interval of convergence?

(b) Find radius and interval of convergence of power series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(x-1)^{2n}}{(n+2)4^n}$.

Solution:

(a) We know that $\displaystyle\sum_{n=0}^{\infty} x^n$ is the Maclaurin series expansion for $\dfrac{1}{1-x}$ about the origin and the interval of convergence is $|x| < 1$.

So $\displaystyle\frac{1}{1-(-2x)} = \sum_{n=0}^{\infty}(-2x)^n$ for $|-2x| < 1$, i.e. $|x| < \dfrac{1}{2}$.

Therefore,

$$\frac{x}{1+2x} = \sum_{n=0}^{\infty} x(-2x)^n = \sum_{n=0}^{\infty}(-2)^n x^{n+1} = \sum_{n=1}^{\infty}(-2)^{n-1}x^n = -\frac{1}{2}\sum_{n=1}^{\infty}(-2)^n x^n$$

for $|x| < \dfrac{1}{2}$.

(b) Consider:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\left|\dfrac{(-1)^{n+1}(x-1)^{2n+2}}{(n+3)4^{n+1}}\right|}{\left|\dfrac{(-1)^n(x-1)^{2n}}{(n+2)4^n}\right|} = \lim_{n\to\infty} \frac{(x-1)^2(n+2)}{4(n+3)} = \frac{(x-1)^2}{4}.$$

By ratio test, series converges if $\dfrac{(x-1)^2}{4} < 1$, i.e. $(x-1)^2 < 4$ which is equivalent to saying $|x-1| < 2$. We now easily find that in the interval $-2 < x-1 < 2$ or $-1 < x < 3$ the series converges. Therefore, radius of convergence is $R = 2$.

For end points (which should be considered separately):

$x = -1 \Rightarrow \displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n(-2)^{2n}}{(n+2)4^n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n+2}$ converges by alternating series test.

$x = 3 \Rightarrow \displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n(2)^{2n}}{(n+2)4^n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n+2}$ converges similarly.

Therefore, interval of convergence is $[-1, 3]$.

6. **(a)** Find $\displaystyle\frac{d}{dx}\left(\int_0^{\sin(x^2)} \frac{dt}{1+t^5}\right)$.

(b) If exists, evaluate $\displaystyle\int_0^2 \frac{x}{x^2-1}dx$.

Solution:

(a) By the fundamental theorem of calculus, we have $\dfrac{d}{dx}\displaystyle\int_{a(x)}^{b(x)} f(t)dt = b'(x)f(b(x)) - a'(x)f(a(x))$.
Therefore:

$$\frac{d}{dx}\left(\int_0^{\sin(x^2)} \frac{dt}{1+t^5}\right) = (\sin x^2)'\frac{1}{1+\sin^5 x^2} - 0 = \frac{2x\cos(x^2)}{1+\sin^5 x^2}.$$

(b) For $x = \pm 1$ the integrand $\dfrac{x}{x^2-1}$ is undefined. Only 1 is in the interval of integration. So,

$$\int_0^2 \frac{x}{x^2-1}dx = \int_0^1 \frac{x}{x^2-1}dx + \int_1^2 \frac{x}{x^2-1}dx = \lim_{t\to 1^-}\int_0^t \frac{x}{x^2-1}dx + \lim_{t\to 1^+}\int_t^2 \frac{x}{x^2-1}dx.$$

For $u = x^2 - 1$, we get $du = 2xdx$, and $0 < x < 2$ implies that $-1 < u = x^2 - 1 < 3$.
Therefore:

$$\lim_{t\to 1^-}\int_0^t \frac{x}{x^2-1}\,dx + \lim_{t\to 1^+}\int_t^2 \frac{x}{x^2-1}\,dx = \lim_{t\to 1^-}\int_{-1}^{t^2-1} \frac{du}{2u} + \lim_{t\to 1^+}\int_{t^2-1}^{3} \frac{du}{2u}$$

$$= \lim_{t\to 1^-}\left[\frac{\ln|u|}{2}\right]_{-1}^{t^2-1} + \lim_{t\to 1^+}\left[\frac{\ln|u|}{2}\right]_{t^2-1}^{3} = \lim_{t\to 1}\left[\frac{\ln|t^2-1|}{2} - \frac{\ln 1}{2}\right] + \lim_{t\to 1}\left[\frac{\ln 3}{2} - \frac{\ln|t^2-1|}{2}\right]$$

$$= \lim_{t\to 1}\frac{\ln|t^2-1|}{2} + \frac{\ln 3}{2} - \frac{\ln|t^2-1|}{2} = \frac{1}{2}\lim_{t\to 1}\ln\frac{|t^2-1|}{|t^2-1|} + \frac{\ln 3}{2} = \ln 1 + \frac{\ln 3}{2} = \frac{\ln 3}{2}.$$

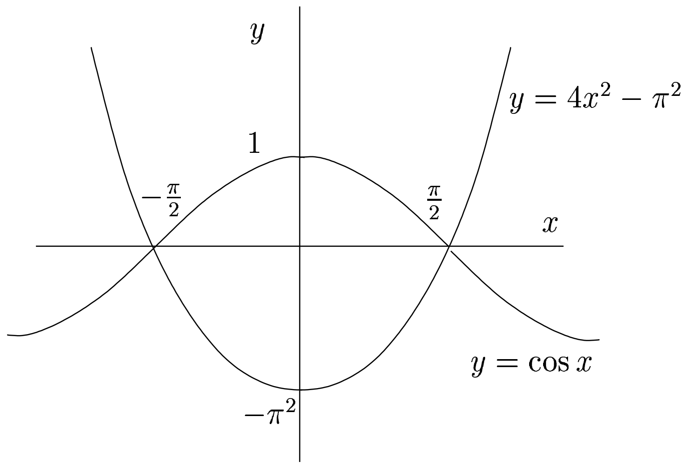
B U Department of Mathematics

Math 101 Calculus I

Spring 2001 Final Exam

1. Sketch the region bounded by the curves $y = \cos x$ and $y = 4x^2 - \pi^2$ and find its area.

Solution:



Intersection points:
$\cos x = 0$ when $x = \pm\frac{\pi}{2}$ and $4x^2 - \pi^2 = 0$ when $x = \pm\frac{\pi}{2}$.

$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x - (4x^2 - \pi^2))dx$ or by symmetry,

$A = 2\int_0^{\frac{\pi}{2}} (\cos x - 4x^2 + \pi^2)dx = 2(\sin x - \frac{4x^3}{3} + \pi^2 x)\Big|_0^{\frac{\pi}{2}} = 2 + \frac{2\pi^3}{3}.$

2. Find the integrals below:

(a) $\int \frac{x^2 + 5x + 2}{(x+1)(x^2+1)}dx$

(b) $\int \frac{e^{2x}}{e^{x+3}}dx$

Solution:

(a) $\frac{x^2 + 5x + 2}{(x+1)(x^2+1)}dx = \frac{A}{x+1} + \frac{Bx + C}{x^2+1}$

$x^2 + 5x + 2 = A(x^2+1) + (Bx + C)(x+1)$

$x = -1 \Rightarrow A = -1$, $x = 0 \Rightarrow C = 3$ and $x = 1 \Rightarrow B = 2$

$\Rightarrow I = \int \frac{x^2 + 5x + 2}{(x+1)(x^2+1)}dx = \int \frac{-dx}{x+1} + \int \frac{2xdx}{x^2+1} + \int \frac{3dx}{x^2+1}$

By substitution let $u = x + 1 \Rightarrow du = dx$ and let $w = x^2 + 1 \Rightarrow dw = 2xdx$.

So, $I = \int \frac{-du}{u} + \int \frac{dw}{w} + 3\tan^{-1} x + C = -\ln|u| + \ln|w| + 3\tan^{-1} x + C.$

Rewriting in the original variable:

$$I = -\ln|x+1| + \ln(x^2+1) + 3\tan^{-1}x + C.$$

(b) Let $u = e^x + 3 \Rightarrow \dfrac{du}{dx} = e^x = u - 3.$

$$\begin{aligned} \int \frac{e^{2x}}{e^{x+3}}dx &= \int \frac{u-3}{u}du = \int (1 - \frac{3}{u})du \\ &= u - 3\ln|u| + C = e^x + 3 - 3\ln|e^x+3| + C. \end{aligned}$$

3. Find a nonzero value for the constant k that makes $f(x) = \begin{cases} \dfrac{\tan kx}{x} & \text{if } x < 0 \\ 3x + 2k^2 & \text{if } x \geq 0 \end{cases}$ continuous.

Solution:

We should have, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0).$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \dfrac{\tan kx}{x} = \dfrac{0}{0} = \lim_{x\to 0^-} \dfrac{k\sec^2 kx}{1} = k$, after applying L'Hopital's rule.
$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (3x^2 + 2k^2) = 2k^2.$
Now equating these two limits: $k = 2k^2 \Rightarrow k - 2k^2 = 0 \Rightarrow k = 0,\ k = 1/2.$
Since $k \neq 0$, k should be $1/2$. Hence, $\lim_{x\to 0} f(x) = 2k^2 = f(0).$

4. Evaluate $\dfrac{d}{dx}\displaystyle\int_1^{2x} \sqrt[3]{t^3+1}dt.$

Solution:

By the fundamental theorem of calculus $\dfrac{d}{dx}\displaystyle\int_a^{h(x)} f(t)dt = f(h(x))h'(x).$

$$\Rightarrow \frac{d}{dx}\int_1^{2x} \sqrt[3]{t^3+1}dt = 2(\sqrt[3]{8x^3+1}).$$

5. Determine whether the following series converges or diverges:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2+3^{-k}}$

(b) $\displaystyle\sum_{k=1}^{\infty} (\sqrt{k} - \sqrt{k-1})^k$

Solution:

(a) By the $k-$th term test:

$$\lim_{k\to\infty} \frac{1}{2+3^{-k}} = \frac{1}{2} \neq 0.$$

Hence the series diverges.
(b) Apply the root test: $\rho = \lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} (\sqrt{k} - \sqrt{k-1})$ which is of the form $(\infty - \infty)$.

Hence we multiply and divide by the conjugate:

$$\begin{aligned}\rho &= \lim_{k\to\infty} \frac{(\sqrt{k}-\sqrt{k-1})(\sqrt{k}+\sqrt{k-1})}{(\sqrt{k}+\sqrt{k-1})} = \lim_{k\to\infty} \frac{k-(k-1)}{\sqrt{k}+\sqrt{k-1}} \\ &= \lim_{k\to\infty} \frac{1}{\sqrt{k}+\sqrt{k-1}} = 0 < 1.\end{aligned}$$

Thus, the series converges.

6. Find the interval of convergence of the power series $\displaystyle\sum_{k=2}^{\infty}\left(\frac{k}{k-1}\right)\frac{x+2^k}{2^k}$.

Solution:

Apply ratio test:

$$\begin{aligned}\rho &= \lim_{k\to\infty}\left|\frac{k+1}{k}\frac{(x+2)^{k+1}}{2^{k+1}}\frac{k-1}{k}\frac{2^k}{(x+2)^k}\right| \\ &= \lim_{k\to\infty}\left|\frac{k^2-1}{k^2}\frac{x+2}{2}\right| = \left|\frac{x+2}{2}\right| < 1.\end{aligned}$$

So we have obtained the open interval of convergence to be:

$$|x+2| < 2 \Rightarrow -2 < x+2 < 2 \Rightarrow -4 < x < 0.$$

Check the endpoints:

$x = -4 \Rightarrow \displaystyle\sum_{k=2}^{\infty}\frac{k}{k-1}(-1)^k$ which is an alternating series of the form $\sum(-1)^k a_k$ where $a_k = \dfrac{k}{k-1} \to 1$. Hence the series diverges by alternating series test.

$x = 0 \Rightarrow \displaystyle\sum_{k=2}^{\infty}\frac{k}{k-1}$ diverges by $k-$th term test since $\displaystyle\lim_{k\to\infty}\frac{k}{k-1} = 1 \neq 0$.

So the interval of convergence is $(-4, 0)$ and the radius of convergence is 2.

7. Sketch the curve $f(x) = \dfrac{x}{(x+3)^2}$ by examining (if any) (a) the $x-$ and $y-$ intercepts, (b) the domain, (c) all asymptotes, (d) all necessary limits, (e) maximum, minimum and inflection points, (f) increasing and decreasing intervals, (g) concavity and (h) tabulating your data.

Solution:

Domain contains all real numbers except $x = -3$.

$(0, 0)$ is both an $x-$ and $y-$intercept.

$f'(x) = \dfrac{-x+3}{(x+3)^3} = 0 \Rightarrow x = 3$ is a critical point.

$f''(x) = \dfrac{2x-12}{(x+3)^4} = 0 \Rightarrow x = 6$ is an inflection point since f'' changes sign at this point.

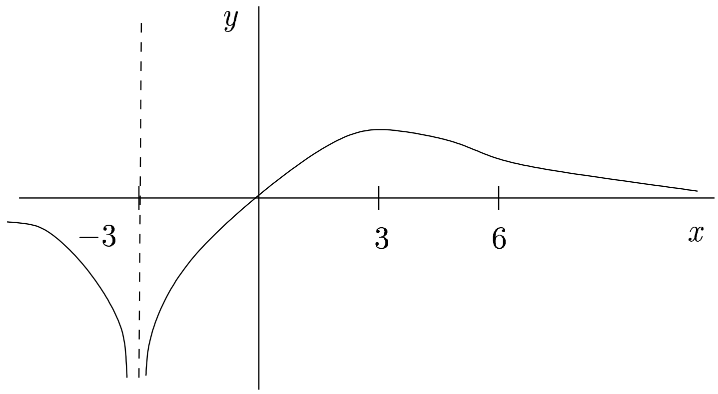
$\displaystyle\lim_{x\to\pm\infty} f(x) = 0^{\pm} \Rightarrow$ there is a horizontal asymptote at $y = 0$.

Moreover, $x = -3$ is a vertical asymptote since:

$$\lim_{x \to -3^+} \frac{x}{(x+3)^2} = -\infty = \lim_{x \to -3^-} \frac{x}{(x+3)^2}.$$

Since $f''(3) < 0$ there is a relative maximum at the point (3,12).

x		-3	3	6
f'	$-$	$+$	$-$	$-$
f''	$-$	$-$	$-$	$+$
f	$\searrow\cap$	$\nearrow\cap$	$\searrow\cap$	$\searrow\cup$



B U Department of Mathematics

Math 101 Calculus I

Spring 2002 Final Exam

1. Let $f(x) = x + \dfrac{\sin x}{2x - \frac{12}{x-1}}$. Find the values of x (if any) at which f is not continuous, and determine whether each such value is a removable discontinuity.

Solution:

We perform a couple of algebraic manipulations:

$$\begin{aligned} f(x) &= x + \frac{\sin x}{2x - \frac{12}{x-1}}, \quad x \neq -2, 1, 3 \\ &= x + \frac{(x-1)\sin x}{2(x^2 - x - 6)} = x + \frac{(x-1)\sin x}{2(x-3)(x+2)}, \quad x \neq -2, 3. \end{aligned}$$

At $x = 1$, we have removable discontinuity, since we can redefine f by:

$$f = x + \frac{(x-1)\sin x}{2(x-3)(x+2)}, \quad x \neq -2, 3$$

which gives a regular value at $x = 1$. The singularities at $x = -2$ and $x = 3$ cannot be removed.

2. If $\displaystyle\int_0^{x^2} f(t)dt = x \cos \pi x$, for $x \geq 0$, calculate $f(4)$.

Solution:

$\displaystyle\int_0^{x^2} f(t)dt = x\cos(\pi x)$ is given. To extract f from this integral, we use the fundamental theorem of calculus. Differentiating both sides with respect to x:

$$\frac{d}{dx}\int_0^{x^2} f(t)dt = \frac{d}{dx}(x \cos \pi x) \iff 2xf(x^2) = \cos \pi x - \pi x \sin \pi x.$$

To compute $f(4)$ we need to substitute $x = 2$:

$$4f(4) = \cos 2\pi - 2 \sin 2\pi \implies f(4) = \frac{1}{4}.$$

3. Evaluate the improper integral $\displaystyle\int_0^1 \frac{dx}{\sqrt{x^2 + 2x}}$.

Solution:

We first remove the singular boundary point and take limit:

$$\int_0^1 \frac{dx}{\sqrt{x^2+2x}} = \lim_{a\to 0+} \int_a^1 \frac{dx}{\sqrt{(x+1)^2-1}}$$

Now performing the change of variable: $\sec\theta = x+1$ which implies $\sec\theta\tan\theta d\theta = dx$ we rewrite the indefinite integral and evaluate:

$$\begin{aligned} \int \frac{dx}{\sqrt{(x+1)^2-1}} &= \int \frac{\sec\theta\tan\theta}{\tan\theta} d\theta \\ &= \ln|\sec\theta+\tan\theta| = \ln|(x+1)+\sqrt{(x+1)^2-1}|. \end{aligned}$$

Putting the boundaries:

$$\begin{aligned} \int_a^1 \frac{dx}{\sqrt{(x+1)^2-1}} &= \ln|(x+1)+\sqrt{(x+1)^2-1}|_a^1 \\ &= \ln|2+\sqrt{3}| - \ln|(a+1)+\sqrt{a^2+2a}|. \end{aligned}$$

Evaluating the limit:

$$\lim_{a\to 0+} \int_a^1 \frac{dx}{\sqrt{(x+1)^2-1}} = \ln(2+\sqrt{3}) - \ln 1 = \ln(2+\sqrt{3}).$$

4. Evaluate $\lim\limits_{x\to\infty}(1-2^{-x})^x$.

Solution:

First check if there is an indeterminacy: $\lim\limits_{x\to\infty}(1-2^{-x})^x = [1^\infty]$ which is indeterminate. Set $y=(1-2^{-x})^x$ and consider $\ln y = x\ln(1-2^{-x})$ instead of y itself. Then:

$$\begin{aligned} \lim_{x\to\infty}(\ln y) &= \lim_{x\to\infty} \frac{\ln(1-2^{-x})}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\ln(\frac{2^x-1}{2^x})}{\frac{1}{x}} = \left[\frac{0}{0}\right] \\ &= \lim_{x\to\infty} \frac{\frac{2^x}{2^x-1}(2^{-x}\ln 2)}{-\frac{1}{x^2}} = \lim_{x\to\infty} -\frac{x^2\ln 2}{2^x-1} = \left[\frac{\infty}{\infty}\right] \\ &= \lim_{x\to\infty} \frac{-2x\ln 2}{2^x\ln 2} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{-2}{2^x\ln 2} = 0. \end{aligned}$$

Hence $\lim\limits_{x\to\infty} y = e^0 = 1$.

5. Decide whether the following series converge or diverge stating the reasons.

(a) $\sum\limits_{n=1}^{\infty}(1-2^{-n})^n$.

Solution:

$\sum\limits_{n=1}^{\infty}(1-2^{-n})^n$ diverges by nth term test, since $\lim a_n = 1 \neq 0$ for $a_n = (1-2^{-n})^n$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{5^n}{4^n+3}$.

Solution:

Let $a_n = \dfrac{5^n}{4^n+3}$. Then $a_n \geq \dfrac{5^n}{4^n+4^n} = \dfrac{1}{2}\left(\dfrac{5}{4}\right)^n \geq 0$.
Since $\sum \frac{1}{2}\left(\frac{5}{4}\right)^n$ diverges (Geometric Series for $x = \frac{5}{4} > 1$), the original series $\sum a_n$ diverges by Comparison Test.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n}$.

Solution:

First remark is $a_n = \dfrac{\cos n\pi}{5^n} \to 0$. Now we make easy comparisons:

$$0 \leq \left|\frac{\cos n\pi}{5^n}\right| \leq \frac{1}{5^n}.$$

On the other hand $\sum \frac{1}{5^n}$ is a convergent geometric series. Being smaller than a convergent series, the given series is absolutely convergent, hence convergent.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{8\tan^{-1} n}{1+n^2}$.

Solution:

The general term tends to 0: $a_n = \dfrac{8\tan^{-1} n}{1+n^2} \longrightarrow 0$ as $n \to \infty$. We furthermore have the following comparison:

$$0 \leq a_n \leq \frac{8(\frac{\pi}{2})}{1+n^2} \leq 4\pi\left(\frac{1}{n^2}\right).$$

But $\sum \frac{1}{n^2}$ is convergent as it is a $p-$series with $p = 2 > 1$. Then $\sum 4\pi\left(\frac{1}{n^2}\right)$ is convergent too. Thus $\sum a_n$ converges by Comparison Test.

6. Given the infinite series $\displaystyle\sum_{n=3}^{\infty} \frac{4}{4n^2-12n+5}$.

(a) Find the partial sum of the series.

Solution:

Use partial fractions

$$\frac{4}{4n^2-12n+5} = \frac{4}{(2n-1)(2n-5)} = \frac{A}{2n-1} + \frac{B}{2n-5},$$

we need to solve $4 = A(2n-5) + B(2n-1)$. It is easily found that $A = -1$ and $B = 1$, so that the general term becomes:

$$a_n = \frac{1}{2n-5} - \frac{1}{2n-1}.$$

The partial sum of the series is: $S_n = a_3 + a_4 + \cdots + a_n$. Writing terms of this partial sum explicitly:

$$\begin{aligned} S_n &= \left(\frac{1}{1} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{7}\right) + \left(\frac{1}{5} - \frac{1}{9}\right) + \left(\frac{1}{7} - \frac{1}{11}\right) + \cdots + \left(\frac{1}{2n-5} - \frac{1}{2n-1}\right) \\ &= 1 + \frac{1}{3} - \frac{1}{2n-3} - \frac{1}{2n-1}. \end{aligned}$$

Note that this is a telescoping series.

(b) Using part **(a)** find the sum of the series.

Solution:

Sum of the series is nothing but the limit of the partial sum as $n \to \infty$, namely:

$$\lim_{n\to\infty} S_n = 1 + \frac{1}{3} = \frac{4}{3}.$$

7. **(a)** Find the radius and interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} 3^n \frac{(x-1)^n}{n}$.

Solution:

Let $a_n = 3^n \dfrac{(x-1)^n}{n}$ and apply the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = 3\left(\frac{n}{n+1}\right)|x-1|$$

and take limit:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 3|x-1|.$$

Series converges if $3|x-1| < 1$ i.e. if $\dfrac{2}{3} < x < \dfrac{4}{3}$ and diverges if $3|x-1| > 1$. So the radius of convergence is found to be $R = \dfrac{1}{3}$.

We now analyze the endpoints:

$x = \dfrac{2}{3}$: the general term becomes:

$$a_n = 3^n \frac{1}{n}\left(\frac{-1}{3}\right)^n = (-1)^n \frac{1}{n}$$

and $\sum (-1)^n \dfrac{1}{n}$ is an alternating harmonic series and hence convergent.

$x = \dfrac{4}{3}$: the general term becomes:

$$a_n = 3^n \frac{1}{n}\left(\frac{1}{3}\right)^n = \frac{1}{n}$$

and $\sum \dfrac{1}{n}$ is a harmonic series and hence diverges.

So interval of convergence $\left[\dfrac{2}{3}, \dfrac{4}{3}\right)$.

(b) Calculate the value of $\dfrac{1}{e}$ within an error of 0.01 by using the first few terms of an appropriate series.

Solution:

The Maclaurin Series for e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{(x)^n}{n!}.$$

For $x = -1$ we have,

$$\frac{1}{e} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

The terms are strictly alternating in sign and decrease in absolute value from $n = 1$:

$$1 > \frac{1}{2!} > \frac{1}{3!} > \quad \ldots$$

Also,

$$\frac{1}{n!} \to 0$$

Therefore, The Alternating Series Estimation theorem guarantees that, the error 0.01 for the nth partial sum is less than a_{n+1}.

Hence, $0.01 < \dfrac{1}{(n+1)!}$ and $(n+1)! < 100$

So, we should take $(n+1)$ to be at least 4 or n to be at least 3:

$$\frac{1}{e} = 1 + (-1) + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} = \frac{1}{3}$$

8. **(a)** Using the Maclaurin series for $\dfrac{1}{1-x}$ find the Maclaurin series for $f(x) = \dfrac{1}{2x-3}$ stating the radius of convergence.

Solution:

First recall that $\dfrac{1}{1-x} = \sum\limits_{0}^{\infty} x^n$, valid for $|x| < 1$.

Manipulate the given function to make use of this fact:

$$\frac{1}{2x-3} = \frac{1}{-3\left(1 - \left(\frac{2}{3}x\right)\right)} = \left(-\frac{1}{3}\right)\sum_{0}^{\infty}\left(\frac{2}{3}x\right)^n = \sum_{0}^{\infty} -\frac{2^n}{3^{n+1}}x^n.$$

Now this series converges if $\left|\dfrac{2}{3}x\right| < 1$, i.e. $|x| < \dfrac{3}{2}$.

(b) Using part **(a)** find the Maclaurin series for $\frac{1}{(2x-3)^2}$.

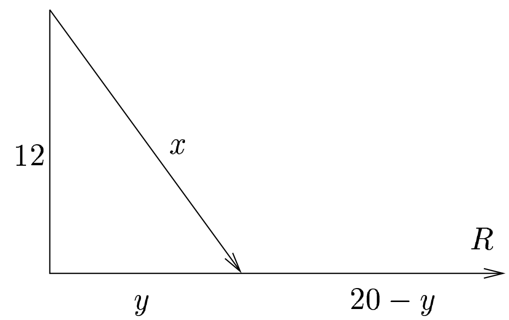
Solution:

$$(\frac{1}{2x-3})' = \frac{-2}{(2x-3)^2}$$

So termwise differentiation gives

$$(\frac{1}{2x-3}) = \frac{1}{2}\sum_0^\infty \frac{2^n}{3^{n+1}}\, n\, x^{n-1} \quad = \sum_0^\infty (\frac{2^{n-1}}{3^{n+1}} n)\, x^{n-1}, \quad |x| < \frac{3}{2}$$

9. A plan is drawn for the piping that will connect a drilling rig 12 km. offshore to a refinery on shore 20 km. down the cost (see the figure). What values of x and y will give the least expensive connection if underwater pipe costs \$5000 per km. and land-based pipe costs \$3000 per km?



Solution:

$x^2 = 144 + y^2$ from the figure. Writing the cost function and using the relation between x and y we get:

$$C(y) = 5.10^4 x + (20-y)3.10^4 = 10^4[5\sqrt{144+y^2} + 3(20-y)].$$

We look for critical points of this cost function:

$$\frac{dC}{dy} = 10^4\left[\frac{5y}{\sqrt{144+y^2}} - 3\right] = 0 \Longrightarrow 5y = 3\sqrt{144+y^2}.$$

Solving this equation:

$$25y^2 = 9(144+y^2) \Longrightarrow 16y^2 = 9.144 \Longrightarrow 16y^2 = 9.9.16 \Longrightarrow y = 9.$$

To decide whether this is indeed a minimum we look at derivative's sign: $\frac{dC}{dy} > 0$ if $y^2 > 81$ i.e. if $(y-9)(y+9) > 0$. This $C(y)$ decreases to and increases from $y = 9$. Hence it is a minimum. So cost is minimum when $y = 9$. Computing this cost we receive:

$$C(9) = 5.10^4\sqrt{144+81} + 11.3.10^4 = 10^4(5\sqrt{225} + 33) = 10^4(75+33) = 10^4(108).$$

B U Department of Mathematics

Math 101 Calculus I

Spring 2003 Final Exam

1. Given the function $f(x) = (1+x)e^{-x}$

a) Determine the interval(s) on which f is increasing or decreasing
Solution:

$$f'(x) = e^{-x} - (1+x)e^{-x} = -xe^{-x}$$

. . . . .0 . . .

x		
$-x$	+	-
e^{-x}	+	+
$f'(x)$	+	-
$f(x)$	$\nearrow$	$\searrow$

So f is increasing on $(-\infty, 0)$, and decreasing on $(0, \infty)$

b) Find and classify the local extrema of f, if any.
Solution:

$f'(x) = -xe^{-x} = 0 \quad \Leftrightarrow \quad x = 0$
f is defined everywhere, so $x = 0$ is the only critical pt.
By the help of table we see that it is local maximum

c) Determine the interval(s) on which f is concave up or concave down.
Solution:

$f' = -xe^{-x}$
$f'' = -e^{-x} + xe^{-x} = (x-1)e^{-x} = 0 \quad \Leftrightarrow \quad x = 1$

. . . . .1 . . .

x		
$x-1$	-	+
e^{-x}	+	+
$f''(x)$	-	+
$f(x)$	$\cap$	$\cup$

f is concave down on $(-\infty, 1)$
f is concave up on $(1, \infty)$

d) Find the inflection points of f, if any.
Solution:

By part c, 1 is an inflection point of f.

e) Find the horizontal, vertical or slant asymptotes of f, if any.
Solution:

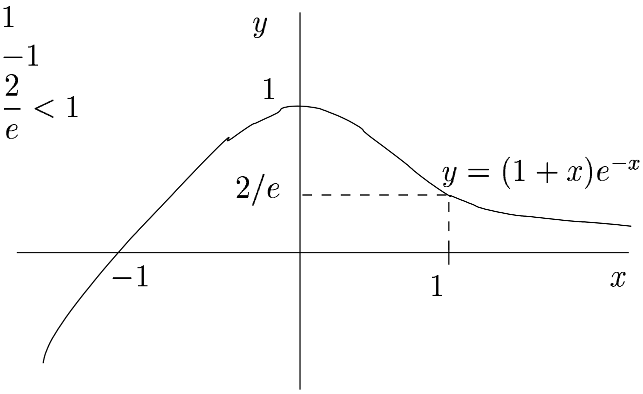
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}(1+x)e^{-x} = \lim_{x\to\infty}\frac{1+x}{e^x} = \lim_{x\to\infty}\frac{1}{e^x} = 0$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}(1+x)e^{-x} = -\infty$$

$y=0$ is horizontal asymptote, no vertical asymptote.

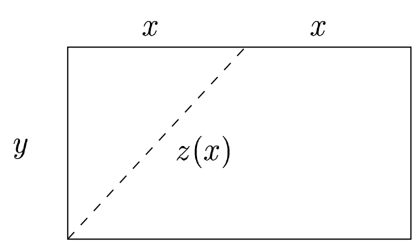
f) Sketch the graph of f
Solution:

$x=0 \Rightarrow y=1$
$y=0 \Rightarrow x=-1$
$x=1 \Rightarrow y=\dfrac{2}{e}<1$



2. A rectangle is to have an area of $64m^2$. Find its dimensions so that the distance from one corner to the midpoint of a non-adjacent side is a minimum.
(You do not need to verify that this is a minimum.)

Solution:



$$2x\cdot y = 64 \quad\Rightarrow\quad y=\frac{32}{x}$$
$$z(x)=\sqrt{x^2+\frac{32^2}{x^2}}=\frac{1}{x}\sqrt{x^4+32^2}$$
Minimize $z(x)$: $z'(x) = -\frac{1}{x^2}\sqrt{x^4+32^2}+\frac{1}{x}\frac{4x^3}{2\sqrt{x^4+32^2}}=0$

$$\Rightarrow\quad -2(x^4+32^2)+4x^3=0 \quad\Rightarrow\quad x^4+32^2-2x^3=0$$

3. Find a function $f(x)$ such that

$$x^2 = 1+\int_1^x \sqrt{1+(f(t))^2}\,dt$$

Solution:

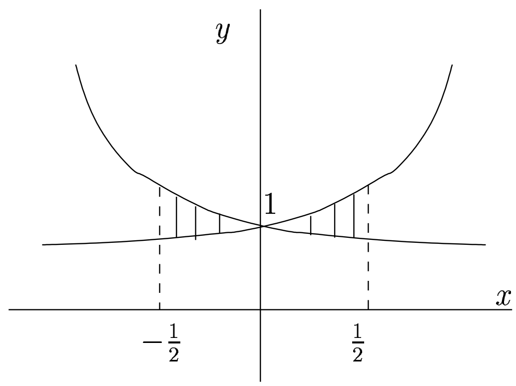
By using Fundamental Theorem of Calculus:

$$\begin{aligned} 2x &= \sqrt{1+(f(x))^2} \\ 4x^2 &= 1+(f(x))^2 \\ 4x^2+1 &= f(x)^2 \end{aligned}$$

So $f(x) = \sqrt{4x^2-1}$

4. Find the area of the region bounded between the curves $y = e^{2x}$ and $y = e^{-2x}$, for $-1/2 \le x \le 1/2$. Sketch the region.

Solution:



$$\int_{-1/2}^{0} e^{-2x} - e^{2x}\,dx + \int_{0}^{1/2} e^{2x} - e^{-2x}\,dx$$

$$= 2\int_{0}^{1/2} e^{2x} - e^{-2x}\,dx = 2\int_{0}^{1/2} e^{2x}\,dx - 2\int_{0}^{1/2} e^{-2x}\,dx$$

$$= \left|e^{2x} + e^{-2x}\right|_0^{1/2} = e^1 + e^{-1} - (e^0 + e^0) = e + \frac{1}{e} - 2$$

5. Evaluate the following integrals:

a) $\displaystyle\int \tan^5\theta \sec\theta\,d\theta$

Solution:

$$\int \tan^5\theta \sec\theta\,d\theta$$

$$= \int \tan^4\theta\,\tan\theta\,\sec\theta\,d\theta = \int (\sec^2\theta - 1)^2\,\tan\theta\,\sec\theta\,d\theta$$

Use $u = \sec\theta \quad \Rightarrow \quad du = \tan\theta\,\sec\theta\,d\theta$

$$= \int (u^2-1)^2 du = \int (u^4 - 2u^2 + 1)\,du = \frac{u^5}{5} - \frac{2u^3}{3} + u + C$$

$$= \frac{\sec^5\theta}{5} - \frac{2\sec^3\theta}{3} + \sec\theta + C$$

b) $\displaystyle\int \frac{(e^x+1)\,dx}{e^{2x}-e^x+2}$

Solution:

$$\int \frac{(e^x+1)\,dx}{e^{2x}-e^x+2} = \underbrace{\int \frac{e^x\,dx}{e^{2x}-e^x+2}}_{A} + \underbrace{\int \frac{dx}{e^{2x}-e^x+2}}_{B}$$

For the part A use $u = e^x \quad \Rightarrow \quad du = e^x\,dx$

$$A = \int \frac{du}{u^2-u+2} = \int \frac{du}{(u-1/2)^2+(\sqrt{3}/2)^2}$$

Then use $v = u - 1/2 \quad \Rightarrow \quad du = dv$

$$A = \int \frac{du}{(u-1/2)^2+(\sqrt{3}/2)^2} = \int \frac{dv}{v^2+(\sqrt{3}/2)^2}$$

$$A = \frac{2}{\sqrt{3}}\arctan(\frac{2v}{\sqrt{3}}) + C$$

$$A = \frac{2}{\sqrt{3}}\arctan(\frac{2u-1}{\sqrt{3}}) + C$$

$$A = \frac{2}{\sqrt{3}}\arctan(\frac{2e^x-1}{\sqrt{3}}) + C$$

For the part B use $u = e^x \quad \Rightarrow \quad du = e^x\,dx$

$$B = \int \frac{dx}{e^{2x}-e^x+2} = \int \frac{du}{u(u^2-u+2)}$$

First we find the rational parts

$$\frac{1}{u(u^2-u+2)} = \frac{Mu+N}{u^2-u+2} + \frac{P}{u}$$

$Mu^2 + Nu + Pu^2 - Pu + 2P = 1$

So we have $P = 1/2,\ N - P = 0,\ M - P = 0$

So we have $M = -1/2,\ N = 1/2,\ P = 1/2$

$$\frac{1}{u(u^2-u+2)} = \frac{-1/2u+1/2}{u^2-u+2} + \frac{1/2}{u}$$

So

$$B = \int \frac{du}{u(u^2-u+2)} = \int \frac{(-1/2)u+1/2}{u^2-u+2}\,du + \int \frac{1/2}{u}\,du$$

$$B = (-1/4)\int \frac{2u-2}{u^2-u+2}\,du + \int \frac{1/2}{u}\,du$$

$$B = (-1/4)\int \frac{2u-1}{u^2-u+2}\,du + (1/4)\int \frac{du}{u^2-u+2} + \int \frac{1/2}{u}\,du$$

$$B = (-1/4)\ln|u^2-u+2| + (1/4)\frac{2}{\sqrt{3}}\arctan(\frac{2e^x-1}{\sqrt{3}}) + (1/2)\ln|u| + C$$

$$B = (-1/4)\ln|e^{2x}-e^x+2| + (1/4)\frac{2}{\sqrt{3}}\arctan(\frac{2e^x-1}{\sqrt{3}}) + (1/2)\ln|e^x| + C$$

So

$$A + B = (-1/4)\ln|e^{2x}-e^x+2| + (5/4)\frac{2}{\sqrt{3}}\arctan(\frac{2e^x-1}{\sqrt{3}}) + (1/2)\ln|e^x| + C$$

6. Determine whether the series given below is convergent or divergent

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln\, n)^4}$$

Solution:

$(\ln\, n)^4 \geq n \quad$ where $n \geq 2$

So $0 \leq \dfrac{1}{n(\ln\, n)^4} \leq \dfrac{1}{n \cdot n} = \dfrac{1}{n^2}$

Since $\sum \dfrac{1}{n^2}$ converges, we have $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln\, n)^4}$

7. Find the MacLaurin series of the function $y = x^3 \cos 2x$; give the answer in sigma notation. (You do not have to check the convergence of the series)

Solution:

$(\cos x)' = -\sin x$

$(\cos x)'' = -\cos x$

$(\cos x)''' = \sin x$

$(\cos x)^{iv} = \cos x$

So we get a repetition with period 4

To find MacLaurin series of $\cos x$ we use the terms above

$$\sum_{n=0}^{\infty} f^n(x_0)\frac{(x-x_0)^n}{n!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$$

So for $\cos 2x$ we have $\quad 1 - \dfrac{(2x)^2}{2!} + \dfrac{(2x)^4}{4!} - \dfrac{(2x)^6}{6!} \cdots$

$$x^3\ \cos 2x = \sum_{k=0}^{\infty} x^3 \frac{(-1)^k x^{2k}}{(2k)!}$$

8. Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-3)^n}{n3^n}$$

Solution:

$$\lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}(x-3)^{n+1}}{(n+1)3^{n+1}}}{\dfrac{(-1)^{n+1}(x-3)^n}{n3^n}} \right| = \lim_{n\to\infty} \frac{|x-3|n}{3(n+1)} = \frac{|x-3|}{3}$$

So $\dfrac{|x-3|}{3} < 1 \quad \Rightarrow \quad |x-3| < 3$

So radius $= 3$

$$x = 3 \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

$$x = 6 \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges}$$

So the interval of convergence is $(0, 6]$.

B U Department of Mathematics

Math 101 Calculus I

Spring 2004 Final Exam

1. Evaluate the following limits, if they exist (justify your answer).

a) $\lim_{n\to\infty} \frac{1}{n}(e^{1/n} + e^{2/n} + ... + e^{\frac{(n-1)}{n}} + e^{n/n})$, (Hint: Think of Riemann sums)

Solution:

For $f(x) = e^x, \qquad 0 \le x \le 1$

$P_n = (0, 1/n, 2/n, ..., n/n)$ regular n-partition of [0,1]

$\frac{1}{n}(e^{1/n} + e^{2/n} + ... + e^{n/n}) = \sum_{i=1}^{n} f(x_i)\frac{1}{n}, \qquad x_i = \frac{i}{n}$ a Riemann sum.

So, $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\frac{1}{n} = \int_0^1 e^x dx = e - 1$

b) $\lim_{x\to 2^+} \frac{\ln(x-1)}{(x-2)^2}$

Solution:

We have $\frac{0}{0}$ indeterminacy. By L'Hôpital:

$$\lim_{x\to 2^+} \frac{\ln(x-1)}{(x-2)^2} = \lim_{x\to 2^+} \frac{\frac{1}{x-1}}{2(x-2)} = \lim_{x\to 2^+} \frac{1}{2(x-1)(x-2)} = \infty.$$

2. Evaluate the integrals

a) $\int \sqrt{1-e^x}dx$ **b)** $\int \sqrt{x(6-x)}dx$

Solution:

a) Put $u = \sqrt{1-e^x}$. So, $u^2 = 1 - e^x$ and $2udu = -e^x dx$ give $\frac{-2udu}{1-u^2} = dx$.
Hence we get:

$$\int \sqrt{1-e^x}dx = -2\int \frac{u^2}{1-u^2}du = 2\int \left(1 + \frac{1}{u^2-1}\right) du = 2u + \int \frac{2}{u^2-1}du.$$

Since $\frac{2}{u^2-1} = \frac{1}{u-1} - \frac{1}{u+1}$, we have:

$$\begin{aligned}\int \sqrt{1-e^x}dx &= 2u + \int \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du \\ &= 2u + \ln\left|\frac{u-1}{u+1}\right| + C \\ &= 2\sqrt{1-e^x} + \ln\left|\frac{\sqrt{1-e^x}-1}{\sqrt{1-e^x}+1}\right| + C.\end{aligned}$$

b) $$\begin{aligned}\sqrt{x(6-x)}dx &= \int \sqrt{-(x^2-6x+9)+9}dx \\ &= \int \sqrt{9-(x-3)^2}dx \\ &= \int 3\cos\theta(3\cos\theta)d\theta && 3\sin\theta = x-3 \Rightarrow 3\cos\theta d\theta = dx \\ &= 9\int \frac{1}{2}(1+\cos 2\theta)d\theta && 9-(x-3)^2 = 9\cos^2\theta \\ &= \frac{9}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C \\ &= \frac{9}{2}(\theta + \sin\theta\cos\theta) + C \\ &= \frac{9}{2}\left(\arcsin\left(\frac{x-3}{3}\right) + \frac{x-3}{3}\sqrt{6x-x^2}\right) + C.\end{aligned}$$

3. Find radius and interval of convergence of the power series $\sum_{n=0}^{\infty}(\frac{3}{4})^n(x+5)^n$.

Solution:

For $x_n = \left(\frac{3}{4}\right)^n (x+5)^n$,

$$\left|\frac{x_{n+1}}{x_n}\right| = \left(\frac{3}{4}\right)|x+5| \to \frac{3}{4}|x+5| \qquad \text{when} \qquad n \to \infty.$$

So by Ratio Test series converges if $|x+5| < \frac{4}{3}$

and series diverges if $|x+5| > \frac{4}{3}$

$\Rightarrow R = \frac{4}{3}$ (radius of convergence).

End-pts. checking:

For $x = -\frac{19}{3}$, $\quad \sum_{n=0}^{\infty} x_n = \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n\left(-\frac{4}{3}\right)^n = \sum_{n=0}^{\infty}(-1)^n$

and $x_n = (-1)^n \nrightarrow 0$ $\qquad$ when $\quad n \to \infty$.

So series diverges by general term test.

For $x = -\frac{11}{3}$, $\quad \sum_{n=0}^{\infty} x_n = \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n\left(\frac{4}{3}\right)^n = \sum_{n=0}^{\infty}(1)$.

Again this series diverges by general term test.

Hence $\left(-\frac{19}{3}, -\frac{11}{3}\right)$ is the interval of convergence.

4. Using Geometric series find power series expansion of $f(x)$ about 0 for $f(x) = \frac{5x}{2x^2-x-3}$.

Solution:

$$f(x) = \frac{5x}{2x^2-x-3} = \frac{5x}{(2x-3)(x+1)} = \frac{A}{2x-3} + \frac{B}{x+1}$$

$5x = A(x+1) + B(2x-3)$

$x = 0 \Rightarrow 0 = A - 3B$
$x = 1 \Rightarrow 5 = 2A - B$
$\Rightarrow A = 3, B = 1$

$$f(x) = \frac{3}{2x-3} + \frac{1}{x+1} = \frac{-1}{1-(\frac{2}{3})^x} + \frac{1}{1-(-x)}$$

For $\frac{1}{1-x} = \sum_{n=0}^{\infty}(x^n)$, $\quad |x| < 1 \quad$ (geometric series)

$x \leftrightarrow -x \qquad \frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n \qquad |x| < 1$

$x \leftrightarrow \frac{2}{3}x \qquad \frac{1}{1-\frac{2}{3}x} = \sum_{n=0}^{\infty}(\frac{2}{3})^n x^n \qquad |x| < \frac{3}{2}$

Then, for $|x| < 1$, the power series expansion is:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty}(-1)^n x^n - \sum_{n=0}^{\infty}(\frac{2}{3})^n x^n \\ &= \sum_{n=0}^{\infty}\left[(-1)^n - (\frac{2}{3})^n\right] x^n \\ &= \sum_{n=0}^{\infty}\left[\frac{(-3)^n - 2^n}{3^n}\right] x^n \end{aligned}$$

5. Use any method to determine whether the series converge.

a) $\sum_{1}^{\infty}(1-\frac{1}{n})^n$

Solution:

Since $x_n = \left(1 + \frac{(-1)}{n}\right)^n \to \frac{1}{e} \neq 0$ as $n \to \infty$, the series diverges by the n^{th} term test.

b) $\sum_{1}^{\infty}\sin(\frac{1}{n})$

Solution:

Observe that $\frac{x_n}{y_n} = \frac{\sin(\frac{1}{n})}{\frac{1}{n}} \to 1$ as $n \to \infty$. Since the harmonic series $\sum \frac{1}{n}$ diverges, given series diverges by Limit Comparison Test.

c) $\sum_{1}^{\infty}(-1)^{n+1}\frac{n+4}{n^2+n}$

Solution:

This is an alternating series because

- $\dfrac{n+4}{n^2+n} > 0$ for all n
- $\lim\limits_{n\to\infty} \dfrac{n+4}{n^2+n} = 0$
- For $f(x) = \dfrac{x+4}{x^2+x}$, we have $f'(x) = \dfrac{-(x^2+8x+4)}{(x^2+x)^2} < 0, \quad \forall x \geq 1.$

Hence $(\dfrac{n+4}{n^2+n})$ is decreasing. Thus series converges by Alternating Series Test.

6. Use Mean Value Theorem to show that:

a) $x - \sin x \geq 0, \quad$ for $0 \leq x \leq \dfrac{\pi}{2}$;

b) $f(x) = x\sin x - \dfrac{1}{2}\sin^2 x$ satisfies that $0 \leq f(x) \leq \dfrac{1}{2}(\pi - 1)$ for $0 \leq x \leq \dfrac{\pi}{2}$.

Solution:

a) For $g(x) = x - \sin x$ and $0 \leq x \leq \dfrac{\pi}{2}$, $g'(x) = 1 - \cos x > 0$. Hence g increases on $[0, \dfrac{\pi}{2}]$ by M.V.T. As $g(0) = 0$, it follows that $g(x) \geq 0$ for $0 \leq x \leq \dfrac{\pi}{2}$.

b) $f'(x) = \sin x + x\cos x - \dfrac{1}{2}2\sin x\cos x = \sin x + \cos x(x - \sin x)$.
By part (a), $(x - \sin x) \geq 0$.
Therefore $f'(x) > 0$ for $0 < x < \dfrac{\pi}{2}$, that is, f increases on $[0, \dfrac{\pi}{2}]$. It follows that $f(0) \leq f(x) \leq f(\dfrac{\pi}{2})$ for $x \in [0, \dfrac{\pi}{2}]$.
Since $f(0) = 0$ and $f(\dfrac{\pi}{2}) = \dfrac{1}{2}(\pi - 1)$, the result follows.

7. Given $\sin^2(xy) = \dfrac{1}{4}$, find $\dfrac{dy}{dx}$ at $x = 1$ and $y = \dfrac{\pi}{6}$.

Solution:

$2\sin(xy)\cos(xy)\big(y + x\dfrac{dy}{dx}\big) = 0$
For x=1 and $y = \dfrac{\pi}{6}$,
$2\sin(\dfrac{\pi}{6})\cos(\dfrac{\pi}{6})\big(\dfrac{\pi}{6} + \dfrac{dy}{dx}\big) = 0.$
$\sin\dfrac{\pi}{6} = \dfrac{1}{2}$ and $\cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2} \Rightarrow \dfrac{dy}{dx} = -\dfrac{\pi}{6}$

8. If $F(x) = \displaystyle\int_1^x f(t)dt$ where $f(t) = \displaystyle\int_1^{t^2} \frac{\sqrt{1+u^4}}{u}du$ then find $F''(2)$.

Solution:

$F'(x) = f(x) = \displaystyle\int_1^{x^2} \frac{\sqrt{1+u^4}}{u}du$ and $\Rightarrow F''(x) = f'(x) = \dfrac{\sqrt{1+x^8}}{x^2} \cdot 2x.$
So $F''(2) = \dfrac{\sqrt{1+2^8}}{4} \cdot 4 = \sqrt{257}.$

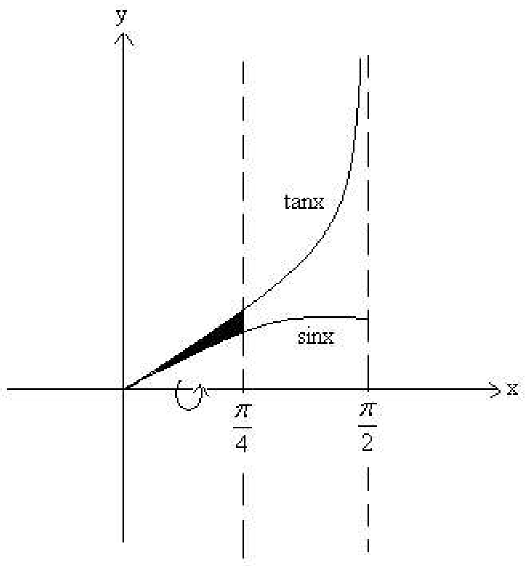
B U Department of Mathematics

Math 101 Calculus I

Spring 2005 Final Exam

Question 1 Find the volume of the solid obtained by revolving the region enclosed by the graphs of $f(x) = \tan x$, $g(x) = \sin x$ and $x = \dfrac{\pi}{4}$ about the $x-$axis.

Solution.



$\tan x = \sin x$ holds for $x = 0, k\pi, k = \pm 1, \pm 2, \ldots$ but only for $x = 0$ both graphs enclose a region with $x = \dfrac{\pi}{4}$. Then $V = \pi \displaystyle\int_0^{\pi/4} (\tan^2 x - \sin^2 x)dx$. Use $\tan^2 x = \sec^2 x - 1$ and $\sin^2 x = \dfrac{1 - \cos 2x}{2}$.

$$\begin{aligned}
V &= \pi \int_0^{\pi/4} (\sec^2 x - 1 - \frac{1 - \cos 2x}{2})dx \\
&= \pi \int_0^{\pi/4} (\sec^2 x - \frac{3}{2} + \frac{\cos 2x}{2})dx \\
&= \pi \left[\tan x - \frac{3}{2}x + \frac{\sin 2x}{4} \right]_0^{\pi/4} \\
&= \pi \left[\tan \frac{\pi}{4} - \frac{3}{2}\frac{\pi}{4} + \frac{\sin 2 \cdot \pi/4}{4} - (\tan 0 - 0 + \frac{\sin 0}{4}) \right] \\
&= \pi \left[1 - \frac{3\pi}{8} + \frac{1}{4} \right] \\
&= \pi \left[\frac{5}{4} - \frac{3\pi}{8} \right]
\end{aligned}$$

Question 2
Show that the function $f(x) = x^4 + 2x^3 - 2$ has *exactly* one zero in $[0, 1]$.

Solution. First note that $f(x)$ is everywhere continuous and differentiable being a polynomial.
$\left.\begin{array}{lcl} f(0) & = & -2 \\ f(1) & = & 1 \end{array}\right\} f(0) = 0$ and $f(1) > 0$ hence f, being continuous, has at least one zero in $(0, 1)$.
To have at least two zeros, $f' = 0$ must hold at some point in $(0, 1)$ as f is differentiable, and further f' must change sign.
Now check $f' = 4x^3 + 6x^2 = 0 \Rightarrow x = 0$ or $x = -3/2$.
Hence f', which is also continuous, never changes sign in $(0, 1)$. Namely f is either always increasing or decreasing. Now using the fact that $f(0) < 0$, $f(1) > 0$ we understand f is increasing on $[0, 1]$. Thus f has exactly one zero in $[0, 1]$.

Question 3 If $x \sin \pi x = \int_{\sqrt{x}}^{x^2} f(t)dt$, where f is a continuous function and $x > 0$, find $f(1)$.

Solution. We differentiate both sides by using the Fundamental Theorem of Calculus:
$\sin \pi x + \pi x \cos \pi x = f(x^2)2x - f(\sqrt{x})\frac{1}{2\sqrt{x}}$. Substituting $x = 1$: $\pi \cos \pi = 2f(1) - \frac{1}{2}f(1)$
$\Rightarrow -\pi = \frac{3}{2}f(1)$
$\Rightarrow f(1) = \frac{-2\pi}{3}$.

Question 4 Evaluate the following integrals:
(a) $\int \frac{x+4}{x^3+4x}\,dx$

Solution. Let $I = \int \frac{x+4}{x^3+4x}dx$.
$\frac{x+4}{x^3+4x} = \frac{x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$ by partial fractions
$\Rightarrow A(x^2+4) + (Bx+C)x = Ax^2 + 4A + Bx^2 + Cx = x + 4$ for each x
$\Rightarrow$

$$\begin{array}{rcl} A + B & = & 0 \\ C & = & 1 \\ 4A & = & 4 \Rightarrow A = 1 \Rightarrow B = -1 \end{array}$$

If $u = x^2 + 4$, then $2xdx = du$ and if $x = 2\tan\theta$, then $dx = 2\sec^2\theta$ and hence

$$I = \int (\frac{1}{x} - \frac{x}{x^2+4} + \frac{1}{x^2+4})dx = \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} + \int \frac{2\sec^2\theta d\theta}{4\sec^2\theta d\theta}$$

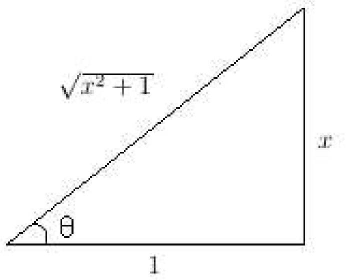
$$= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2}\theta + c$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\arctan\frac{x}{2} + c$$

$$= \ln\frac{|x|}{\sqrt{x^2+4}} + \frac{1}{2}\arctan\frac{x}{2} + c.$$

(b) $\displaystyle\int \frac{dx}{x\sqrt{1+x^2}}$

Solution. Let $I = \displaystyle\int \frac{dx}{x\sqrt{1+x^2}}$.



Let $x = \tan\theta$. Then $dx = \sec^2\theta$.

$$I = \int \frac{\sec^2\theta d\theta}{\tan\theta\sec\theta} = \int \frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}d\theta = \int \csc\theta d\theta = -\ln|\csc\theta + \cot\theta| + c.$$

$$\left.\begin{aligned} \csc\theta &= \frac{1}{\sin\theta} = \frac{\sqrt{x^2+1}}{x} \\ \cot\theta &= \frac{1}{x} \end{aligned}\right\} \Rightarrow I = -\ln\left|\frac{\sqrt{x^2+1}}{x} + \frac{1}{x}\right| + c.$$

Question 5 Evaluate the following definite integrals:

(a) $\displaystyle\int_0^{1/2} \frac{\arcsin x}{\sqrt{1+x}}\,dx$

Solution.
Call the integral expression I and integrate by parts. Let $u = \arcsin x$ and $dv = \frac{dx}{\sqrt{1+x}}$. Then $du = \dfrac{1}{\sqrt{1-x^2}}\,dx$ (valid since $0 \le x \le 1/2$; $v = 2\sqrt{1+x}$. So we get:

$$\begin{aligned} I &= uv|_0^{1/2} - \int_0^{1/2} v\,du \\ &= 2\sqrt{1+x}\arcsin x|_0^{1/2} - \int_0^{1/2} \frac{2\sqrt{1+x}}{\sqrt{1-x^2}}dx \\ &= 2\sqrt{\frac{3}{2}}\frac{\pi}{6} - \int_0^{1/2} \frac{2}{\sqrt{1-x}}dx \\ &= \sqrt{6}\frac{\pi}{6} + 4\sqrt{1-x}|_0^{1/2} \\ &= \frac{\pi}{\sqrt{6}} + 4\sqrt{\frac{1}{2}} - 4\sqrt{1} \\ &= \frac{\pi}{\sqrt{6}} + \frac{4}{\sqrt{2}} - 4. \end{aligned}$$

(b) $\int_1^\infty \frac{\ln x}{x^2}\,dx$ This is an improper integral equal to $= \lim_{A\to\infty} \int\limits_1^A \frac{\ln x}{x^2}$. Let $\ln x = u$, $dv = \frac{dx}{x^2}$, then $du = \frac{1}{x}dx$, $v = \frac{-1}{x}$. So by the method of integration by parts:

$$= \lim_{A\to\infty}\left[\frac{-1}{x}\ln x\Big|_1^A + \int\limits_1^A \frac{1}{x}\frac{1}{x}dx\right] = \lim_{A\to\infty}\left[\frac{-1}{A}\ln A - \frac{1}{x}\Big|_1^A\right] = \lim_{A\to\infty}\left[-\frac{\ln A}{A} - \frac{1}{A} + 1\right] = 1$$

since $\frac{\ln A}{A} \to 0$ and $\frac{1}{A} \to 0$ as $A \to \infty$.

Question 6 Determine whether the following series are convergent or divergent:

(a) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$

Take $\sum_{n=1}^{\infty} \frac{1}{n}$. Now $\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n}\sqrt[n]{n}} = 1$ where $1 \neq 0$ and $\neq \infty$. Hence both series converge and diverge together. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, so is $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$.

(b) $\sum_{n=0}^{\infty} \frac{(n+1)!}{3^n (n!)^2}$

This is typical ratio test: $a_n = \frac{(n+1)!}{3^n(n!)^2} > 0$

$$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{(n+2)!}{3^{n+1}(n+1)!(n+1)!}\cdot\frac{3^n n! n!}{(n+1)!} = \frac{n+2}{3(n+1)(n+1)} \to 0 \text{ as } n \to \infty.$$

Since $0 < 1$ the series is convergent.

(c) $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{2^n}$. If this series is convergent, find its sum.

This is sum of two geometric series one with $r = \frac{1}{2}$, the other with $r = -\frac{1}{2}$ hence for both r: $|r| < \frac{1}{2}$ is satisfied; therefore the series is convergent.

$$\sum_{n=1}^{\infty} \frac{2+(-1)^n}{2^n} = \sum_{n=1}^{\infty} \frac{2}{2^n} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} = \sum_{n=1}^{\infty} \frac{2}{2^n} + \sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^n$$

$$= 2\left[\sum_{n=0}^{\infty} \frac{1}{2^n} - 1\right] + \left[\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n - 1\right] = 2\left[\frac{1}{1-\frac{1}{2}} - 1\right] + \left[\frac{1}{1+\frac{1}{2}} - 1\right]$$

$$= 2(2-1) + \left(\frac{2}{3} - 1\right) = 2 - \frac{1}{3} = \frac{5}{3}.$$

Question 7

(a) Find the Taylor series of $f(x) = \ln(x+1)$ around the point $x = 0$.

Taylor series of $\ln(1+x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$ about $x = 0$.

$$\left.\begin{array}{rcl} f(0) & = & 0 \\ f'(0) & = & \left.\dfrac{1}{x+1}\right|_{x=0} = 1 \\ f''(0) & = & \left.\dfrac{-1}{(x+1)^2}\right|_{x=0} = -1 \\ f'''(0) & = & \left.\dfrac{2}{(x+1)^3}\right|_{x=0} = 2 \\ \vdots & & \\ f^{(n)}(0) & = & \left.\dfrac{(-1)^{n-1}\cdot(n-1)!}{(x+1)^n}\right|_{x=0} = \dfrac{(-1)^{n-1}\cdot(n-1)!}{1} \end{array}\right\} \Rightarrow \sum_{n=1}^{\infty}(-1)^{n-1}(n-1)!\frac{x^n}{n!} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

is the required Taylor series near $x = 0$.

(b) For which values of x is the Taylor series found above convergent?

First we apply absolute convergence test (ratio test):

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n}\right| = |x|\frac{n}{n+1} \to |x| \text{ as } n \to \infty.$$

$\Rightarrow$ The series is convergent when $|x| < 1$ and divergent when $|x| > 1$ by ratio test.
For $|x| = 1$, i.e. $x = \pm 1$ this test is inconclusive.

$\underline{x = -1}$: $$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(-1)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{2n-1}}{n} = -\underbrace{\sum_{n=1}^{\infty}\frac{1}{n}}_{\text{divergent harmonic series}}$$

Hence at $x = -1$ the series is divergent.

$\underline{x = +1}$: $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(1)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ is an alternating series with $\frac{1}{n} > 0$ and $\frac{1}{n} \to 0$ as $n \to \infty$.

Hence at $x = 1$ the series is convergent.

$\Rightarrow$ Interval of convergence is $(-1, 1]$.

B U Department of Mathematics

Math 101 Calculus I

Spring 2006 Final Exam

1. (a) State the Mean Value Theorem.

   (b) Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(t) = \begin{cases} 2 & \text{if} \quad t \leq -2 \\ t & \text{if} \quad t > -2 \end{cases}$

   Is the Mean Value Theorem satisfied in $[-2, 2]$? Explain.

   Solution:

   (a) For a function f, which is continuous on $[a, b]$ and differentiable on (a, b), there exits $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

   (b) Clearly

$$g'(t) = \begin{cases} 0 & \text{if} \quad t < -2 \\ 1 & \text{if} \quad t > -2. \end{cases}$$

   So, for all $t \in (-2, 2)$, $g'(t) = 1$. On the other hand, $\dfrac{g(2) - g(-2)}{2 - (-2)} = 0$, but there is no $c \in (-2, 2)$ for which $g'(c) = 0$. So the Mean Value Theorem is not satisfied for g.

   The reason for this fact is that g is not continuous at -2.

2. Let f be a function differentiable at $x = 0$ satisfying the relation $f(x + y) = f(x)f(y)$ and $f(0) = 1$. Find $f'(x)$ and $f''(x)$ in terms of $f'(0)$ and $f(x)$.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x)f(h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x)(f(h) - 1)}{h} \\ &= f(x) \lim_{h\to 0} \frac{f(h) - 1}{h} = f(x) \lim_{h\to 0} \frac{f(h) - f(0)}{h} = f(x)f'(0). \end{aligned}$$

$$\begin{aligned} f''(x) &= \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h\to 0} \frac{f(x+h)f'(0) - f(x)f'(0)}{h} \\ &= f'(0) \lim_{h\to 0} \frac{f(x+h) - f(x))}{h} = f'(0)f'(x) = f'(0)f(x)f'(0) \\ &= f(x)(f'(0))^2. \end{aligned}$$

3. Let

$$F(x) = \int_0^x \frac{1}{1+t^2}dt + \int_0^{1/x} \frac{1}{1+t^2}dt, \;\; x > 0.$$

Show that $F(x)$ is constant on $(0, \infty)$ and evaluate this constant value.

Solution:

Consider F'. Using the fundamental theorem of calculus

$$F'(x) = \frac{1}{1+x^2} - \frac{1}{x^2}\frac{1}{1+\frac{1}{x^2}} = \frac{1}{1+x^2} - \frac{1}{x^2+1} = 0.$$

Hence, $F'(x) = 0$ implies that $F(x)$ is constant, say C.
Integrating F we get that

$$F(x) = \arctan x + \arctan\frac{1}{x} = C = F(1).$$

But we have, $F(1) = \arctan 1 + \arctan 1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$. Therefore

$$F(x) = \frac{\pi}{2} \qquad \text{for all } x \in (0, \infty).$$

4. (a) Evaluate $\lim\limits_{x\to 1} \dfrac{1}{\ln x} - \dfrac{1}{x-1}$

Solution:

$$\lim_{x\to 1} \frac{1}{\ln x} - \frac{1}{x-1} = \lim_{x\to 1} \frac{(x-1)-\ln x}{(\ln x)(x-1)} = \lim_{x\to 1} \frac{1-\frac{1}{x}}{\frac{1}{x}(x-1)+\ln x} = \lim_{x\to 1} \frac{1-\frac{1}{x}}{1-\frac{1}{x}+\ln x} =$$
$$\lim_{x\to 1} \frac{\frac{1}{x^2}}{\frac{1}{x^2}+\frac{1}{x}} = \lim_{x\to 1} \frac{\frac{1}{x^2}}{\frac{1+x}{x^2}} = \lim_{x\to 1} \frac{1}{1+x} = \frac{1}{2}$$ by applying L'Hospital Rule twice.

(b) Test the convergence (absolute and conditional) of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sqrt{n}}{n+1}$

Solution:

Clearly, $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}\sqrt{n}}{n+1} \right| = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$. Now, we will apply limit comparison test with $\sum \frac{1}{\sqrt{n}}$. Since

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}}{\frac{\sqrt{n}}{n+1}} = \lim_{n\to\infty} \frac{n}{n+1} = 1,$$

and since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent, $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$ is divergent. Thus the series is not absolutely convergent.

Next, consider $f(x) = \dfrac{\sqrt{x}}{x+1}$. Then $f'(x) = \dfrac{\frac{1}{2\sqrt{x}}(x+1)-\sqrt{x}}{(x+1)^2} = \dfrac{x+1-2x}{2\sqrt{x}(x+1)^2} = \dfrac{1-x}{2\sqrt{x}(x+1)^2} \leqslant 0$ if $x \geqslant 0$. Hence $f(n) = \dfrac{\sqrt{n}}{n+1}$ is decreasing. Also $\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = 0$ and $\dfrac{\sqrt{n}}{n+1} > 0$, so by the alternating series test, the given series is convergent.

5. Evaluate

(a) $\int x^{-2} \ln x dx$

Solution:

Let $I = \int x^{-2} \ln x dx$. Taking $u = \ln x$, $dv = \dfrac{dx}{x^2}$, we have $du = \dfrac{1}{x} dx$ and $v = -\dfrac{1}{x}$. Hence, using integration by parts formula we get

$$I = -\frac{\ln x}{x} + \int \frac{1}{x}\frac{1}{x} dx = -\frac{\ln x}{x} - \frac{1}{x} + C.$$

(b) $\int \dfrac{x^2}{\sqrt{9 - x^2}} dx$

Solution:

As above, let $I = \int \dfrac{x^2}{\sqrt{9 - x^2}} dx$. Using the trigonometric substitution $\sin\theta = \dfrac{x}{3}$, with $3\cos\theta d\theta = dx$, we get

$$\begin{aligned} I &= \int \frac{9\sin^2\theta}{\sqrt{9 - 9\sin^2\theta}} 3\cos\theta d\theta = \int 9\sin^2\theta d\theta \\ &= \int 9\left(\frac{1 - \cos 2\theta}{2} d\theta\right) = \frac{9}{2}\theta - \frac{9}{2}\frac{\sin 2\theta}{2} + C. \end{aligned}$$

Since $\sin 2\theta = 2\sin\theta\cos\theta$, substituting back, we get

$$I = \frac{9}{2}\arcsin\frac{x}{3} - \frac{9}{2}\frac{x}{3}\frac{\sqrt{9 - x^2}}{3} + C = \frac{9}{2}\arcsin\frac{x}{3} - \frac{x}{2}\sqrt{9 - x^2} + C$$

6. Find the sum of the following series

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!}$

Solution:

$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \left(\frac{\pi}{4}\right)^{2n+1} \frac{1}{(2n+1)!} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$. Since the MacLaurin series of $\sin x$ is given by $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$.

(b) $\displaystyle\sum_{k=2}^{\infty} \frac{2}{4k^2 - 8k + 3}$

Solution:

Consider

$$\frac{2}{4k^2 - 8k + 3} = \frac{2}{(2k-1)(2k-3)} = \frac{A}{2k-1} + \frac{B}{2k-3}$$
$$A(2k-3) + B(2k-1) = 2.$$

Now, the coefficient of k is $2(A+B) = 0$ and the constant term is $-3A - B = 2$. Solving these two equations for A and B we get $A = -1$, $B = 1$. Hence

$$\frac{2}{4k^2 - 8k + 3} = -\frac{1}{2k-1} + \frac{1}{2k-3}$$

See that $s_n = \left(-\frac{1}{3} + 1\right) + \left(-\frac{1}{5} + \frac{1}{3}\right) + \left(-\frac{1}{7} + \frac{1}{5}\right) + \cdots + \left(-\frac{1}{2n-1} + \frac{1}{2k-3}\right)$, hence cancelling same terms, we get

$$s_n = 1 - \frac{1}{2n-1}.$$

Therefore,

$$\sum_{k=2}^{\infty} \frac{2}{4k^2 - 8k + 3} = \lim_{n\to\infty} s_n = 1.$$

7. Test the convergence (absolute and conditional) of the following series. In each case give a reason for your decision.

(a) $\sum_{k=0}^{\infty} (-1)^k \frac{k}{\sqrt{1+k+k^2}}$

Solution:

Since $\lim_{k\to\infty} (-1)^k \frac{k}{\sqrt{1+k+k^2}} \neq 0$ the series diverges by the n'th term test.

(b) $\sum_{k=0}^{\infty} 2^{-k} \sin^2(e^{2k})$

Solution:

Note that $\frac{1}{2^k} \sin^2\left(e^{2k}\right) \leqslant \frac{1}{2^k}$ and $\sum_k \frac{1}{2^k}$ is convergent since it is a geometric series as $\frac{1}{2} < 1$. Thus by the comparison test, the given series converges.

(c) $\sum_{k=0}^{\infty} \left(1 + (-1)^k\right) \frac{3^k}{(k-1)!}$

Solution:

Since $\lim_{k\to\infty} \frac{3^{k+1}}{k!} \frac{(k-1)!}{3^k} = \lim_{k\to\infty} \frac{3}{k} = 0 < 1$, $\sum_k \frac{3^k}{(k-1)!}$ is convergent. Thus $\sum_k (-1)^k \frac{3^k}{(k-1)!}$ is absolutely convergent, and so $\sum_k \left(1 + (-1)^k\right) \frac{3^k}{(k-1)!}$ is convergent.

8. Find the interval of convergence of the power series

$$\sum_{k=0}^{\infty} \frac{(-1)^k (x-2)^k}{2^k (k+1)^{3/4}}$$

Solution:

We have $\lim_{k\to\infty} \left| \frac{(x-2)^{k+1}}{2^{k+1}(k+2)^{3/4}} \frac{2^k(k+1)^{3/4}}{(x-2)^k} \right| = \lim_{k\to\infty} \left| \frac{(x-2)}{2} \frac{(k+1)^{3/4}}{(k+2)^{3/4}} \right| = \frac{|x-2|}{2} < 1.$

Thus the series converges absolutely for $|x-2| < 2$, that means for $-2 < x-2 < 2$, hence for $0 < x < 4$.

Now we will check the boundaries $x = 0$ and $x = 4$.

For $x = 0$, $\sum_k \frac{(-1)^k(-2)^k}{2^k(k+1)^{3/4}} = \sum_k \frac{1}{(k+1)^{3/4}}$, which is divergent by the limit comparison test with $\sum_k \frac{1}{k^{3/4}}$. To see this, consider $\lim_{k\to\infty} \frac{\frac{1}{k^{3/4}}}{\frac{1}{(k+1)^{3/4}}} = 1$. As $\frac{3}{4} < 1$, $\sum_k \frac{1}{k^{3/4}}$ is divergent, hence so is $\sum_k \frac{1}{(k+1)^{3/4}}$.

For $x = 4$, $\sum_k \frac{(-1)^k 2^k}{2^k(k+1)^{3/4}} = \sum_k \frac{(-1)^k}{(k+1)^{3/4}}$. Let $f(x) = \frac{1}{(x+1)^{3/4}}$. Then $f'(x) = -\frac{3}{4}(x+1)^{-1/4} < 0$, so f is decreasing. Also $\lim_{k\to\infty} \frac{1}{(k+1)^{3/4}} = 0$. Hence by the alternating series test $\frac{(-1)^k}{(k+1)^{3/4}}$ converges.

Therefore the interval of convergence is $(0, 4]$.

B U Department of Mathematics

Math 101 Calculus I

Summer 2004 Final Exam

1. Find the length of the curve $y = (\frac{x}{2})^{\frac{2}{3}},\ 0 \le x \le 2.$

Solution:

L=$\displaystyle\int_0^2 \sqrt{1+(y')^2}\,dx$, where $y' = \dfrac{2}{3}\left(\dfrac{x}{2}\right)^{-\frac{1}{3}}\dfrac{1}{2}$. So,

$$L = \lim_{a\to 0^+}\left(\int_a^2 \frac{\sqrt{9x^{\frac{2}{3}}+4^{\frac{1}{3}}}}{3x^{\frac{1}{3}}}\,dx\right)$$

Now take $u = 9x^{\frac{2}{3}} + 4^{\frac{1}{3}}$ and hence $du = 6x^{-\frac{1}{3}}\,dx$. Then,

$$\begin{aligned}
L &= \int \frac{\sqrt{u}}{18}\,du \\
&= \frac{1}{27}u^{\frac{3}{2}} \\
&= \lim_{a\to 0^+}\left(\frac{1}{27}\left(9x^{\frac{2}{3}}+4^{\frac{1}{3}}\right)^{\frac{3}{2}}\Big]_a^2\right) \\
&= \frac{1}{27}\lim_{a\to 0^+}\left(\left(9.2^{\frac{2}{3}}+2^{\frac{2}{3}}\right)^{\frac{3}{2}} - \left(9.a^{\frac{2}{3}}+4^{\frac{1}{3}}\right)^{\frac{3}{2}}\right) \\
&= \frac{1}{27}\left(10^{\frac{3}{2}}.2 - \lim_{a\to 0^+}\left(9a^{\frac{2}{3}}+4^{\frac{1}{3}}\right)^{\frac{3}{2}}\right) \\
&= \frac{1}{27}\left(10^{\frac{3}{2}}.2 - 16\right)
\end{aligned}$$

2. Find $f(x)$ if $\displaystyle\int_1^x f(t)\,dt = x^2 - 2x + 1$

Solution:

Taking the derivative of both sides, we get

$$\begin{aligned}
\frac{d}{dx}\left(\int_1^x f(t)\,dt\right) &= \frac{d}{dx}\left(x^2-2x+1\right) \\
f(x) &= 2x-2
\end{aligned}$$

3. Derive a reduction formula for $\int x^n \cos x\,dx$.

Solution:

Let $u = x^n$ and $dv = \cos x\, dx$, hence $du = nx^{n-1}\, dx$ and $v = \sin x$. Then:

$$\begin{aligned}\int x^n \cos x\, dx &= x^n \sin x - \int \sin x\, nx^{n-1}\, dx \\ &= x^n \sin x - n \int \sin x\, x^{n-1}\, dx\end{aligned}$$

Now take $u = x^{n-1}$ and $dv = \sin x\, dx$, hence $du = (n-1)x^{n-2}\, dx$ and $v = -\cos x$. Then we obtain,

$$\begin{aligned}\int x^n \cos x\, dx &= x^n \sin x - n \int \sin x\, x^{n-1}\, dx \\ &= x^n \sin x - n\left(-x^{n-1}\cos x + \int \cos x (n-1)x^{n-2}\, dx\right) \\ &= x^n \sin x + nx^{n-1}\cos x - n(n-1)\int x^{n-2}\cos x\, dx\end{aligned}$$

4. Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{4^n n! n!}{(2n)!}$ convergent or divergent? Justify your answer.

Solution:

$$\begin{aligned}\frac{a_{n+1}}{a_n} = \frac{4^{n+1}(n+1)!(n+1)!(2n)!}{(2n+2)!4^n n! n!} &= \frac{4^n \times 4 \times (n+1) \times n!(n+1) \times n!(2n)!}{4^n \times n!n!(2n+2) \times (2n+1) \times (2n)!} \\ &= \frac{4(n+1)^2}{(2n+2)(2n+1)} \longrightarrow 1\end{aligned}$$

as $n \to \infty$. So ratio test does not apply.

Observe that:

$$\frac{a_{n+1}}{a_n} = \frac{4(n+1)^2}{(2n+2)(2n+1)} = \frac{4n^2+8n+4}{4n^2+6n+2} = \frac{4n^2+6n+2+2n+2}{4n^2+6n+2} = 1 + \frac{2n+2}{4n^2+6n+2} > 1$$

for $n \geq 1$. Therefore $a_{n+1} \geq a_n$ and $a_1 = 2$.
So $\lim_{n\to\infty} a_n \neq 0$ and series diverges by nth term test.

5. **(a)** Suppose $x^3 - 2x + 4 = a_0 + a_1(x-2) + a_2(x-2)^2 + \cdots + a_n(x-2)^n + \ldots\ldots$
Find $a_0, a_1, \ldots$. Where does the series converge?

Solution:

$f(x) = x^3 - 2x + 4 = a_0 + a_1(x-2) + a_2(x-2)^2 + \ldots\ldots$
Then $a_n = \dfrac{f^{(n)}(2)}{n!}$.

$$a_0 = f(2) = 8, \quad a_1 = f'(2) = 10, \quad a_2 = \frac{f''(2)}{2!} = 6, \quad a_3 = \frac{f'''(2)}{3!} = 1$$

$a_n = 0$ if $n \geq 4$.
The series converges everywhere since $f(x)$ is a polynomial.

(b) Suppose $\dfrac{x^2}{(1-x)^2} = b_0 + b_1 x + b_2 x^2 + \ldots$. Find $b_0, b_1, \ldots$. Where does the series converge?

Solution:

We know that,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \qquad |x| < 1$$

By differentiating, we get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots \qquad |x| < 1.$$

Multiplying with x^2 gives,

$$\frac{x^2}{(1-x)^2} = x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + \ldots \qquad |x| < 1.$$

So, $a_0 = 0, \quad a_1 = 0, \quad a_2 = 1, \quad a_3 = 2, \quad a_4 = 3, \quad \ldots \quad a_n = n - 1, \quad$ for $n \geq 1$.

Power series converges for $|x| < 1$.

B U Department of Mathematics

Math 101 Calculus I

Summer 1999 Final Exam

1. Using the definition of the derivative evaluate $f'(0)$ if

$$f(x) = \begin{cases} xe^{-1/x^2} & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$$

Solution:

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{he^{-1/h^2} - 0}{h} \\ &= \lim_{h\to 0} \frac{1}{e^{1/h^2}} \\ &= 0 \end{aligned}$$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that, for all $x, y \in \mathbb{R}$

$$f(x + y) = f(x) + f(y).$$

a) Show that $f(0) = 0$.

b) Show that if f is continuous at 0, then f must be continuous at every x in $\mathbb{R}$.

Solution:

a) $f(0) = f(0 + 0) = f(0) + f(0) \Rightarrow f(0) = f(0) - f(0) = 0$.

b) It is enough to show that for any $x_0 \in \mathbb{R}$, $f(x_0) = \lim_{x\to x_0} f(x)$. Put $y = x - x_0$ then

$$\begin{aligned} \lim_{x\to x_0} f(x) &= \lim_{y\to 0} f(y + x_0) \\ &= \lim_{y\to 0} [f(y) + f(x_0)] \\ &= \left[\lim_{y\to 0} f(y)\right] + f(x_0) \end{aligned}$$

Since f is continuous at $x = 0$, $\lim_{y\to 0} f(y) = f(0) = 0$. So $\lim_{x\to x_0} f(x) = f(x_0)$ for all x_0. Hence f is continuous everywhere.

3. Prove that if $0 \leq x \leq 1$ then $\ln(1 + x) \leq \arctan(x)$. Hint : You can express each function as a definite integral over the interval $[0, x]$.

Solution:

$\arctan x = \int_0^x \frac{1}{1+x^2}dx$ and $\ln(1+x) = \int_0^x \frac{1}{1+x}dx$. Then,

$$\begin{aligned} \arctan x - \ln(1+x) &= \int_0^x \left[\frac{1}{1+x^2} - \frac{1}{1+x}\right] dx \\ &= \int_0^x \frac{x-x^2}{(1+x^2)(1+x)}dx \end{aligned}$$

Since $\frac{x-x^2}{(1+x^2)(1+x)} \geq 0$ on $[0,1]$, $\int_0^x \frac{x-x^2}{(1+x^2)(1+x)}dx \geq 0$ when $x \in [0,1]$.
So $\arctan x \geq \ln(1+x)$ on [0,1].

4. Determine the convergence of the following series :

a) $\sum_{n=1}^{\infty} \sin(\frac{1}{n})$

b) $\sum_{n=1}^{\infty} (\frac{1}{2}-1)(\frac{1}{3}-1)\ldots(\frac{1}{n}-1)$

c) $\sum_{n=1}^{\infty} n^{-n}$

Solution:

a) Since $\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, then by limit comparison test $\sum_{n=1}^{\infty} \sin(\frac{1}{n})$ is divergent.

b) $\sum_{n=1}^{\infty} (\frac{1}{2}-1)(\frac{1}{3}-1)\ldots(\frac{1}{n}-1) = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n}$.

The series is alternating. We use the alternating series test:

i) $\frac{1}{n}$ is decreasing and

ii) $\lim_{n\to+\infty} \frac{1}{n} = 0$.

So the series converges.

c) Since $\frac{1}{n^n} \leq \frac{1}{2^n}$ when $n \geq 2$ and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent, $\sum_{n=1}^{\infty} n^{-n}$ is convergent.

5. Find the interval of convergence of the power series $\sum_{k=2}^{\infty} \frac{(-1)^k x^k}{k \ln k}$.

Solution:

By ratio test :

$$\lim_{k\to\infty}\left|\frac{\frac{(-1)^{k+1}x^{k+1}}{(k+1)\ln(k+1)}}{\frac{(-1)^k x^k}{(k)\ln(k)}}\right| = \lim_{k\to\infty}\frac{k\ln(k)}{(k+1)\ln(k+1)}|x|$$

$\displaystyle\lim_{k\to\infty}\frac{k\ln(k)}{(k+1)\ln(k+1)} = \lim_{k\to\infty}\frac{\ln(k)+1}{\ln(k+1)+1} = \lim_{k\to\infty}\frac{k+1}{k} = 1$ by l'Hôpital's rule. So series converges when $|x| < 1$.

Now look at the end points: At $x = -1$, by integral test, the series diverges and at $x = 1$ it converges absolutely so converges, by alternating series test.

Then Series converges on (-1,1].

6. Find the MacLaurin series for $f(x) = \ln(4 + x)$ and determine its radius and interval of convergence.

Solution:

$$\begin{aligned} f'(x) &= (x+4)^{-1} \\ f''(x) &= -(x+4)^{-2} \\ f'''(x) &= 2(x+4)^{-3} \\ &\vdots \\ f^{(n)}(x) &= (n-1)!(-1)^{n+1}(x+4)^{-n} \end{aligned}$$

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n &= \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(4)^{-n}}{n}x^n \\ &= \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n4^n}(x)^n \end{aligned}$$

The equation

$$\rho = \lim_{n\to\infty}\frac{\left|\frac{(-1)^{n+2}x^{n+1}}{(n+1)4^{n+1}}\right|}{\left|\frac{(-1)^{n+1}x^n}{n4^n}\right|} = \lim_{n\to\infty}\frac{n}{4(n+1)}|x| = \frac{|x|}{4}$$

and $\rho < 1$ gives that $|x| < 4$. Moreover, at $x = 4$ the series diverges and at $x = -4$ it converges. Hence the interval of convergence is $(-4, 4]$.